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Diffusion with nonlocal Dirichlet boundary conditions on unbounded domains

by

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Abstract. We consider a second order differential operator $\mathscr{A}$ on an open and Dirichlet regular set $\Omega \subset \mathbb{R}^d$ (typically unbounded) and subject to nonlocal Dirichlet boundary conditions of the form

$$u(z) = \int_\Omega u(x)\, \mu(z, dx) \qquad \text{for } z \in \partial\Omega.$$

Here, $\mu : \partial\Omega \to \mathscr{M}(\Omega)$ takes values in the probability measures on Ω and is continuous in the weak topology $\sigma(\mathscr{M}(\Omega), C_b(\Omega))$. Under suitable assumptions on the coefficients of $\mathscr{A}$, which may be unbounded, we prove that a realization A_μ of $\mathscr{A}$ subject to the above nonlocal boundary condition generates a (not strongly continuous) semigroup on $L^\infty(\Omega)$. We establish a sufficient condition for this semigroup to be Markovian and prove that in this case, it enjoys the strong Feller property. We also study the asymptotic behavior of the semigroup.

1. Introduction. There is a well-known connection between Markov processes on the one hand and parabolic partial differential equations and Markovian semigroups on the other. Starting with the seminal work of Feller [14, 15], who studied the one-dimensional situation, this connection has developed into a rich and active field of scientific research. In this article, we seek to combine two aspects of this field which, over time, have received much attention: *nonlocal boundary conditions* and *unbounded coefficients.*

We shall consider second order differential operators $\mathscr{A}$ on an open subset Ω of $\mathbb{R}^d$, formally given by

$$\mathscr{A}u := \sum_{i,j=1}^d a_{ij} D_i D_j u + \sum_{j=1}^d b_j D_j u. \tag{1.1}$$

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In the typical applications we have in mind, the set Ω is unbounded and the coefficients a_{ij} and b_j are functions on Ω which may be unbounded as $|x| \to \infty$ within Ω. We will study a realization A_μ of $\mathscr{A}$ subject to nonlocal Dirichlet boundary conditions of the form

$$u(z) = \int_\Omega u(x)\,\mu(z,dx) \tag{1.2}$$

for all $z \in \partial\Omega$. In this equation, for every $z \in \Omega$ we are given a probability measure $\mu(z,\cdot)$ on Ω. Nonlocal boundary conditions of this form arise naturally in applications, e.g. in financial mathematics [23], for entropy in models of thermoelasticity [13], for heat conduction in "well-stirred liquids" [8], or in functional differential equations [33].

This boundary condition has a clear probabilistic interpretation. Whenever a diffusing particle reaches the boundary of Ω at a point z, it immediately jumps back to the interior of Ω. The point to which it jumps is chosen randomly according to the probability measure $\mu(z,\cdot)$. Thus, this boundary condition models what Feller [15] called an *instantaneous return process.*

On bounded domains, nonlocal boundary conditions of this form were considered by several authors, using different approaches [3, 6, 7, 17, 34, 35, 36]. We should point out that these boundary conditions fall into the so-called "nontransversal case" where the nonlocal term is of highest order in the boundary condition, since only terms of order 0 appear in the boundary condition. As a consequence, we cannot expect to obtain a strongly continuous semigroup on the space $C_b(\overline{\Omega})$ of bounded continuous functions on $\overline{\Omega}$. Thus, to use strongly continuous semigroups, one has to either work on the L^p-scale (as in [6, 7]) or to consider a closed subspace (heavily depending on the measure μ) of the space of bounded continuous functions (as was the case in [17, 34, 35]). The drawback of both approaches is that it is not clear how to extract transition probabilities from these semigroups. In [3] we proved generation of an analytic semigroup on $L^\infty(\Omega)$. This semigroup is not strongly continuous but it enjoys the strong Feller property (so that in particular the semigroup is given through transition probabilities). We point out that in the case of nonlocal Robin boundary conditions (which, due to the presence of the normal derivative in the boundary condition which is of order 1, fall into the "transversal case") we do obtain strong continuity and analyticity of the semigroup on the space of bounded continuous functions [4].

In contrast to the situation on *bounded domains* we cannot expect analyticity of the semigroup for differential operators with unbounded coefficients on unbounded domains. This can already be seen in the prototype example of the Ornstein–Uhlenbeck semigroup [11]. Thus, in this article, one of the main obstacles to overcome is the choice of an appropriate semigroup setting, in which we can handle semigroups that are neither strongly continuous nor analytic. To that end, we will introduce the concept of a $*$-*semigroup* (see Section 2). Even though such semigroups consist of adjoint operators, they are a priori not adjoint semigroups in the sense of [30]. This is due to the fact that the orbits need not be weak*-continuous at 0, so that such a semigroup need not be the adjoint of a strongly continuous semigroup. While the semigroups we will consider have no continuity at 0, the regularity of the orbits for $t > 0$ is quite good as a consequence of the strong Feller property (see Theorem 3.7).

Our basic strategy to tackle the problem on unbounded domains is the same as in [28], namely, we approximate the elliptic problem on unbounded domains by problems on bounded domains. This was done in [28] for operators on all of $\mathbb{R}^d$. In the case of unbounded domains also Dirichlet condition [16] and Neumann conditions [9, 10] were considered. We point out that in the cited articles the parabolic problem for $\mathscr{A}$ was treated independently of the elliptic problem, heavily using Schauder theory for parabolic equations on bounded domains. However, in the Schauder approach higher regularity of the boundary and of the coefficients is needed. Even worse, in Schauder boundary estimates also Hölder regularity of the boundary data is needed. In our situation, these boundary data are given via equation (1.2). If u is continuous in the interior of Ω, then the boundary data are also continuous. However, Hölder continuity cannot be expected.

Thus, in this article we use a different approach which is abstract and closer in spirit to semigroup theory in that we obtain all information about the parabolic equation by studying the corresponding elliptic problem. Our main tool is a monotone convergence theorem for $*$-semigroups (Proposition 2.12).

Let us now specify our assumptions and state our main result. We refer to Section 4 for unexplained terminology.

Hypothesis 1.1. Throughout, let $\emptyset \neq \Omega \subset \mathbb{R}^d$ be an open and Dirichlet regular set. Concerning the coefficients in (1.1), we assume that $a_{ij} \in C(\overline{\Omega})$ and $b_j \in L^\infty_{\mathrm{loc}}(\overline{\Omega})$ are real-valued for $i, j = 1, \ldots, d$. The diffusion coefficients a_{ij} are assumed to be symmetric (i.e. $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, d$) and strictly elliptic in the sense that there is a function $\eta \in C(\overline{\Omega})$ with $\eta(x) > 0$ for every $x \in \overline{\Omega}$ such that for all $\xi \in \mathbb{R}^d$ we have

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \eta(x)|\xi|^2 \quad \text{for every } x \in \overline{\Omega}.$$

In addition, we assume that either

(i) the coefficients a_{ij} are locally Dini-continuous for $i, j = 1, \ldots, d$, or
(ii) for every $n \in \mathbb{N}$ the set $\Omega \cap B_n(0)$ satisfies the uniform exterior cone condition. Here, $B_n(0)$ denotes the Euclidean ball of radius n centered at 0.

Above, $L^\infty_{\mathrm{loc}}(\overline{\Omega})$ is the space of all functions that are essentially bounded on compact subsets of $\overline{\Omega}$. Thus the drift coefficients b_j and the diffusion coefficients a_{ij} may be unbounded as $|x| \to \infty$, but they may *not* explode near $\partial\Omega$. Likewise, the ellipticity constant η may degenerate to 0 as $|x| \to \infty$, but not near the boundary.

Next, we specify our assumptions concerning the boundary condition. We denote the Borel σ-algebra on Ω by $\mathscr{B}(\Omega)$ and the space of (signed) Borel measures on Ω by $\mathscr{M}(\Omega)$.

HYPOTHESIS 1.2. We let $\mu : \partial\Omega \times \mathscr{B}(\Omega) \to \mathscr{M}(\Omega)$. We will sometimes write

$$\mu(z) := \mu(z, \cdot) \in \mathscr{M}(\Omega).$$

We assume that

(i) $\mu(z)$ is a probability measure for every $z \in \partial\Omega$;
(ii) the map $z \mapsto \mu(z)$ is $\sigma(\mathscr{M}(\Omega), C_b(\Omega))$-continuous.

Here, $\sigma(\mathscr{M}(\Omega), C_b(\Omega))$ is the weak topology on $\mathscr{M}(\Omega)$ induced by the bounded continuous functions. Thus, condition (ii) is equivalent to asking that the map

$$z \mapsto \int_\Omega f \, d\mu(z)$$

is continuous for every $f \in C_b(\Omega)$.

As in [3], given an open set $U \subset \mathbb{R}^d$, we set

$$W(U) := \bigcap_{1<p<\infty} W^{2,p}_{\mathrm{loc}}(U).$$

By elliptic regularity [22, Lemma 9.16], we have $u \in W(\Omega)$ whenever $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for some $1 < p < \infty$ and $\mathscr{A}u \in L^\infty_{\mathrm{loc}}(\Omega)$. We now complement our differential operator $\mathscr{A}$ with nonlocal boundary conditions of the form (1.2). To that end, we define the *maximal domain* $D_{\max}$ by

$$D_{\max} := \Big\{ u \in C_b(\overline{\Omega}) \cap W(\Omega) : \mathscr{A}u \in L^\infty(\Omega), \\ u(z) = \int_\Omega u(x)\, \mu(z, dx) \ \forall z \in \partial\Omega \Big\}.$$

Our main result is as follows.

THEOREM 1.3. *Assume Hypotheses* 1.1 *and* 1.2. *Then there is a subspace* $D(A_\mu)$ *of* $D_{\max}$ *such that the operator* $A_\mu : D(A_\mu) \to L^\infty(\Omega)$, $A_\mu u = \mathscr{A}u$, *has the following properties:*

(a) $(0, \infty) \subset \rho(A_\mu)$ *and* $R(\lambda, A_\mu)$ *is a positive operator on* $L^\infty(\Omega)$ *which satisfies* $\|\lambda R(\lambda, A_\mu)\| \le 1$ *for all* $\lambda > 0$.

(b) *For every* $\lambda > 0$ *and* $f \in L^\infty(\Omega)_+$ *the function* $u := R(\lambda, A_\mu)f$ *is the smallest positive solution of the equation* $\lambda u - \mathscr{A}u = f$ *in* $D_{\max}$.

(c) A_μ *generates a positive and contractive* $*$*-semigroup* $T_\mu = (T_\mu(t))_{t\geq 0}$ *on* $L^\infty(\Omega)$.

(d) $D(A_\mu) = D_{\max}$ *if and only if* $\mathbb{1} \in D(A_\mu)$. *In this case the semigroup* T_μ *enjoys the strong Feller property.*

(e) *If* $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$ *then there is at most one invariant probability measure for the semigroup* T_μ. *If there is such a measure* $\nu^\star$, *then for every* $f \in L^\infty(\Omega)$ *we have*

$$\lim_{t\to\infty} T_\mu(t)f = \int_{\overline{\Omega}} f\, d\nu^\star \cdot \mathbb{1}$$

uniformly on compact subsets of $\overline{\Omega}$, *whereas for the adjoint semigroup* T'_μ *on* $\mathscr{M}(\overline{\Omega})$ *we have, for every* $\nu \in \mathscr{M}(\overline{\Omega})$,

$$\lim_{t\to\infty} T'_\mu(t)\nu = \nu(\overline{\Omega})\nu^\star$$

in total variation norm.

As we are dealing with elliptic equations with unbounded coefficients, we cannot expect uniqueness for the solution of the associated elliptic equation in general, so that we may have several solutions of the elliptic equation $\lambda u - \mathscr{A}u = 0$ in $D_{\max}$. As A_μ is a bijection between $D(A_\mu)$ and $L^\infty(\Omega)$, part (d) of Theorem 1.3 characterizes unique solvability. As is to be expected, we can establish this unique solvability making use of an appropriate Lyapunov function (see Corollary 6.6). We point out that our assumptions on the Lyapunov function in Corollary 6.6 do not involve the boundary condition (though we have to additionally make a weak concentration assumption on the measures μ) so that Lyapunov functions can be constructed as in [28], imposing suitable growth conditions on the coefficients.

Lyapunov functions can also be used to establish existence of an invariant measure. However, typically the assumptions on such a Lyapunov function are more restrictive than in the case where we merely want to establish uniqueness for the elliptic equation. In our situation, we need to involve the boundary condition in our requirements on the Lyapunov function to ensure existence of an invariant measure (see Theorem 8.3).

In Section 9, we present concrete examples where we can construct Lyapunov functions and thus establish existence of an invariant probability measure. In these examples, Ω is an outer domain and the differential operator $\mathscr{A}$ has coefficients which grow polynomially.

This article is organized as follows. In Section 2 we introduce the notion of a $*$-semigroup on the dual of a separable Banach space and prove some results to be used later on. Section 3 is concerned with the notion of "kernel operator" and the strong Feller property. These two sections might also be of independent interest and are presented in an abstract framework. After recalling in Section 4 some results concerning diffusions with nonlocal boundary conditions on bounded domains, we study the elliptic equation $\lambda u - \mathscr{A} u = f$ in Section 5. There we prove parts (a) and (b) of Theorem 1.3. In Section 6 we address the unique solvability of the elliptic equation. Parts (c) and (d) are proved in Section 7, and our results concerning the asymptotic behavior of the semigroup (in particular the proof of (e)) are found in Section 8. In Section 9, we present our examples.

2. Semigroups on the dual of a separable space. As already mentioned, we will consider semigroups on $L^\infty(\Omega)$. It follows from a result of Lotz [27] that a strongly continuous semigroup on $L^\infty(\Omega)$ is already norm continuous and thus has a bounded generator. To handle semigroups that are not strongly continuous, we will introduce the notion of a $*$-semigroup. At first, the only structural property of $L^\infty(\Omega)$ that we will use is that it is the dual space of the separable space $L^1(\Omega)$. We have therefore decided to treat semigroups on the dual of a separable space in general, as the results obtained here might also be of interest in other situations. We also mention that some of the results presented here can be obtained from the more general theory of "semigroups on norming dual pairs" [25, 26]. However, the situation of a dual space is easier to handle and proofs simplify. Thus, for the convenience of the reader, we will give a self-contained exposition and complete proofs.

Throughout this section, X denotes a separable Banach space and X^* its dual space. If T is an adjoint operator, say $T = S^*$ for some bounded linear operator $S \in \mathscr{L}(X)$, then clearly T is a bounded operator on X^* which is also weak*-continuous. Conversely, if T is a weak*-continuous linear map on X^*, then T is an adjoint operator, thus in particular bounded. To shorten notation, we write $\sigma^* := \sigma(X^*, X)$ for the weak*-topology on X^* and $\mathscr{L}(X^*, \sigma^*)$ for the space of weak*-continuous operators on X^*.

LEMMA 2.1. *Let X be a separable Banach space and $T : X^* \to X^*$ be a bounded linear operator. Then T is weak*-continuous if and only if T is sequentially weak*-continuous.*

Proof. Clearly, every continuous mapping is sequentially continuous. So assume that T is sequentially weak*-continuous. By definition of the weak*-topology it suffices to show that for every $x \in X$ the linear mapping $\phi_x : X^* \to \mathbb{R}$ given by $\phi_x(x^*) := \langle Tx^*, x\rangle$ is continuous. This, in turn, is equivalent to $\ker \phi_x$ being weak*-closed. By the Krein–Shmul'yan theorem, it suffices to show that $\ker \phi_x \cap \overline{B}_r(0)$ is closed for each $x \in X$ and $r > 0$, where $\overline{B}_r(0)$ denotes the norm-closed ball of radius $r > 0$ in X^*. As X is separable, the weak*-topology is metrizable on norm-bounded sets, whence it suffices to check that $\ker\phi_x \cap \overline{B}_r(0)$ is sequentially closed for each $x \in X$ and $r > 0$. This, however, follows immediately from our assumption since each ϕ_x is sequentially weak*-continuous. ∎

DEFINITION 2.2. Let $T = (T(t))_{t>0} \subseteq \mathscr{L}(X^*, \sigma^*)$ be a family of operators such that $T(t+s) = T(t)T(s)$ for all $t, s > 0$, and for all $x^* \in X^*$ and $x \in X$ the mapping $t \mapsto \langle T(t)x^*, x\rangle$ is measurable. Then T is called a *$*$-semigroup on X^**. If $\|T(t)\| \le 1$ for all $t > 0$, then T is called *contractive*. Moreover, T is *injective* if $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

Next, for a contractive $*$-semigroup $T = (T(t))_{t>0}$ and $\operatorname{Re}\lambda > 0$ we define the operator $R(\lambda)$ on X^* by

$$\langle R(\lambda)x^*, x\rangle := \int_0^\infty e^{-\lambda t}\langle T(t)x^*, x\rangle\,dt, \tag{2.1}$$

i.e. $R(\lambda)$ is the *Laplace transform* of $t \mapsto T(t)x^*$, computed by means of the weak*-integral. This is well-defined as the right-hand side of (2.1) defines a bounded linear functional on X in view of the boundedness of T.

PROPOSITION 2.3. *For a contractive $*$-semigroup $T = (T(t))_{t>0}$, the following assertions hold:*

(a) $R(\lambda) \in \mathscr{L}(X^*, \sigma^*)$ *for all* $\operatorname{Re}\lambda > 0$.

(b) *For* $\lambda, \mu \in \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ *we have*

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu), \tag{2.2}$$

i.e. $(R(\lambda))_{\operatorname{Re}\lambda>0}$ *is a* pseudoresolvent.

Proof. (a) In view of Lemma 2.1, this follows immediately from (2.1) and the dominated convergence theorem.

(b) We first show that for each $\operatorname{Re}\lambda > 0$, $x^* \in X^*$ and $h > 0$,

$$\int_0^h e^{-\lambda t}T(t)x^*\,dt = R(\lambda)x^* - e^{-\lambda h}T(h)R(\lambda)x^*, \tag{2.3}$$

where the integral is understood as a weak*-integral as before. To see this, fix $x \in X$ and let $S(t) \in \mathscr{L}(X)$ be such that $S(t)^* = T(t)$. Then

$$\begin{aligned}\langle T(h)R(\lambda)x^*, x\rangle &= \langle R(\lambda)x^*, S(h)x\rangle = \int_0^\infty e^{-\lambda t}\langle T(t)x^*, S(h)x\rangle\,dt \\ &= \int_0^\infty e^{-\lambda t}\langle T(t+h)x^*, x\rangle\,dt = e^{ht}\int_h^\infty e^{-\lambda t}\langle T(t)x^*, x\rangle\,dt \\ &= e^{\lambda h}\Big(\langle R(\lambda)x^*, x\rangle - \int_0^h e^{-\lambda r}\langle T(r)x^*, x\rangle\,dr\Big).\end{aligned}$$

As $x \in X$ was arbitrary, (2.3) is proved. Now let $0 < \operatorname{Re}\mu < \operatorname{Re}\lambda$. Then

$$\begin{aligned}(\mu-\lambda)R(\lambda)R(\mu)x^* &= (\mu-\lambda)\int_0^\infty e^{-\lambda t}T(t)R(\mu)x^*\,dt\\ &= (\mu-\lambda)\int_0^\infty e^{(\mu-\lambda)t}e^{-\mu t}T(t)R(\mu)x^*\,dt\\ &= (\mu-\lambda)\int_0^\infty e^{(\mu-\lambda)t}\Big(R(\mu)x^* - \int_0^t e^{-\mu r}T(r)x^*\,dr\Big)\,dt\\ &= -R(\mu)x^* - (\mu-\lambda)\int_0^\infty e^{-\mu r}T(r)x^*\int_r^\infty e^{(\mu-\lambda)t}\,dt\,dr\\ &= R(\lambda)x^* - R(\mu)x^*.\end{aligned}$$

Here, the third equality uses (2.3), and the fourth uses Fubini's theorem and the inequality $\operatorname{Re}\mu < \operatorname{Re}\lambda$. Of course, each integral in this calculation is to be understood in the weak* sense.

Finally, assume that $0 < \operatorname{Re}\mu = \operatorname{Re}\lambda$. We set $\lambda_n := \lambda + n^{-1}$, so that $0 < \operatorname{Re}\mu < \operatorname{Re}\lambda_n$. It follows from (2.1) and dominated convergence that $R(\lambda_n)x^*$ weak*-converges to $R(\lambda)x^*$ for every $x^* \in X^*$. By the above,

$$R(\lambda_n)x^* - R(\mu)x^* = (\mu-\lambda_n)R(\lambda_n)R(\mu)x^*$$

for every $x^* \in X^*$ and $n \in \mathbb{N}$. Letting $n \to \infty$ we obtain (2.2). ∎

Our next goal is to prove that the Laplace transform $(R(\lambda))_{\operatorname{Re}\lambda>0}$ determines the semigroup $(T(t))_{t>0}$ uniquely. To this end, we use the following lemma [1, Lemma 3.16.5].

LEMMA 2.4. *Let $N \subseteq (0,\infty)$ be a set of Lebesgue measure* 0 *and assume that $t, s \notin N$ implies $t+s \notin N$. Then $N = \emptyset$.*

THEOREM 2.5. *Let $T_1 = (T_1(t))_{t>0}$ and $T_2 = (T_2(t))_{t>0}$ be contractive $*$-semigroups on X^* with Laplace transforms $(R_1(\lambda))_{\operatorname{Re}\lambda>0}$ and $(R_2(\lambda))_{\operatorname{Re}\lambda>0}$, respectively. If there exists $\lambda_0 \geq 0$ such that $R_1(\lambda) = R_2(\lambda)$ for all $\lambda > \lambda_0$, then $T_1 = T_2$.*

Proof. Let $\lambda_0 \geq 0$ be such that

$$\int_0^\infty e^{-\lambda t}\langle T_1(t)x^*, x\rangle\,dt = \int_0^\infty e^{-\lambda t}\langle T_2(t)x^*, x\rangle\,dt$$

for all $x^* \in X^*$, $x \in X$ and $\lambda > \lambda_0$. By the uniqueness theorem for Laplace transforms [1, Theorem 1.7.3], there is a null set $N(x^*, x)$ such that

$$\langle T_1(t)x^*, x\rangle = \langle T_2(t)x^*, x\rangle \quad \text{for all } t \notin N(x^*, x).$$

Now pick a dense sequence $(x_n) \subseteq X$ and define $N(x^*) := \bigcup_{n\in\mathbb{N}} N(x^*, x_n)$. Then each $N(x^*)$ is a null set and $T_1(t)x^* = T_2(t)x^*$ for all $t \notin N(x^*)$ since (x_n) separates the points of X^*. As X is separable, we may find a norming sequence $(x_n^*) \subseteq X^*$ and put $N := \bigcup_{n\in\mathbb{N}} N(x_n^*)$. Let $S_1(t), S_2(t) \in \mathscr{L}(X)$ be such that $S_1^*(t) = T_1(t)$ and $S_2^*(t) = T_2(t)$ for all $t > 0$. Since

$$\langle x_n^*, S_1(t)x\rangle = \langle T_1(t)x_n^*, x\rangle = \langle T_2(t)x_n^*, x\rangle = \langle x_n^*, S_2(t)x\rangle$$

for all $t \notin N$, $x \in X$ and $n \in \mathbb{N}$ and since the norming set $\{x_n^* : n \in \mathbb{N}\}$ separates the points of X, we have $S_1(t) = S_2(t)$ and thus also $T_1(t) = T_2(t)$ for all $t \notin N$. Now consider $M := \{t > 0 : T_1(t) \neq T_2(t)\}$. Then $M \subseteq N$ is a null set and it follows from the semigroup law that $t, s \notin M$ implies that $t + s \notin M$. Hence, $M = \emptyset$ by Lemma 2.4. ∎

Example 2.6. Without the assumption that X be separable, the Laplace transform does not determine the semigroup uniquely, even if X is a Hilbert space. Indeed, consider the counting measure ζ on $\mathbb{R}$. The corresponding L^2-space is $\ell^2(\mathbb{R})$ and consists of functions of the form $f(x) = \sum \alpha_n \mathbb{1}_{\{x_n\}}$, where (x_n) is a sequence of real numbers and (α_n) is a square-summable sequence. Now consider the shift semigroup $T = (T(t))_{t>0}$ given by $T(t)f(x) = f(x+t)$. Then, given $f, g \in \ell^2(\mathbb{R})$, we have $\langle T(t)f, g\rangle = 0$ for all but at most countably many values of t. Consequently, the Laplace transform is $R(\lambda) \equiv 0$, whereas the semigroup is *not* the zero semigroup.

Next, we want to associate a generator to a $*$-semigroup, i.e. an operator whose resolvent is the Laplace transform of the semigroup. However, in order to do so, the Laplace transform has to consist of injective operators, which is not always the case.

Since $(R(\lambda))_{\operatorname{Re}\lambda>0}$ is a pseudoresolvent by Proposition 2.3, the kernel of $R(\lambda)$ for $\lambda \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is independent of λ. Moreover, if $\ker R(\lambda) = \{0\}$ for some/all $\lambda \in \mathbb{C}_+$, then there exists an operator A with $\mathbb{C}_+ \subseteq \rho(A)$ and $R(\lambda, A) = R(\lambda)$ for all $\lambda \in \mathbb{C}_+$ (see [1, Proposition B.6]). In that case, $D(A) = \operatorname{rg} R(\lambda)$ and $A = \lambda - R(\lambda)^{-1}$. The proof of Theorem 2.5 shows that $\ker R(\lambda) = \{0\}$ for some/all $\lambda \in \mathbb{C}_+$ if and only if the semigroup T is injective. We may thus introduce

Definition 2.7. Let $T = (T(t))_{t>0} \subseteq \mathscr{L}(X^*, \sigma^*)$ be an injective and contractive $*$-semigroup. The unique operator A such that

$$R(\lambda, A)x^* = \int_0^\infty e^{-\lambda t} T(t)x^*\, dt$$

for all $\operatorname{Re}\lambda > 0$ is called the *generator* of T.

We can now characterize the generator of an injective and contractive $*$-semigroup:

Proposition 2.8. *Let $T = (T(t))_{t>0} \subseteq \mathscr{L}(X^*, \sigma^*)$ be an injective and contractive $*$-semigroup with generator A. Then for all $y^*, z^* \in X^*$ the following are equivalent:*

(i) $y^* \in D(A)$ *and* $Ay^* = z^*$.
(ii) $\int_0^t T(s)z^*\,ds = T(t)y^* - y^*$ *for all* $t > 0$.

Proof. (i)⇒(ii). For fixed $t > 0$ and $x \in X$ we define the holomorphic functions $f, g : \mathbb{C} \to \mathbb{C}$ by

$$f(\lambda) := \lambda \int_0^t e^{-\lambda s} \langle T(s)y^*, x\rangle\,ds - \int_0^t e^{-\lambda s}\langle T(s)z^*, x\rangle\,ds,$$
$$g(\lambda) := \langle y^*, x\rangle - e^{-\lambda t}\langle T(t)y^*, x\rangle.$$

For $\operatorname{Re}\lambda > 0$, it follows from (2.3) with $x^* = \lambda y^* - z^*$ that

$$\int_0^t e^{-\lambda s} T(s)x^*\,ds = R(\lambda)x^* - e^{-\lambda t}T(t)R(\lambda)x^* = y^* - e^{-\lambda t}T(t)y^*$$

since $R(\lambda)x^* = y^*$. This shows that $f(\lambda) = g(\lambda)$ for all $\operatorname{Re}\lambda > 0$ and thus, by the uniqueness theorem for holomorphic functions, for all $\lambda \in \mathbb{C}$. In particular, $f(0) = g(0)$ and this implies (ii) as $x \in X$ was arbitrary.

(ii)⇒(i). If (ii) holds, it follows from Fubini's theorem that

$$\begin{aligned}\lambda R(\lambda)y^* - y^* &= \int_0^\infty \lambda e^{-\lambda t}(T(t)y^* - y^*)\,dt = \int_0^\infty \lambda e^{-\lambda t}\int_0^t T(s)z^*\,ds\,dt\\ &= \int_0^\infty \int_s^\infty \lambda e^{-\lambda t} T(s)z^*\,dt\,ds = \int_0^\infty e^{-\lambda s}T(s)z^*\,ds = R(\lambda)z^*.\end{aligned}$$

This shows that $y^* = R(\lambda)(\lambda y^* - z^*) \in D(A)$ and $Ay^* = z^*$. ∎

Recall that the semigroups we consider here are not strongly continuous (not even weak*-continuous). Nevertheless, we can expect some continuity of the orbits.

Corollary 2.9. *Let $T = (T(t))_{t>0} \subset \mathscr{L}(X^*, \sigma^*)$ be an injective and contractive $*$-semigroup with generator A. Then for $x^* \in \overline{D(A)}$ the orbit $t \mapsto T(t)x^*$ is $\|\cdot\|$-continuous. In particular, T is strongly continuous if and only if $\overline{D(A)} = X^*$.*

Proof. For $x^* \in D(A)$ we see from Proposition 2.8 that

$$\|T(t)x^* - T(s)x^*\| = \left\|\int_s^t T(r)Ax^*\,dr\right\| \le |t-s|\,\|Ax^*\| \to 0$$

as $t \to s$. A 3ε-argument making use of the uniform boundedness of the operators shows that this remains true for $x^* \in \overline{D(A)}$. ∎

We now add an additional structure to X: we assume that X is a Banach lattice. We denote the positive cone of X by X_+. The dual cone in X^* is denoted by X_+^*. Note that $x^* \in X_+^*$ if and only if $\langle x^*, x\rangle \geq 0$ for all $x \in X_+$. An operator T on X^* is called *positive* if $Tx^* \in X_+^*$ whenever $x^* \in X_+^*$. This defines an ordering on $\mathscr{L}(X^*)$ by setting $T_1 \leq T_2$ if and only if $T_2 - T_1 \geq 0$. We call a $*$-semigroup $(T(t))_{t>0}$ *positive* if $T(t)$ is positive for every $t > 0$.

Proposition 2.10. *Let X be a separable Banach lattice and $T_1 = (T_1(t))_{t>0}$ and $T_2 = (T_2(t))_{t\geq 0}$ be contractive $*$-semigroups on X^* with Laplace transforms $(R_1(\lambda))_{\operatorname{Re}\lambda>0}$ and $(R_2(\lambda))_{\operatorname{Re}\lambda>0}$, respectively, and suppose that T_1 is positive. Then $T_1(t) \leq T_2(t)$ for all $t > 0$ if and only if there exists $\lambda_0 \geq 0$ such that $R_1(\lambda) \leq R_2(\lambda)$ for all $\lambda > \lambda_0$. In particular, T_2 is positive if and only if $R_2(\lambda)$ is positive for all real λ large enough.*

Proof. If $T_1(t) \leq T_2(t)$ for all $t > 0$, then clearly $R_1(\lambda) \leq R_2(\lambda)$ for all $\lambda > 0$. Now suppose that $R_1(\lambda) \leq R_2(\lambda)$ for all $\lambda > \lambda_0$. Let $x \in X_+$, $x^* \in X_+^*$ and define $r_{x,x^*} : (\lambda_0, \infty) \to [0, \infty)$ by

$$r_{x,x^*}(\lambda) := \langle R_2(\lambda)x^* - R_1(\lambda)x^*, x\rangle = \int_0^\infty e^{-\lambda t}\langle T_2(t)x^* - T_1(t)x^*, x\rangle\, dt.$$

It follows from the resolvent equation (2.2) that r_{x,x^*} is infinitely differentiable with

$$\frac{d^n}{d\lambda^n} r_{x,x^*}(\lambda) = (-1)^n n!\langle R_2^{n+1}(\lambda)x^* - R_1^{n+1}(\lambda)x^*, x\rangle.$$

Now the Post–Widder inversion formula [1, Proposition 1.7.7] implies that there is a null set $N(x^*, x) \subset (0, \infty)$ such that

$$\langle T_2(t)x^* - T_1(t)x^*, x\rangle = \lim_{n\to\infty} (-1)^n \frac{1}{n!}\left(\frac{n}{t}\right)^{n+1} \frac{d^n}{d\lambda^n} r_{x,x^*}\left(\frac{n}{t}\right) \geq 0$$

for all $t \in (0, \infty) \setminus N(x^*, x)$. Now we proceed similarly to the proof of Theorem 2.5. Since X is separable, we find a sequence $(x_n) \subseteq X_+$ dense in the positive cone X_+ and a sequence $(x_n^*) \subseteq X_+^*$ norming in X. Set $N := \bigcup_{n,m\in\mathbb{N}} N(x_n^*, x_m)$. Since $\{x_n^* : n \in \mathbb{N}\}$ is weak*-dense in X_+^*, it follows from

$$\langle T_2(t)x_n^* - T_1(t)x_n^*, x_m\rangle \geq 0$$

for all $n, m \in \mathbb{N}$ and $t \in (0, \infty) \setminus N$ that $T_2(t) - T_1(t)$ is positive for all $t \in (0, \infty) \setminus N$. Now $M := \{t > 0 : T_1(t) \not\leq T_2(t)\}$ is contained in N and thus is a null set. Moreover, for $t, s \notin M$ the positivity of T_1 yields

$$T_1(t + s) = T_1(t)T_1(s) \leq T_1(t)T_2(s) \leq T_2(t)T_2(s) = T_2(t + s),$$

i.e. $t + s \notin M$. Thus, Lemma 2.4 implies $T_1(t) \leq T_2(t)$ for all $t \in (0, \infty)$. ∎

Recall that a Banach lattice is called a *KB-space* if every increasing and norm-bounded net of positive vectors converges in norm (cf. [29, Definition 2.4.11]). For instance, every L^1-space has this property.

Lemma 2.11. *Let X be a separable KB-space and let $(U_n) \subset \mathscr{L}(X^*, \sigma^*)$ be an increasing sequence of positive and contractive operators.*

(1) *There is a positive contraction $U \in \mathscr{L}(X^*, \sigma^*)$ such that*
$$Ux^* = \sup_{n\in\mathbb{N}} U_n x^* \quad \text{for all } x^* \in X^*_+.$$
We write $U_n \uparrow U$.

(2) *If $(V_n) \subset \mathscr{L}(X^*, \sigma^*)$ is another increasing sequence of positive contractions with $V_n \uparrow V$, then $U_nV_n \uparrow UV$.*

Proof. (1) Pick $S_n \in \mathscr{L}(X)$ such that $S_n^* = U_n$. For $x \in X_+$ the sequence $S_n x$ is increasing and norm-bounded. Since X is a KB-space, the limit $\tilde{S}x := S_n x$ exists. Obviously, $\tilde{S}$ is additive and positively homogeneous on X_+. Consequently, it uniquely extends to a positive linear operator S on X (cf. [29, Lemma 1.3.3]). It follows that in the ordering of $\mathscr{L}(X)$ we have $S = \sup_{n\in\mathbb{N}} S_n$ and so $U := S^* = \sup_{n\in\mathbb{N}} S_n^* = \sup_{n\in\mathbb{N}} U_n$ is an adjoint operator. That U is a positive contraction is obvious.

(2) Clearly, $U_nV_n \le UV$ and so $\sup_{n\in\mathbb{N}} U_nV_nx^* \le UVx^*$ for all $x^* \in X^*_+$. On the other hand, for fixed $m \in \mathbb{N}$ we have
$$\sup_{n\in\mathbb{N}} U_nV_nx^* \ge \sup_{n\in\mathbb{N}} U_nV_mx^* = UV_mx^*$$
for all $x^* \in X^*_+$. As X is a KB-space, it is a band in its bi-dual X^{**} (see [29, Theorem 2.4.12]). Thus, by [29, Proposition 1.4.15], the elements of X are precisely the order continuous linear functionals on X^*, whence an adjoint operator on X^* is automatically order continuous. Consequently, for $x^* \in X^*_+$ we have
$$\sup_{m\in\mathbb{N}} UV_mx^* = UVx^*,$$
so that altogether we also obtain $\sup_{n\in\mathbb{N}} U_nV_nx^* \ge UVx^*$. ∎

We can now prove the following monotone convergence theorem for positive contractive $*$-semigroups.

Proposition 2.12. *Let X be a separable KB-space and let $(T_n(t))_{t>0} \subseteq \mathscr{L}(X^*, \sigma^*)$ denote an increasing sequence of positive contractive $*$-semigroups with Laplace transforms $(R_n(\lambda))_{\operatorname{Re}\lambda>0}$. Then $\sup_{n\in\mathbb{N}} T_n(t)$ defines a positive contractive $*$-semigroup with Laplace transform $\sup_{n\in\mathbb{N}} R_n(\lambda)$ for all real $\lambda > 0$.*

Proof. By Lemma 2.11(1), $T(t) := \sup_{n\in\mathbb{N}} T_n(t)$ defines a positive contraction in $\mathscr{L}(X^*, \sigma^*)$ for every $t > 0$. By Lemma 2.11(2), for $t, s > 0$,

$$T(t+s) = \sup_{n\in\mathbb{N}} T_n(t+s) = \sup_{n\in\mathbb{N}} T_n(t)T_n(s) = T(t)T(s),$$

so that $T = (T(t))_{t>0}$ satisfies the semigroup law. Since for each $x^* \in X^*_+$ and $x \in X_+$ the function $t \mapsto \langle T(t)x^*, x\rangle = \sup_{n\in\mathbb{N}}\langle T_n(t)x^*, x\rangle$ is measurable, this shows that $T = (T(t))_{t>0}$ is a $*$-semigroup. Clearly, T is contractive. Finally, by monotone convergence,

$$\begin{aligned}\sup_{n\in\mathbb{N}}\langle R_n(\lambda)x^*, x\rangle &= \sup_{n\in\mathbb{N}} \int_0^\infty e^{-\lambda t}\langle T_n(t)x^*, x\rangle\, dt \\ &= \int_0^\infty e^{-\lambda t}\langle T(t)x^*, x\rangle\, dt\end{aligned}$$

for all $\lambda > 0$, $x^* \in X^*_+$ and $x \in X_+$. By linearity, this shows that the Laplace transform of T is given by $\sup_{n\in\mathbb{N}} R_n(\lambda)$ for all $\lambda > 0$. ∎

3. Kernel operators and the strong Feller property. In the previous section we have established tools that will allow us to prove that a realization A_μ of our operator $\mathscr{A}$ subject to the nonlocal boundary condition (1.2) generates an injective $*$-semigroup on $L^\infty(\Omega)$ which consists of positive contractions. From the point of view of Markov processes, however, it is more natural to work on the space $B_b(\overline{\Omega})$ of bounded, Borel-measurable functions on the set $\overline{\Omega}$. It is particularly important that the operators involved are *kernel operators*, since then we can extract the transition probabilities of the associated stochastic process from these operators.

In this section, we recall the relevant notions concerning kernel operators. We will also recall the strong Feller property, which is an important tool in studying the asymptotic behavior of transition semigroups of Markov operators. As we will see, the strong Feller property for semigroups also entails nice continuity properties.

In this section, we set $E := \overline{\Omega}$. Note that everything remains valid if E is replaced with a general complete, separable metric space. We denote by $\mathscr{B}(E)$, $B_b(E)$, $C_b(E)$ and $\mathscr{M}(E)$ the Borel σ-algebra, the space of bounded Borel-measurable functions, the space of bounded continuous functions and the space of signed measures on E, respectively.

A *bounded kernel* on E is a map $k : E \times \mathscr{B}(E) \to \mathbb{C}$ such that

(i) the map $x \mapsto k(x, A)$ is Borel-measurable for every $A \in \mathscr{B}(E)$;
(ii) the map $A \mapsto k(x, A)$ is a complex measure for every $x \in E$;
(iii) $\sup_{x\in E} |k|(x, E) < \infty$, where $|k|(x, \cdot)$ denotes the total variation of the measure $k(x, \cdot)$.

An operator $K \in \mathscr{L}(B_b(E))$ is called a *kernel operator* if there exists a kernel k such that

$$(3.1) \qquad (Kf)(x) = \int_E f(y)\, k(x, dy)$$

for every $f \in B_b(E)$ and $x \in E$. As there is at most one k satisfying (3.1), we call it the *kernel associated with* K, and conversely, K is the *operator associated with* k. Likewise, we can associate an operator $K' \in \mathscr{L}(\mathscr{M}(E))$ with k by setting

$$K'\nu(A) := \int_E k(x, A)\, d\nu(x)$$

for $A \in \mathscr{B}(E)$. It turns out that a bounded linear operator K on $B_b(E)$ is a kernel operator if and only if the norm adjoint K^*: $B_b(E)^* \to B_b(E)^*$, defined by $K^*\varphi := \varphi \circ K$ for any norm-continuous linear functional $\varphi : B_b(E) \to \mathbb{C}$, leaves the space $\mathscr{M}(E)$ invariant. In this case, $K^*|_{\mathscr{M}(E)} = K'$. For us, the following characterization is more important.

LEMMA 3.1. *Let $K \in \mathscr{L}(B_b(E))$. Then K is a kernel operator if and only if it is* pointwise continuous*, i.e. whenever $(f_n) \subset B_b(E)$ is a bounded sequence converging pointwise to $f \in B_b(E)$, then Kf_n converges pointwise to Kf.*

Proof. If K is pointwise continuous, then setting $k(x, A) := (K\mathbb{1}_A)(x)$, we see that k is a kernel. By linearity and the density of simple functions in $B_b(E)$ it follows that k is associated with K. The converse follows by dominated convergence. ∎

DEFINITION 3.2. An operator $K \in \mathscr{L}(B_b(E))$ is called a *strong Feller operator* if it is a kernel operator and $Kf \in C_b(E)$ for all $f \in B_b(E)$.

Let us now assume that $(E, \mathscr{B}(E))$ is additionally endowed with a measure m with full support, i.e. for every $x \in E$ and $r > 0$ we have $m(B_r(x)) > 0$. This is certainly the case in our intended application, where $E = \overline{\Omega}$ and m is Lebesgue measure on Ω. If m has full support, then two continuous functions which are equal almost everywhere, are equal everywhere. In particular, an element of $L^\infty(E, m)$ may have at most one continuous representative. Suppose now that $\tilde{K} \in \mathscr{L}(L^\infty(E, m))$ is such that for every $f \in L^\infty(E, m)$ the image $\tilde{K}f$ has a continuous representative. In this case, we will say that $\tilde{K}$ *takes values in* $C_b(E)$. In view of the closed graph theorem, we may then consider $\tilde{K}$ as a bounded operator from $L^\infty(E, m)$ to $C_b(E)$. Let $\iota : B_b(E) \to L^\infty(E, m)$ be the canonical injection which maps a bounded, measurable function to its equivalence class modulo equality almost everywhere. If $\tilde{K} \in \mathscr{L}(L^\infty(E, m))$ takes values in $C_b(E)$, then $K := \tilde{K} \circ \iota$ defines a map from $B_b(E)$ to $C_b(E)$. It is a natural question whether $\tilde{K}$ is a strong

Feller operator. Unfortunately, this is not true without further assumptions, as $\tilde{K} \circ \iota$ may fail to be a kernel operator (cf. [3, Example 5.4]).

However, making use of Lemma 3.1, we easily obtain the following characterization.

LEMMA 3.3. *Let $\tilde{K} \in \mathscr{L}(L^\infty(E,m))$ take values in $C_b(E)$ and $\iota : B_b(E) \to L^\infty(E,m)$ be as above. Then $\tilde{K} \circ \iota$ is a kernel operator if and only if for every bounded sequence $(f_n) \subset L^\infty(E,m)$ that converges almost everywhere to f we have*

$$\tilde{K} f_n(x) \to \tilde{K} f(x) \quad \text{for all } x \in E.$$

Slightly abusing notation, we define the strong Feller property also for operators on $L^\infty(E,m)$.

DEFINITION 3.4. An operator $\tilde{K} \in \mathscr{L}(L^\infty(E,m))$ is called a *strong Feller operator* if

(i) $\tilde{K}$ takes values in $C_b(E)$;
(ii) for every bounded sequence $(f_n) \subset L^\infty(E,m)$ converging pointwise almost everywhere to f, we have $\tilde{K} f_n \to \tilde{K} f$ pointwise.

In what follows we will not distinguish between strong Feller operators $\tilde{K}$ on $L^\infty(E,m)$ and the strong Feller operators $K := \tilde{K} \circ \iota$ on $B_b(E)$. In particular, given a strong Feller operator $\tilde{K}$ on $L^\infty(E,m)$, we can consider the operator $\tilde{K}' \in \mathscr{M}(E)$ (which of course should be identified with $(\tilde{K} \circ \iota)'$).

A strong Feller operator in the sense of Definition 3.4 is always an adjoint operator:

LEMMA 3.5. *Let $K \in \mathscr{L}(L^\infty(E,m))$ be a strong Feller operator. Then K is an adjoint operator.*

Proof. In view of Lemma 2.1 it suffices to prove that K is sequentially weak*-continuous. Fix $x \in E$ and put $\varphi_x(f) := Kf(x)$ for $f \in L^\infty(E,m)$. By the closed graph theorem, $\varphi_x \in L^\infty(E,m)^*$. We now make use of the continuity condition (ii) from Definition 3.4 to prove that $\varphi_x(f) = \langle f, g_x \rangle$ for some $g_x \in L^1(E,m)$. To that end, let (A_n) be a sequence of pairwise disjoint Borel subsets of E. Then

$$f_n := \mathbb{1}_{\bigcup_{k=1}^n A_k} \uparrow f := \mathbb{1}_{\bigcup_{k=1}^\infty A_k}$$

almost everywhere, whence (ii) implies that $\nu_x(A) := \varphi_x(\mathbb{1}_A)$ defines a σ-additive measure on E. If $m(A) = 0$, then $\mathbb{1}_A = 0$ almost everywhere, whence $K\mathbb{1}_A \equiv 0$. Thus ν_x is absolutely continuous with respect to m. By the Radon–Nikodym theorem, ν_x has a density $g_x \in L^1(E,m)$.

Now let $(f_n) \subset L^\infty(E,m)$ with $f_n \rightharpoonup^* f$. By the above,

$$Kf_n(x) = \langle f_n, g_x \rangle \to \langle f, g_x \rangle = Kf(x)$$

for all $x \in E$. In view of the uniform boundedness principle, (f_n) is uniformly bounded and hence (Kf_n) is bounded. It now follows from the dominated convergence theorem that $Kf_n \rightharpoonup^* Kf$ as $n \to \infty$. ∎

The importance of the strong Feller property in the study of asymptotic behavior and continuity properties of transition semigroups stems from the following fact.

LEMMA 3.6 ([31, §1.5]). *Let K, L be positive strong Feller operators. Then the product $K \cdot L$ is* ultra Feller, *i.e. it maps bounded subsets of $B_b(E)$ to equicontinuous subsets of $C_b(E)$.*

Thus, if K and L are positive strong Feller operators, then it follows from the Arzelà–Ascoli theorem that given a bounded sequence (f_n), we can extract a subsequence (f_{n_k}) such that KLf_{n_k} converges locally uniformly, i.e. with respect to the compact-open topology. In our setting, it is preferable to use another topology.

The *strict topology* β_0 on $C_b(E)$ is defined as follows. We let $\mathscr{F}_0$ be the space of all functions φ on E that vanish at infinity, i.e. given $\varepsilon > 0$ we can find a compact set $K \subset E$ such that $|\varphi(x)| \leq \varepsilon$ for all $x \in E \setminus K$. The locally convex topology β_0 on $C_b(E)$ is defined by the family $\{q_\varphi : \varphi \in \mathscr{F}_0\}$ of seminorms $q_\varphi : f \mapsto \|\varphi f\|_\infty$.

On norm-bounded subsets of $C_b(E)$, β_0 coincides with the compact-open topology (see [24, Theorem 2.10.4]); thus if K and L are positive strong Feller operators and (f_n) is a bounded sequence, then KLf_n has a β_0-convergent subsequence. The main advantage of considering β_0 instead of the compact-open topology is that β_0 is *consistent with the duality* between $C_b(E)$ and $\mathscr{M}(E)$, i.e. $(C_b(E), \beta_0)' = \mathscr{M}(E)$. In fact, β_0 is even the Mackey topology of the pair $(C_b(E), \mathscr{M}(E))$ [32, Theorems 4.5 and 5.8], i.e. the finest locally convex topology on $C_b(E)$ consistent with the duality. From this one can infer that an operator on $C_b(E)$ is β_0-continuous if and only if it is $\sigma(C_b(E), \mathscr{M}(E))$-continuous. It is not difficult to see that the latter is the case if and only if the operator is associated with a bounded kernel [26, Proposition 3.5].

We now obtain the following result about continuity properties of strong Feller semigroups.

THEOREM 3.7. *Let $(T(t))_{t>0} \subset \mathscr{L}(L^\infty(E, m), \sigma^*)$ be an injective and contractive $*$-semigroup with generator A. Assume furthermore that every operator $T(t)$ is a positive strong Feller operator.*

(a) *If $(t_n) \subset (0, \infty)$ converges to $t > 0$ and $f \in L^\infty(E)$, then $T(t_n)f \to T(t)f$ locally uniformly.*
(b) *For every $f \in L^\infty(E, m)$ the map $(0, \infty) \times E \ni (t, x) \mapsto (T(t)f)(x)$ is continuous.*

Proof. (a) By Lemma 3.6 and the semigroup property, T consists of ultra Feller operators. Let $s := \inf\{t_n : n \in \mathbb{N}\} > 0$. The sequence $(T(t_n - s)f)$ is bounded and thus mapped to an equicontinuous set by the ultra Feller operator $T(s)$. Thus $f_n := T(t_n)f$ has a subsequence (f_{n_k}) which β_0-converges, say to $g \in C_b(E)$. In particular, f_{n_k} weak*-converges to g in $L^\infty(E, m)$.

We now have

$$R(\lambda)g = \lim_{k\to\infty} R(\lambda)f_{n_k} = \lim_{k\to\infty} T(t_{n_k})R(\lambda)f = T(t)R(\lambda)f = R(\lambda)T(t)f.$$

Here, the first equality is the weak*-continuity of $R(\lambda)$. The second and the last equalities follow from the fact that $R(\lambda)$ commutes with every operator $T(s)$ for $s > 0$. The third equality follows from Corollary 2.9 since $R(\lambda)$ takes values in $D(A)$. As $R(\lambda)$ is injective, we must have $g = T(t)f$.

In the same fashion we see that every subsequence of $T(t_n)f$ has a subsequence which β_0-converges to $T(t)f$. Hence the whole sequence converges.

(b) Let $(t_n, x_n) \to (t, x)$. By (a), $T(t_n)f \to T(t)f$ uniformly on the compact set $\{x\} \cup \{x_n : n \in \mathbb{N}\}$. ■

4. Preliminary results on bounded domains. We recall that a set $\Omega \subset \mathbb{R}^d$ is called *Dirichlet regular* if for every point $z \in \partial\Omega$ there exists a *barrier at* z, i.e. there is a radius $r > 0$ and a function $w \in C(\overline{\Omega \cap B_r(z)})$, where $B_r(z)$ denotes the open ball of radius r centered at z, such that

$$\Delta w \le 0 \quad \text{in } \mathscr{D}(\Omega \cap B_r(z)), \qquad w(z) = 0, \qquad w(x) > 0 \quad \text{for } x \in \Omega \cap B_r(z).$$

By classical results (see e.g. [22, Theorem 2.14]), a bounded open set Ω is Dirichlet regular if and only if the classical Dirichlet problem is well-posed, i.e. for every $\varphi \in C(\partial\Omega)$ we can find a harmonic function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $u = \varphi$ on $\partial\Omega$. We point out that every bounded Lipschitz domain is Dirichlet regular; more generally, every bounded domain that satisfies the uniform exterior cone condition is Dirichlet regular. In $\mathbb{R}$, every open set is Dirichlet regular. In $\mathbb{R}^2$, every open and simply connected subset is Dirichlet regular. For proofs and more information, we refer the reader to [12].

We will now recall some results from [3] concerning diffusion operators subject to nonlocal boundary conditions on *bounded* sets. Throughout, U will be a bounded subset of $\mathbb{R}^d$. We will later apply these results to certain subsets U of Ω.

A function $g : \overline{U} \to \mathbb{R}$ is called *Dini-continuous* if the modulus of continuity

$$\omega_g(t) := \sup\{|g(x) - g(y)| : x, y \in \overline{U},\ |x - y| \le t\}$$

satisfies

$$\int_0^1 \frac{\omega_g(t)}{t}\, dt < \infty.$$

Clearly, every Hölder-continuous function is Dini-continuous.

We now recall from [3] some results concerning the situation on bounded subsets of $\mathbb{R}^d$. We make the following assumptions.

Hypothesis 4.1. Let $U \subset \mathbb{R}^d$ be a bounded, Dirichlet regular set and assume that we are given functions $\alpha_{ij} \in C(\overline{U})$ and $\beta_j \in L^\infty(U)$ which are real-valued for $i, j = 1, \ldots, d$. The diffusion coefficients α_{ij} are assumed to be symmetric and strictly elliptic in the sense that there exists a constant $\kappa > 0$ such that for all $x \in \overline{U}$ and $\xi \in \mathbb{R}^d$ we have

$$\sum_{i,j=1}^{d} \alpha_{ij}(x)\xi_i\xi_j \geq \kappa|\xi|^2.$$

Finally, we assume that either

(i) the coefficients α_{ij} are Dini-continuous, or
(ii) U satisfies the uniform exterior cone condition.

We will then set

$$\mathscr{B}u := \sum_{i,j=1}^{d} \alpha_{ij} D_i D_j u + \sum_{j=1}^{d} \beta_j D_j u \quad \text{for } u \in W(U).$$

Also, on bounded domains U we consider a measure-valued function on the boundary which will give us our boundary condition. In contrast to the situation on unbounded domains, we here also allow subprobability measures. This will be important in our approximation scheme in the next section. We make the following assumptions.

Hypothesis 4.2. We let $\gamma : \partial U \times \mathscr{B}(U) \to \mathscr{M}(U)$. We will occasionally write $\gamma(z) := \gamma(z, \cdot) \in \mathscr{M}(U)$. We assume that

(1) for every $z \in \partial U$ the measure $\gamma(z)$ is positive and satisfies $0 \leq \gamma(z, U) \leq 1$;
(2) the map $z \mapsto \gamma(z)$ is $\sigma(\mathscr{M}(U), C_b(U))$-continuous.

We now define the operator B on $L^\infty(U)$ as follows. We set

$$D(B) := \Big\{ u \in C_b(\overline{U}) \cap W(U) : \mathscr{B}u \in L^\infty(U), \\ u(z) = \int_U u(x)\, \gamma(z, dx)\ \forall z \in \partial U \Big\}.$$

From [3] we infer the following properties of the operator B.

Proposition 4.3. *Assume Hypotheses* 4.1 *and* 4.2 *and let B be defined as above. Then the following hold true:*

(a) $(0, \infty) \subset \rho(B)$. *For $\lambda > 0$ the resolvent $R(\lambda, B) \in \mathscr{L}(L^\infty(U))$ is a positive operator that satisfies $\|\lambda R(\lambda, B)\| \leq 1$.*
(b) *B is the generator of an analytic semigroup $S = (S(t))_{t>0}$ which is positive and contractive.*

(c) *The operators* $R(\lambda, B)$ $(\lambda > 0)$ *and* $S(t)$ $(t > 0)$ *are strongly Feller in the sense of Definition* 3.4.

Proof. It was seen in [3, Theorem 4.8] that B generates an analytic semigroup on $L^\infty(U)$ which is positive. In [3, Proposition 4.12] it was proved that this semigroup is also contractive. This shows (b). Inspecting the proofs, we see that actually all statements of (a) were proved along the way. Part (c) was established in [3, Proposition 5.7] for $R(\lambda, B)$ and in [3, Corollary 5.8] for the semigroup. ∎

Remark 4.4. We should point out that we can view the semigroup S generated by B also as a contractive and injective $*$-semigroup on $L^\infty(U)$. Indeed, being analytic, the semigroup S is norm-continuous and the resolvent can be computed from the semigroup via an $\mathscr{L}(L^\infty(U))$-valued Bochner integral. This implies the weaker measurability and integrability conditions in Definition 2.2. The only thing which is not obvious is that we are dealing with adjoint operators. This, however, follows from Lemma 3.5 in view of the strong Feller property.

We now collect some appropriate maximum principles for our situation.

Lemma 4.5 ([5, Lemma 3.2]). *Assume Hypothesis* 4.1. *Let* $u \in W(U)$ *be a complex-valued function such that* $\mathscr{B}u$ *coincides almost everywhere with a continuous function on* U *and assume that* $|u(x)| \leq |u(x_0)|$ *in a neighborhood of* x_0. *Then*

$$\operatorname{Re}|\overline{u(x_0)}\mathscr{B}u(x_0)| \leq 0.$$

In particular, if u *is real-valued and* $u(x_0) > 0$, *then* $\mathscr{B}u(x_0) \leq 0$.

The next lemma relates the position of possible maxima to the boundary condition. Here and subsequently, we use the following notation. Given a measure γ on U and a function $u \in C(\overline{U})$, we define $\langle u, \gamma\rangle$ by

$$\langle u, \gamma\rangle := \int_U f\, d\gamma.$$

Lemma 4.6 ([3, Lemma 4.10]). *Assume Hypothesis* 4.2. *Let* $u \in C(\overline{U})$ *be a real-valued function such that* $u(z) \leq \langle u, \gamma(z)\rangle$ *for all* $z \in \partial U$. *If* $c := \max_{x\in\overline{U}} u(x) > 0$, *then there is a point* $x_0 \in U$ *such that* $u(x_0) = c$.

In the proof of [3, Lemma 4.10], the boundedness of U is only used to infer that, by compactness, there is some $x_0 \in \overline{U}$ with $u(x_0) = \max_{x\in\overline{U}} u(x)$. Then it is proved that x_0 cannot lie on the boundary ∂U. However, inspecting the proof, we obtain the following version for *unbounded* domains:

Lemma 4.7. *Assume Hypothesis* 1.2 *and let* $u \in C_b(\overline{\Omega})$ *be a real-valued function such that* $u(z) \leq \langle u, \mu(z)\rangle$ *for all* $z \in \partial\Omega$. *If* $S := \sup_{x\in\overline{\Omega}} u(x) > 0$, *then* $u(z) < S$ *for all* $z \in \partial\Omega$.

We can now establish a maximum principle for our differential operator that involves the boundary condition.

Lemma 4.8. *Assume Hypotheses* 4.1 *and* 4.2 *and let* $\lambda > 0$. *If* $u \in C(\overline{U}) \cap W(U)$ *is a real-valued function such that* $\mathscr{B}u$ *coincides almost everywhere on* U *with a continuous function, and if* $(\lambda - \mathscr{B})u \le 0$ *on* U *and* $u \le \langle u, \gamma\rangle$ *on* ∂U, *then* $u \le 0$.

Proof. If there exists $x \in U$ with $u(x) > 0$, then $c := \sup_{x\in\overline{U}} u(x) > 0$. As $u \le \langle u, \gamma\rangle$, it follows from Lemma 4.6 that there is some $x_0 \in U$ with $u(x_0) = c$. By Lemma 4.5 we have $\mathscr{B}u(x_0) \le 0$. Consequently,

$$0 \le \lambda u(x_0)^2 \le u(x_0)\mathscr{B}u(x_0) \le 0,$$

in contradiction to $u(x_0) > 0$. This proves that $u \le 0$. ∎

5. The elliptic equation. We are now ready to tackle the solvability of the elliptic equation

$$\begin{cases} \lambda u - \mathscr{A}u = f & \text{on } \Omega, \\ u(z) = \langle u, \mu(z)\rangle & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

for $\lambda > 0$ and $f \in L^\infty(\Omega)$. From now on, we are again in the situation of Hypotheses 1.1 and 1.2. In particular, Ω may be an unbounded set and the coefficients of the operator $\mathscr{A}$ may be unbounded.

The main idea to construct solutions to (5.1) is the same as in [28], namely to consider approximate problems on bounded domains and to show that the solutions of these approximate problems converge, in a suitable sense, to a solution of (5.1). To that end, we set $\Omega_n := \Omega \cap B_{n+1}(0)$, where, as before, $B_r(x)$ denotes the open ball of radius r centered at x. Note that as an intersection of two Dirichlet regular sets, the set Ω_n is again Dirichlet regular (cf. [2, Lemma 3.5]). We also recall that in the case where the diffusion coefficients a_{ij} are merely assumed to be continuous, we have explicitly required in (ii) of Hypothesis 1.1 that Ω_n satisfies the uniform outer cone condition. Altogether, we see that Hypothesis 4.1 is satisfied for $U = \Omega_n$, $\alpha_{ij} = a_{ij}|_{\Omega_n}$ and $\beta_j = b_j|_{\Omega_n}$.

To define approximate boundary conditions on $\partial\Omega_n$, we proceed as follows. We fix functions $\rho_n \in C(\mathbb{R}^d)$ satisfying $\mathbb{1}_{B_n(0)} \le \rho_n \le \mathbb{1}_{B_{n+1}(0)}$ and define $\mu_n : \partial\Omega_n \times \mathscr{B}(\Omega_n) \to \mathscr{M}(\Omega_n)$ by setting

$$\mu_n(z, A) = \begin{cases} \rho_n(z) \int_A \rho_n(x)\, \mu(z, dx) & \text{for } z \in \partial\Omega_n \cap \partial\Omega, \\ 0 & \text{for } z \in \partial\Omega_n \setminus \partial\Omega \subset \partial B_{n+1}(0). \end{cases} \tag{5.2}$$

As before, we occasionally write $\mu_n(z) := \mu_n(z, \cdot)$.

Lemma 5.1. *For the measures* μ_n *defined above, the following assertions hold true:*

(a) *Every* $\mu_n(z)$ *is a positive measure satisfying* $0 \le \mu_n(z, \Omega_n) \le 1$.
(b) *The map* $z \mapsto \mu_n(z)$ *is* $\sigma(\mathscr{M}(\Omega_n), C_b(\Omega_n))$*-continuous.*
(c) *For every* $z \in \partial\Omega_n \cap \partial\Omega_{n+1}$ *we have* $\mu_n(z) \le \mu_{n+1}(z)$.

Proof. (a) This follows directly from the inequalities $0 \le \rho_n \le 1$ and the fact that every $\mu(z)$ is a probability measure.

(b) Let $(z_k) \subset \partial\Omega_n$ be such that $z_k \to z$. If $|z| < n+1$, then also $|z_k| < n+1$ for all but finitely many k. We may thus assume that $(z_k) \subset \partial\Omega_n \cap \partial\Omega$ converges to $z \in \partial\Omega_n \cap \partial\Omega$. Let $f \in C_b(\Omega_n)$. Extending the function $f \cdot \rho_n$ by zero outside Ω_n we obtain a bounded continuous function on all of Ω. Thus,

$$\langle f, \mu_n(z_k)\rangle = \rho_n(z_k)\langle f\rho_n, \mu(z_k)\rangle \to \rho_n(z)\langle f\rho_n, \mu(z)\rangle = \langle f, \mu_n(z)\rangle$$

as $k \to \infty$, by the continuity of $z \mapsto \mu(z)$ and ρ_n.

If, on the other hand, $|z| = n+1$, then the convergence $\langle f, \mu_n(z_k)\rangle \to 0 = \langle f, \mu_n(z)\rangle$ follows from the boundedness of the integrals $\int f\rho_n\, d\mu(z_k)$ and the fact that $\rho_n(z) \to 0$ as $z \to \partial B_{n+1}(0)$.

(c) This follows immediately from the definition, on noting that the functions ρ_n are pointwise increasing. ■

It follows that the measures $\gamma = \mu_n$ satisfy Hypothesis 4.2. Thus, we can define the operator A_n on $L^\infty(\Omega)$ by setting $A_n u = \mathscr{A} u$ for $u \in D(A_n)$, where

$$D(A_n) := \Big\{u \in C(\overline{\Omega_n}) \cap W(\Omega_n) : \mathscr{A} u \in L^\infty(\Omega_n), \\ u(z) = \int_{\Omega_n} u(x)\,\mu_n(z, dx)\ \forall z \in \partial\Omega_n\Big\}.$$

It follows from Proposition 4.3 that $(0,\infty) \subset \rho(A_n)$, and for $\lambda > 0$ the operator $R(\lambda, A_n)$ is positive and satisfies $\|\lambda R(\lambda, A_n)\| \le 1$. Given $f \in L^\infty(\Omega)$, we set $u_n := R(\lambda, A_n)f$. Here, by a slight abuse of notation, we have identified f with its restriction to Ω_n. We will do so also in what follows.

Note that $u_n \in D(A_n)$ so that, by the definition of the measure μ_n, we have $u_n(z) = 0$ for all $z \in \partial\Omega_n \backslash \partial\Omega \subset \partial B_{n+1}(0)$. Thus, setting $\tilde u_n(x) = u_n(x)$ for $x \in \overline{\Omega_n}$ and $\tilde u_n(x) = 0$ for $x \in \overline{\Omega} \setminus \overline{\Omega_n}$ we obtain a continuous function on all of $\overline{\Omega}$. In what follows, we will not distinguish between u_n and its extension $\tilde u_n$ to $\overline{\Omega}$.

We will show that the approximate solutions u_n converge to a solution of problem (5.1) on the unbounded domain Ω. First we prove two lemmas in which the fact that u_n is the resolvent of A_n applied to f is not important. We therefore formulate them in greater generality.

LEMMA 5.2. *Assume Hypothesis* 1.1 *and let* $u_n \in C(\overline{\Omega_n}) \cap W(\Omega_n)$ *be a uniformly bounded sequence, say* $\|u_n\|_\infty \le M$ *for all* $n \in \mathbb{N}$, *such that for every* $m \in \mathbb{N}$ *the sequence* $(\mathscr{A} u_n)_{n\ge m}$ *is uniformly bounded on the set* Ω_m.

Moreover assume that there exists a function $g : \Omega \to \mathbb{R}$ such that for every $m \in \mathbb{N}$ the sequence $(\mathscr{A} u_n)_{n\geq m}$ converges pointwise almost everywhere on Ω_m to g. Then (u_n) has a subsequence that converges locally uniformly and in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for all $p \in (1,\infty)$ to a function $u \in C_b(\Omega) \cap W(\Omega)$ such that $\mathscr{A} u = g$.

Proof. For any $U \Subset \Omega$ we may choose $n_0 \in \mathbb{N}$ such that $U \Subset \Omega_{n_0}$ and thus conclude from [3, Proposition 3.4] that there is a constant $C = C(U)$ such that

$$\begin{aligned}\|u_n\|_{C^1(U)} &\leq C(\|u_n - \mathscr{A} u_n\|_{L^\infty(\Omega_{n_0})} + \|u_n|_{\partial\Omega_n}\|_{C(\partial\Omega_{n_0})}) \\ &\leq C(2M + \|\mathscr{A} u_n\|_{L^\infty(\Omega_{n_0})}) < \infty\end{aligned}$$

for all $n \geq n_0$. By exhausting Ω with increasing sets $U \Subset \Omega$, it follows from the Arzelà–Ascoli theorem and a diagonal argument that a subsequence of u_n (which, for simplicity of notation, we denote by u_n again) converges locally uniformly on Ω to some $u \in C_b(\Omega)$. Moreover, given $p \in (1,\infty)$ we deduce from [22, Theorem 9.11] that there is a constant $C = C(p, U, \mathscr{A}, n_0)$ such that

$$\|u_n\|_{W^{2,p}(U)} \leq C(\|\mathscr{A} u_n\|_{L^p(\Omega_{n_0})} + \|u_n\|_{L^p(\Omega_{n_0})})$$

for all $n \geq n_0$. Applying this estimate to $u_n - u_m$, by dominated convergence we see that (u_n) is a Cauchy sequence in $W^{2,p}(U)$. Since U and $p \in (1,\infty)$ were arbitrary, it now follows that $u \in W(\Omega)$ and that (u_n) along with its first and second derivatives converge in $L^p_{\mathrm{loc}}(\Omega)$ for any $p \in (1,\infty)$. By the structure of $\mathscr{A}$, this shows that also $\mathscr{A} u_n \to \mathscr{A} u$ in $L^p_{\mathrm{loc}}(\Omega)$ and therefore $\mathscr{A} u = g$. ∎

Lemma 5.2 allows us to prove that the solutions of our auxiliary problems converge locally uniformly and in $W^{2,p}_{\mathrm{loc}}(\Omega)$ to a function in $C_b(\Omega) \cap W(\Omega)$. It is an important question whether one can extend this limit to a continuous function on the closure $\overline{\Omega}$. The next lemma provides a sufficient condition for this.

LEMMA 5.3. *Let $(u_n) \subset C(\overline{\Omega})$ be a sequence such that $0 \leq u_n \leq u_{n+1}$ and $u_n|_{\Omega_n} \in W(\Omega_n)$ for every $n \in \mathbb{N}$. Define $u(x) := \sup_{n\in\mathbb{N}} u_n(x)$ for $x \in \overline{\Omega}$ and suppose that $u|_\Omega \in C_b(\Omega) \cap W(\Omega)$ and $u|_{\partial\Omega} \in C_b(\partial\Omega)$. Finally, assume that there is $\lambda > 0$ such that $\lambda u_n - \mathscr{A} u_n \leq \lambda u - \mathscr{A} u$ on Ω_n for every $n \in \mathbb{N}$. Then $u \in C_b(\overline{\Omega})$.*

Proof. As a supremum of continuous functions, u is lower semicontinuous. Since u is assumed to be continuous in Ω, it remains to show continuity of u on $\partial\Omega$. To that end, let $z \in \partial\Omega$ and $(z_n) \subset \overline{\Omega}$ be a sequence converging to z. Pick an $m \in \mathbb{N}$ such that $z \in \partial\Omega \cap B_m(0)$, so that Ω_m contains a neighborhood of z.

As an auxiliary result, let us first show that we can find a function $\varphi \in C(\partial\Omega_m)$ with $\varphi \geq u$ on $\partial\Omega_m$ and $\varphi(s) = u(s)$ for all $s \in \partial\Omega_m \cap \overline{B}_m(0)$. We set $\tilde{\varphi}(s) := u(s)$ for all $s \in C_1 := \partial\Omega_m \cap \overline{B}_m(0)$. Since $C_1 \subset \partial\Omega \cap \overline{B}_m(0)$ and $u \in C_b(\partial\Omega)$, it follows that $\tilde{\varphi} \in C_b(C_1)$. Now let $C_2 := \partial\Omega_m \setminus B_{m+1}(0)$ and $M = \max_{x\in\overline{\Omega_m}} u(x)$. For $s \in C_2$ we set $\tilde{\varphi}(s) := M$, thus obtaining a continuous function $\tilde{\varphi}$ on the closed set $C_1 \cup C_2$. Using the Tietze extension theorem, we can extend $\tilde{\varphi}$ to a continuous function on $\partial\Omega_m$. Finally, we define $\varphi(s) := \max\{\tilde{\varphi}(s), u(s)\}$ for all $s \in \partial\Omega_m$. Clearly, φ is continuous in $s \in \partial\Omega_m \setminus B_{m+1}(0)$, as for such s, we can find a neighborhood of s contained in $\partial\Omega_m$ where φ is continuous as the maximum of two continuous functions. Moreover, φ is continuous in $C_2 \setminus \partial\Omega$, as there $\varphi \equiv \tilde{\varphi}$. It remains to consider $s \in \partial\Omega_m \cap \partial B_{m+1}(0)$. Note that $\varphi(s) = M$ for such an s. If $u(s) < M$, then $\varphi = \tilde{\varphi}$ in a neighborhood of s, proving that φ is continuous in s. If, on the other hand, $u(s) = M$, then $u(z) \to u(s)$ as $z \to s$ in $\partial\Omega \cap B_{m+1}(0)$, hence also $\varphi(z) \to \varphi(s)$ as $z \to s$ in $\partial\Omega \cap B_{m+1}(0)$. But this also holds if $z \to s$ in $\partial\Omega_m \setminus B_{m+1}(0)$, as there $\varphi \equiv M$. This shows that, altogether, φ is a continuous function on $\partial\Omega_m$ which, by construction, has all the other desired properties.

By [3, Proposition 3.3] we find $w \in C(\overline{\Omega_m}) \cap W(\Omega_m)$ with $\lambda w - \mathscr{A} w = \lambda u - \mathscr{A} u$ on Ω_m and $w = \varphi$ on $\partial\Omega_m$. Note that for each $n \geq m$ we have $u_n \leq u \leq \varphi$ on $\partial\Omega_m$ and $\lambda u_n - \mathscr{A} u_n \leq \lambda u - \mathscr{A} u$ on Ω_m. It thus follows from Lemma 4.8, applied with $\gamma = 0$, that $u_n \leq w$ on $\overline{\Omega_m}$ and hence $u \leq w$ on $\overline{\Omega_m}$.

Now observe that

$$\limsup_{n\to\infty} u(z_n) \leq \limsup_{n\to\infty} w(z_n) = w(z) = \varphi(z) = u(z).$$

As $z \in \partial\Omega$ was arbitrary, this shows that u is also upper semicontinuous and hence $u \in C(\overline{\Omega})$. ∎

We can now prove the main result of this section.

THEOREM 5.4. *For every $\lambda > 0$, $(R(\lambda, A_n))_{n\in\mathbb{N}}$ is an increasing sequence of positive operators on $L^\infty(\Omega)$ such that for every $f \in L^\infty(\Omega)$ the sequence $R(\lambda, A_n)f$ converges locally uniformly in Ω to a function $R(\lambda)f \in D_{\max}$ satisfying $\|\lambda R(\lambda)f\|_\infty \leq \|f\|_\infty$ and $(\lambda - \mathscr{A})R(\lambda)f = f$. Moreover, $R(\lambda)$ is a positive operator, i.e. $R(\lambda)f \geq 0$ for $f \geq 0$.*

Proof. Fix $f \in L^\infty(\Omega)$ and $\lambda > 0$. First assume that $f \geq 0$. We consider the approximate operators A_n introduced above and set $u_n := R(\lambda, A_n)f$. Extending it by 0 outside Ω_n, we consider u_n as a continuous function on all of $\overline{\Omega}$. It follows from Proposition 4.3 that $u_n \geq 0$ and $\|\lambda u_n\|_\infty \leq \|f\|_\infty$.

We claim that $0 \leq u_n \leq u_{n+1}$ on Ω_n. To see this, put $v = u_n - u_{n+1}$. If $z \in \partial\Omega_n$ satisfies $|z| < n + 1$, then also $z \in \partial\Omega_{n+1}$. As u_n and u_{n+1} satisfy the boundary condition, we find that

$$\begin{aligned} v(z) &= \langle u_n, \mu_n(z)\rangle - \langle u_{n+1}, \mu_{n+1}(z)\rangle \\ &\le \langle u_n, \mu_n(z)\rangle - \langle u_{n+1}, \mu_n(z)\rangle = \langle v, \mu_n(z)\rangle \end{aligned}$$

since $u_{n+1} \ge 0$ and $\mu_n(z) \le \mu_{n+1}(z)$. If $z \in \partial\Omega_n \cap \partial B_{n+1}(0)$, then $\mu_n(z) = 0$ and also in this case we get $v(z) \le \langle v, \mu_n(z)\rangle$. Moreover, on Ω_n we have

$$\lambda v - \mathscr{A} v = \lambda u_n - \mathscr{A} u_n - (\lambda u_{n+1} - \mathscr{A} u_{n+1}) = f - f = 0$$

almost everywhere, which shows that $\mathscr{A} v$ has the continuous representative λv. Consequently, by Lemma 4.8, we have $v \le 0$ as claimed.

We now define $u(x) := \sup_{n\in\mathbb{N}} u_n(x)$ for $x \in \overline{\Omega}$. Since $\mathscr{A} u_n = \lambda u_n - f$ on Ω_n, for every $m \in \mathbb{N}$ the sequence $(\mathscr{A} u_n)_{n\ge m}$ is uniformly bounded on Ω_m and converges pointwise to $\lambda u - f$. Thus, we conclude from Lemma 5.2 that $u \in C_b(\Omega) \cap W(\Omega)$ and $\mathscr{A} u = \lambda u - f$. Now put

$$\varphi(z) := \int_\Omega u(x)\, \mu(z, dx)$$

for $z \in \partial\Omega$. Since $u \in C_b(\Omega)$, the function φ belongs to $C_b(\partial\Omega)$ by the continuity of $z \mapsto \mu(z)$. Moreover, using monotone convergence, we find

$$\begin{aligned} u(z) &= \sup_{n\in\mathbb{N}} u_n(z) = \sup_{n\in\mathbb{N}} \langle u_n, \mu_n(z)\rangle \\ &= \sup_{n\in\mathbb{N}} \rho_n(z)\langle u_n\rho_n, \mu(z)\rangle = \langle u, \mu(z)\rangle = \varphi(z) \end{aligned}$$

for all $z \in \partial\Omega$. In particular, u is continuous on $\partial\Omega$. Therefore, $u \in C_b(\overline{\Omega})$ by Lemma 5.3.

Now let $f \in L^\infty(\Omega)$ be real-valued. We have $f = f^+ - f^-$. Then $R(\lambda, A_n)f = R(\lambda, A_n)f^+ - R(\lambda, A_n)f^-$. By the above, $u_n^\pm := R(\lambda, A_n)f^\pm$ converges locally uniformly to a function $u^\pm \in C(\overline{\Omega}) \cap W(\Omega)$ with $u^\pm(z) = \langle u^\pm, \mu(z)\rangle$ for all $z \in \partial\Omega$, and $\lambda u^\pm - \mathscr{A} u^\pm = f^\pm$. Consequently, $R(\lambda, A_n)f$ converges locally uniformly to $u := u^+ - u^-$, which is in $D_{\max}$ and solves $\lambda u - \mathscr{A} u = f$. The case of a complex-valued f can be handled similarly, by decomposing $f = \operatorname{Re} f + i \operatorname{Im} f$. ■

We next want to define the realization A_μ of the differential operator $\mathscr{A}$ that appears in Theorem 1.3. To that end, we first prove that the operators $R(\lambda)$ constructed in Theorem 5.4 form a pseudoresolvent.

Lemma 5.5. *Let the operator $R(\lambda) \in \mathscr{L}(L^\infty(\Omega))$ for $\lambda > 0$ be given as in Theorem* 5.4. *Then:*

(a) *$R(\lambda)$ is an adjoint operator for each $\lambda > 0$.*
(b) *The family $(R(\lambda))_{\lambda>0}$ is a pseudoresolvent, i.e.*

$$R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1)R(\lambda_1)R(\lambda_2).$$

Proof. (a) Note that the $R(\lambda, A_n)$ are adjoint operators in view of Lemma 3.5 and Proposition 4.3(c). Now (a) follows from Lemma 2.11(1).

(b) We have $R(\lambda_1, A_n) - R(\lambda_2, A_n) = (\lambda_2 - \lambda_1)R(\lambda_1, A_n)R(\lambda_2, A_n)$ for all $\lambda_1, \lambda_2 > 0$. In view of the definition of $R(\lambda_1)$ and $R(\lambda_2)$, part (b) follows immediately from Lemma 2.11(2). ■

Since $(R(\lambda))_{\lambda>0}$ is a pseudoresolvent, the kernel and the range of $R(\lambda)$ are independent of $\lambda > 0$. However, as $(\lambda - \mathscr{A})R(\lambda)f = f$, it follows that $\ker R(\lambda) = \{0\}$ for all $\lambda > 0$. By [1, Proposition B.6], $(R(\lambda))_{\lambda>0}$ is the resolvent of an operator.

Definition 5.6. The operator A_μ is defined as the unique operator for which $R(\lambda, A_\mu) = R(\lambda)$ for all $\lambda > 0$. In particular, $D(A_\mu)$ is the range of $R(\lambda)$.

We can now characterize the domain $D(A_\mu)$ in a different way.

Lemma 5.7. *Let $\lambda > 0$. For $f \geq 0$ the function $R(\lambda, A_\mu)f$ is minimal among the positive solutions of $\lambda u - \mathscr{A}u = f$ in $D_{\max}$.*

Proof. Let $0 \leq u \in D_{\max}$ be such that $\lambda u - \mathscr{A}u = f$. Given $n \in \mathbb{N}$, let $u_n = R(\lambda, A_n)f$. Then $(\lambda - \mathscr{A})(u_n - u) = 0$ on Ω_n. Moreover, for $z \in \partial\Omega_n$,

$$u_n(z) - u(z) = \langle u_n, \mu_n(z)\rangle - \langle u, \mu(z)\rangle \leq \langle u_n - u, \mu_n(z)\rangle.$$

By Lemma 4.8, $u_n \leq u$. Taking the supremum over n, we conclude that $R(\lambda, A_\mu)f = \sup_{n\in\mathbb{N}} u_n \leq u$. ■

Let us now prove that the resolvent of the operator A_μ consists of strong Feller operators.

Lemma 5.8. *For $\lambda > 0$ the operator $R(\lambda, A_\mu)$ is a strong Feller operator.*

Proof. As $R(\lambda, A_\mu)$ takes values in $D_{\max} \subset C_b(\overline{\Omega})$, it remains to prove the continuity condition (ii) in Definition 3.4. As we are dealing with positive operators, it actually suffices to merely consider increasing sequences (cf. [3, Lemma 5.5]).

So let $(f_n) \subset L^\infty(\Omega)$ be an increasing sequence that is uniformly bounded and consists of positive functions. We set $f := \sup_{n\in\mathbb{N}} f_n$. We fix $\lambda > 0$ and set $u_n := R(\lambda, A_\mu)f_n \in D_{\max} \subseteq C_b(\overline{\Omega}) \cap W(\Omega)$. Since $R(\lambda, A_\mu)$ is a positive operator, (u_n) is an increasing and uniformly bounded sequence of positive functions. Let $u(x) := \sup_{n\in\mathbb{N}} u_n(x)$ for $x \in \overline{\Omega}$. Note that $\mathscr{A}u_n = \lambda u_n - f_n$ is uniformly bounded and converges pointwise almost everywhere to $\lambda u - f$. Hence, it follows from Lemma 5.2 that $u \in C_b(\Omega) \cap W(\Omega)$ and $\lambda u - \mathscr{A}u = f$. Consequently, the mapping

$$\varphi(z) := \int_\Omega u(x)\,\mu(z, dx)$$

defines a function $\varphi \in C_b(\partial\Omega)$ by the continuity of $z \mapsto \mu(z)$. Moreover, by monotone convergence we obtain

$$u(z) = \sup_{n\in\mathbb{N}} u_n(z) = \sup_{n\in\mathbb{N}} \langle u_n, \mu(z)\rangle = \langle u, \mu(z)\rangle = \varphi(z)$$

for all $z \in \partial\Omega$. In particular, u is continuous on $\partial\Omega$. Therefore, $u \in C_b(\overline{\Omega})$ by Lemma 5.3. This shows that $u \in D_{\max}$ and $\lambda u - \mathscr{A}u = f$.

As a consequence of Lemma 2.11, $R(\lambda, A_\mu)$ is an adjoint operator, whence $R(\lambda, A_\mu)f_n \rightharpoonup^* R(\lambda, A_\mu)f$. Since $L^1(\Omega)$ separates $C_b(\overline{\Omega})$, we must have $u = R(\lambda, A_\mu)f$. ∎

6. Unique solvability of the elliptic equation. Throughout this section, we assume Hypotheses 1.1 and 1.2. We have seen in Lemma 5.7 that for positive f the function $R(\lambda, A_\mu)f$ is the minimal solution of the elliptic equation $\lambda u - \mathscr{A}u = f$ in $D_{\max}$. It is a natural question when the elliptic equation is uniquely solvable in $D_{\max}$, i.e. when $D(A_\mu) = D_{\max}$. Without further assumptions, this is not the case: see [28, Example 7.10] for an example where $\Omega = \mathbb{R}^d$, i.e. we do not have boundary conditions.

Let us begin with the following lemma.

LEMMA 6.1. *The following are equivalent:*

(i) $\mathbb{1} \in D(A_\mu)$.
(ii) $D(A_\mu) = D_{\max}$.

Proof. Assume that $\mathbb{1} \in D(A_\mu)$. To prove (ii), we only need to show that for some $\lambda > 0$ the operator $\lambda - \mathscr{A}$ is injective on $D_{\max}$. Indeed, $R(\lambda, A_\mu)$ is a bijection between $D(A_\mu)$ and $L^\infty(\Omega)$ and $\lambda - \mathscr{A} : D_{\max} \to L^\infty(\Omega)$ is clearly surjective. Thus, if $\lambda - \mathscr{A}$ is injective on $D_{\max}$, then $R(\lambda, A_\mu)(\lambda - \mathscr{A})$ is a bijection from $D_{\max}$ to $D(A_\mu)$ and $R(\lambda, A_\mu)(\lambda - \mathscr{A})u = u$ for $u \in D_{\max}$.

So fix $\lambda > 0$ and let $u \in D_{\max}$ with $\lambda u - \mathscr{A}u = 0$ be given. We can assume that $-1 \le u(x) \le 1$ for all $x \in \overline{\Omega}$. Then $v := \mathbb{1} - u$ is a positive function which satisfies $\lambda v - \mathscr{A}v = \lambda\mathbb{1}$. As $\mathbb{1} \in D(A_\mu)$, we must have $R(\lambda, A_\mu)\lambda\mathbb{1} = \mathbb{1}$. It follows from Lemma 5.7 that $\mathbb{1} \le v = \mathbb{1} - u$, i.e. $u \le 0$. Similarly, $\tilde{v} := \mathbb{1} + u$ is a positive function with $\lambda\tilde{v} - \mathscr{A}\tilde{v} = \lambda\mathbb{1}$, and with the same arguments we find $u \ge 0$. This proves that $\lambda - \mathscr{A}$ is injective on $D_{\max}$ and finishes the proof of (i)⇒(ii). The converse is trivial. ∎

We will see in the next section that A_μ generates a positive, injective and contractive $*$-semigroup T_μ on $L^\infty(\Omega)$. Noting that $\mathscr{A}\mathbb{1} = 0$, we see that $\mathbb{1} \in D(A_\mu)$ is equivalent to $\mathbb{1} \in \ker A_\mu$, which in view of Proposition 2.8 is equivalent to $T_\mu(t)\mathbb{1} = \mathbb{1}$ for all $t > 0$. Thus, the elliptic equation is uniquely solvable if and only if the semigroup generated by A_μ is Markovian.

We next provide a sufficient condition for $\lambda - \mathscr{A}$ to be injective on $D_{\max}$. This condition involves the existence of a certain *Lyapunov function* for $\mathscr{A}$.

HYPOTHESIS 6.2. There exists a function $V \in C(\overline{\Omega}) \cap W(\Omega)$ such that

(a) $\lim_{|x|\to\infty} V(x) = \infty$;
(b) $\mathscr{A}V$ coincides almost everywhere on Ω with a continuous function that is bounded on bounded subsets of Ω;
(c) there is a radius $r > 0$ such that $(\lambda - \mathscr{A})V \geq 0$ on $\Omega \setminus B_r(0)$.

LEMMA 6.3. *Assume Hypothesis* 6.2. *Let* $\lambda > 0$ *and let* $u \in W(\Omega)\cap C_b(\overline{\Omega})$ *be such that* $\mathscr{A}u$ *has a continuous version and such that* $(\lambda - \mathscr{A})u \leq 0$. *Then*

$$\sup_{x\in\overline{\Omega}} u(x) \leq \sup_{z\in\partial\Omega} u^+(z). \tag{6.1}$$

Proof. In view of Hypothesis 6.2 we may (and do) assume that $(\lambda - \mathscr{A})V \geq 0$ on Ω, since we may replace V by $V + c\mathbb{1}_{\overline{\Omega}}$ if necessary.

For each $n \in \mathbb{N}$ define $u_n := u - \frac{1}{n}V$ and note that by Hypothesis 6.2(a) we may find a constant $C \geq 0$ such that $V \geq -C$. Therefore, $u_n \leq u + \frac{1}{n}C$ on $\overline{\Omega}$ for each $n \in \mathbb{N}$ and in particular u_n is bounded from above. We immediately deduce that

$$\sup_{z\in\partial\Omega} u_n^+(z) \leq \sup_{z\in\partial\Omega} u^+(z) + \frac{1}{n}C$$

for each $n \in \mathbb{N}$. Since u_n converges to u pointwise, it also follows that

$$\lim_{n\to\infty} \sup_{x\in\overline{\Omega}} u_n(x) = \sup_{x\in\overline{\Omega}} u(x). \tag{6.2}$$

Indeed, given $\varepsilon > 0$ we can pick $x_0 \in \overline{\Omega}$ such that $u(x_0) > \sup u - \varepsilon$. Then we may find $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in\overline{\Omega}} u_n(x) \leq \sup_{x\in\overline{\Omega}} u(x) + \varepsilon \leq u(x_0) + 2\varepsilon \leq u_n(x_0) + 3\varepsilon \leq \sup_{x\in\overline{\Omega}} u_n(x) + 3\varepsilon$$

for every $n \geq n_0$, which proves (6.2) as $\varepsilon > 0$ was arbitrary. To show (6.1), it thus suffices to show that

$$\sup_{x\in\overline{\Omega}} u_n(x) \leq \sup_{z\in\partial\Omega} u_n^+(z) \tag{6.3}$$

for every $n \in \mathbb{N}$.

It follows from Hypothesis 6.2(a) that $\lim_{|x|\to\infty} u_n(x) = -\infty$ for any $n \in \mathbb{N}$. Thus we can find $x_n \in \overline{\Omega}$ with $u_n(x_n) = \max_{x\in\overline{\Omega}} u_n(x)$. If $x_n \in \partial\Omega$ then (6.3) holds true, so assume $x_n \in \Omega$. As $\mathscr{A}V$ has a continuous version, so does $\mathscr{A}u_n$ and we conclude from Lemma 4.5 that $\mathscr{A}u_n(x_n) \leq 0$. Since both $(\lambda - \mathscr{A})V \geq 0$ and $(\lambda - \mathscr{A})u \leq 0$, we find $(\lambda - \mathscr{A})u_n \leq 0$ and it follows that $\lambda u_n(x_n) \leq \mathscr{A}u_n(x_n) \leq 0$. Thus, in this case, (6.3) holds trivially. ∎

THEOREM 6.4. *Assume Hypothesis* 6.2 *and that there exist* $N \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that* $\mu(z, \Omega_N) \geq \varepsilon$ *for all* $z \in \partial\Omega$. *Let* $\lambda > 0$ *and* $u \in D_{\max}$ *be such that* $(\lambda - \mathscr{A})u \leq 0$. *Then* $u \leq 0$.

Remark 6.5. The condition that there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\mu(z, \Omega_N) \geq \varepsilon$ is a mild concentration condition for the measures $\mu(z)$. It is in particular satisfied whenever the set $\{\mu(z) : z \in \partial\Omega\}$ is tight. As the map $z \mapsto \mu(z)$ is $\sigma(\mathscr{M}(\Omega), C_b(\Omega))$-continuous, this is in particular the case when $\partial\Omega$ is compact, e.g. for an outer domain. However, this condition is weaker than tightness. For example, if $\Omega = (0, \infty) \times \mathbb{R}$, then we might choose for $z = (0, y) \in \partial\Omega$ the measure $\mu(z) = \frac{1}{2}\delta_{(1,1)} + \frac{1}{2}\delta_{(y,0)}$. These measures satisfy the concentration condition but they are not tight.

Proof of Theorem 6.4. Assume to the contrary that $u(x_0) > 0$ for some $x_0 \in \overline{\Omega}$. By Lemma 6.3, we have

$$\sup_{x \in \overline{\Omega}} u(x) \leq \sup_{z \in \partial\Omega} u^+(z),$$

which implies that $\sup_{z\in\partial\Omega} u^+(z) > 0$. We set $S := \sup_{x\in\overline{\Omega}} u(x) > 0$.

We claim that $\sup_{x\in\overline{\Omega_N}} u(x) < S$. Otherwise, we find some $x_1 \in \overline{\Omega_N}$ such that $\sup_{x\in\overline{\Omega}} u(x) = u(x_1)$. By Lemma 4.7, we must have $x_1 \in \Omega_N$. It now follows from Lemma 4.5 that

$$\lambda u(x_1) \leq \mathscr{A} u(x_1) \leq 0,$$

in contradiction to $u(x_1) > 0$.

Thus, we must have $\sup_{x\in\overline{\Omega_N}} u(x) = \sup_{x\in\Omega_N} u(x) = S - \rho$ for some $0 < \rho \leq S$. Now pick a sequence $(z_n) \subset \partial\Omega$ such that $u(z_n) \to S$ as $n \to \infty$. Using the boundary conditions, we see that for every $n \in \mathbb{N}$,

$$\begin{aligned} u(z_n) &= \int_\Omega u\, d\mu(z_n) = \int_{\Omega_N} u\, d\mu(z_n) + \int_{\Omega\setminus\Omega_N} u\, d\mu(z_n) \\ &\leq (S - \rho)\mu(z_n, \Omega_N) + S\mu(z_n, \Omega \setminus \Omega_N) = S - \rho \cdot \mu(z_n, \Omega_N) \leq S - \varepsilon\rho. \end{aligned}$$

Letting $n \to \infty$ we obtain the contradiction $S \leq S - \varepsilon\rho$. This shows that $u \leq 0$ on $\overline{\Omega}$. ∎

Corollary 6.6. *Assume Hypothesis* 6.2 *and that there are* $N \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that* $\mu(z, \Omega_N) \geq \varepsilon$ *for all* $z \in \partial\Omega$. *Then* $D(A_\mu) = D_{\max}$.

Proof. Let $u \in D_{\max}$ be such that $\lambda u - \mathscr{A} u = 0$. Theorem 6.4 implies that $u = 0$. Thus, $\lambda - \mathscr{A} u$ is injective on $D_{\max}$, whence $D(A_\mu) = D_{\max}$. ∎

We finally determine the kernel of A_μ when Ω is additionally connected.

Corollary 6.7. *Assume Hypothesis* 6.2 *and that there exist* $N \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that* $\mu(z, \Omega_N) \geq \varepsilon$ *for all* $z \in \partial\Omega$. *Moreover, let* Ω *be connected. Then* $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$.

Proof. If $u \in D_{\max}$ satisfies $-\mathscr{A} u \leq 0$, then either u is constant or $u \leq 0$. This can be proved by repeating the proof of Theorem 6.4 till the point where we deduced from $\sup_{x\in\overline{\Omega_N}} u(x) = \sup_{x\in\overline{\Omega}} u(x)$ that there must be some $x_1 \in \Omega_N$ such that $u(x_1) = \sup_{x\in\overline{\Omega}} u(x) > 0$. At this point, the strict maximum principle [22, Theorem 9.6] implies that u is constant. If $\sup_{x\in\overline{\Omega_N}} < \sup_{x\in\overline{\Omega}}$, the proof can be finished as that of Theorem 6.4. ∎

7. The semigroup. After our preparations it is now very easy to establish that A_μ generates a semigroup. Again, we assume Hypotheses 1.1 and 1.2 throughout this section.

THEOREM 7.1. *The operator A_μ generates a positive contractive $*$-semigroup $T_\mu = (T_\mu(t))_{t>0}$ on $L^\infty(\Omega)$.*

Proof. Consider again the operators A_n from Section 5. By Proposition 4.3 the operator A_n generates a contractive, positive, holomorphic semigroup T_n on $L^\infty(\Omega_n)$. We have already remarked that we may also view T_n as an injective contractive $*$-semigroup. Extending T_n and $R(\lambda, A_n)$ (for $\lambda > 0$) by zero outside $\overline{\Omega}_n$, we obtain a (no longer injective) contractive $*$-semigroup with Laplace transform $R(\lambda, A_n)$. By Theorem 5.4 and Proposition 2.10 the semigroups T_n are increasing. The claim now follows from Proposition 2.12. ∎

We should point out that in Theorem 7.1 we only obtain a semigroup on the space $L^\infty(\Omega)$. In that respect, the situation here is very different from that on bounded domains or for the elliptic equation in Section 5 where the operators we obtained always took values in the space of bounded continuous functions. It was this fact that allowed us to "lift" an operator on $L^\infty(\Omega)$ to a bounded linear operator on $B_b(\overline{\Omega})$. Afterwards, we could use Lemma 3.3 to establish that the lifted operator is a kernel operator.

Our next goal is to prove that we can also lift $T_\mu(t) \in \mathscr{L}(L^\infty(\Omega))$ for $t > 0$ to kernel operators on $\overline{\Omega}$. To that end, we will use some results concerning order-theoretic properties of kernel operators from [21]. In particular, we will use the following result which we formulate in the setting of Section 3.

LEMMA 7.2. *Let E be a complete, separable metric space and let k_n be a sequence of sub-Markovian kernel on E, i.e. every k_n is a kernel on E such that $k_n(x,\cdot)$ is a positive measure on $\mathscr{B}(E)$ with $0 \le k_n(x,E) \le 1$ for every $x \in E$. Denote the associated operators on $B_b(E)$ and $\mathscr{M}(E)$ by K_n and K_n' respectively. Put $k(x,A) := \sup_n k_n(x,A)$ for $x \in E$ and $A \in \mathscr{B}(E)$. Then*

(a) *k is a sub-Markovian kernel on E; denote the associated operators on $B_b(E)$ and $\mathscr{M}(E)$ by K and K' respectively;*
(b) *$\sup_n K_n = K$ in $\mathscr{L}(B_b(E))$ and $\sup_n K_n' = K'$ in $\mathscr{L}(\mathscr{M}(E))$;*
(c) *$\sup_n K_n f = Kf$ for every $f \in B_b(E)_+$ and $\sup_n K_n'\nu = K'\nu$ for every $\nu \in \mathscr{M}(E)_+$.*

Proof. (a) follows from [21, Lemma 3.5].

Note that since k_n is sub-Markovian, we have $K_n' \le I$ for every $n \in \mathbb{N}$. It follows from [21, Theorem 3.6] that $\sup K_n'$ exists in $\mathscr{L}(\mathscr{M}(E))$ and is again a kernel operator. The proof of [21, Theorem 3.6] shows that the kernel associated to $\sup K'_n$ is exactly k, and $\sup_n K_n\nu = K\nu$ for every $\nu \in \mathscr{M}(E)_+$. Thus our assertions in (b) and (c) concerning K' hold true. Now note that if $f = \mathbb{1}_A$, then

$$Kf = k(\cdot, A) = \sup_n k_n(\cdot, A) = K_n f.$$

By linearity, the same holds true whenever $f \geq 0$ is a simple function. For a general $f \in B_b(E)_+$, we find, given $\varepsilon > 0$, a simple function $g \geq 0$ with $0 \leq g \leq f$ and $\|f - g\|_\infty \leq \varepsilon$. Since the kernels k_n, thus also k, are sub-Markovian, the operators K_n and K are contractions, whence $\|Kf - Kg\| \leq \varepsilon$ and $\|K_n f - K_n g\| \leq \varepsilon$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}&\|Kf - \sup K_n f\| \\ &\quad \leq \|Kf - Kg\| + \|Kg - \sup K_n g\| + \|\sup K_n g - \sup K_n f\| \leq 2\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, this proves the rest of the assertions. ∎

We obtain

Proposition 7.3. *There is a family of kernel operators $(K_\mu(t))_{t>0}$, associated to sub-Markovian kernels on $\overline{\Omega}$, such that*

(a) *$K_\mu(t)f$ is a version of $T_\mu(t)\iota(f)$ for every $f \in B_b(\overline{\Omega})$;*
(b) *$K_\mu(t + s) = K_\mu(t)K_\mu(s)$ for all $t, s > 0$.*

Proof. We again consider the semigroups T_n generated by the approximate operators A_n, extended to all of $\overline{\Omega}$ by zero. As $T_n(t)$ takes values in $C_b(\overline{\Omega})$, we can consider the operator $K_n(t) := T_n(t) \circ \iota \in \mathscr{L}(B_b(\overline{\Omega}))$ for every $t > 0$. By Proposition 4.3(c) these are kernel operators and as a consequence of Theorem 5.4 this sequence is increasing. Lemma 7.2 shows that $K_\mu(t) := \sup_n K_n(t)$ exists in $\mathscr{L}(B_b(\overline{\Omega}))$ and is a kernel operator.

Since $K_\mu(t)f = \sup_n K_n(t)f$ for all $f \in B_b(\overline{\Omega})_+$ by Lemma 7.2(c), $K_\mu(t)f$ is a version of $T_\mu(t)\iota(f)$ for all $f \geq 0$. By linearity, this is also true for general f, proving (a).

As for (b), first note that for $t, s > 0$ and $n \in \mathbb{N}$, we have $K_n(t)K_n(s) \leq K_\mu(t)K_\mu(s)$, whence

$$K_\mu(t + s) = \sup_n K_n(t + s) = \sup_n K_n(t)K_n(s) \leq K_\mu(t)K_\mu(s).$$

On the other hand, for $f \geq 0$, the sequence $K_n(s)f$ is bounded and converges pointwise to $K_\mu(s)f$. As $K_\mu(t)$ is a kernel operator, Lemma 3.1 yields $\sup_n K_\mu(t)K_n(s)f = K_\mu(t)K_\mu(s)f$. Consequently,

$$\sup_n K_n(t)K_n(s)f \geq K_\mu(t)K_\mu(s)f,$$

which proves the other inequality and thus (b). ∎

REMARK 7.4. Using the monotone convergence theorem, we see that

$$\langle R(\lambda, A_\mu)\iota(f), \nu\rangle = \int_0^\infty e^{-\lambda t}\langle K_\mu(t)f, \nu\rangle\, dt$$

for all $\lambda > 0$, $f \in B_b(\overline{\Omega})_+$, and $\nu \in \mathscr{M}(\overline{\Omega})_+$. By linearity, this also holds for all $f \in B_b(\overline{\Omega})$ and $\nu \in \mathscr{M}(\overline{\Omega})$.

This shows that $(K_\mu(t))_{t>0}$ defines an integrable semigroup on the norming dual pair $(B_b(\overline{\Omega}), \mathscr{M}(\overline{\Omega}))$ in the sense of [26, Definition 5.11]. Its Laplace transform is given by $(R(\lambda, A_\mu) \circ \iota)_{\lambda>0}$, which, of course, is not injective and thus cannot be the resolvent of an operator. However, we may associate a multi-valued generator to the semigroup $K_\mu(t)$. A characterization of this multi-valued generator similar to Proposition 2.8 remains valid: see [26, Proposition 5.7].

THEOREM 7.5. *If $\mathbb{1} \in D(A_\mu)$, then T_μ is Markovian and enjoys the strong Feller property.* (Note that by Lemma 6.1 the condition $\mathbb{1} \in D(A_\mu)$ is equivalent to $D(A_\mu) = D_{\max}$.)

Proof. If $\mathbb{1} \in D(A_\mu)$, then $A_\mu\mathbb{1} = 0$. As A_μ is the generator of T_μ, we must have $T_\mu(t)\mathbb{1} = \mathbb{1}$ for all $t > 0$ in view of Proposition 2.8. We should point out that this is an equality almost everywhere. However, as explained in Remark 7.4, we can apply the corresponding result to the semigroup $(K_\mu(t))_{t>0}$ on $B_b(\overline{\Omega})$ and obtain $K_\mu(t)\mathbb{1} = \mathbb{1}$ everywhere on $\overline{\Omega}$ for every $t > 0$.

Now let $0 \le f \le \mathbb{1}$ be given, so that $K_n(t)f \uparrow K_\mu(t)f$ pointwise. It follows that $K_\mu(t)f$ is lower semicontinuous. On the other hand,

$$\mathbb{1} - K_\mu(t)f = K_\mu(t)(\mathbb{1} - f) = \sup_n K_n(t)(\mathbb{1} - f)$$

is also lower semicontinuous. As $\mathbb{1}$ is continuous, it follows that $K_\mu(t)f$ is upper semicontinuous.

Altogether, we have proved that $K_\mu(t)f$ is continuous whenever $0 \le f \le 1$. Scaling and decomposing a function into the positive and the negative part, we see that $K_\mu(t)$ is a strong Feller operator. ∎

8. Asymptotic behavior. In this section, we will study the asymptotic behavior of the semigroup T_μ under the assumption that $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$. We note that Corollary 6.7 provides a sufficient condition for this to happen. If $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$, then in particular T_μ is Markovian and enjoys the strong Feller property and we can use recent results [19, 21] on the asymptotic behavior of such semigroups. Of particular importance are *invariant probability measures* of the semigroup. We recall that a measure $\nu^\star \in \mathscr{M}(\overline{\Omega})$ is called *invariant* if $T_\mu(t)'\nu^\star = \nu^\star$ for all $t > 0$, i.e. $\nu^\star \in \operatorname{fix}(T_\mu')$.

Theorem 8.1. *Assume that* $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$. *Then there is at most one invariant probability measure for* T_μ. *If* $\nu^\star$ *is such a measure, then for* $f \in L^\infty(\Omega)$ *we have*

$$\lim_{t\to\infty} T_\mu(t)f = \int_{\overline{\Omega}} f\,d\nu^\star \cdot \mathbb{1}$$

uniformly on compact subsets of $\overline{\Omega}$, *and for* $\nu \in \mathscr{M}(\overline{\Omega})$ *we have*

$$\lim_{t\to\infty} T_\mu' \nu = \nu(\overline{\Omega})\nu^\star$$

in total variation norm.

Proof. If $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$, then in particular $\mathbb{1} \in D(A_\mu)$, so that T_μ enjoys the strong Feller property by Theorem 7.5. Moreover, in view of Proposition 2.8, we have $\operatorname{fix}(T_\mu) = \operatorname{span}\{\mathbb{1}\}$. We now have to distinguish the case when the semigroup T_μ is weakly ergodic (in the sense of [20]) and when it is not. As T_μ enjoys the strong Feller property, we infer from [20, Theorems 4.4 and 5.7] that T_μ is weakly ergodic if and only if $\operatorname{fix}(T_\mu')$ separates $\operatorname{fix}(T_\mu)$.

If $\operatorname{fix}(T_\mu')$ separates $\operatorname{fix}(T_\mu)$, then the semigroup is weakly ergodic and it follows from [20, Theorem 4.4] that $\operatorname{fix}(T_\mu)$ separates $\operatorname{fix}(T_\mu')$. As $\operatorname{fix}(T_\mu)$ is one-dimensional, so is $\operatorname{fix}(T_\mu')$. If, on the other hand, $\operatorname{fix}(T_\mu')$ does not separate $\operatorname{fix}(T_\mu)$, then we must have $\operatorname{fix}(T_\mu') = \{0\}$. In either case, there can be at most one invariant probability measure.

Now assume that $\nu^\star$ is an invariant probability measure. Then T_μ is weakly ergodic with ergodic projection $P = \mathbb{1} \otimes \nu^\star$, i.e. $Pf = \int_{\overline{\Omega}} f\,d\nu^\star \cdot \mathbb{1}$. It follows from [18, Corollary 3.7] (a related result can be found in Version 1 of [19] on the arXiv) that for every $\nu \in \mathscr{M}(\overline{\Omega})$ we have $T_\mu'(t)\nu \to P'\nu$ in total variation norm as $t \to \infty$. It easily follows that $T_\mu(t)f \to Pf$ with respect to $\sigma(C_b(\overline{\Omega}), \mathscr{M}(\overline{\Omega}))$ as $t \to \infty$. However, as T_μ enjoys the strong Feller property, for every sequence $t_n \to \infty$ the sequence $T_\mu(t_n)f$ has a β_0-convergent subsequence. But as $T_\mu(t)f \to Pf$ with respect to $\sigma(C_b(\overline{\Omega}), \mathscr{M}(\overline{\Omega}))$, the only possible accumulation point is Pf and we find that $T_\mu(t)f \to Pf$ with respect to β_0 and thus also uniformly on compact subsets of $\overline{\Omega}$. ∎

To establish the existence of an invariant probability measure, again the existence of a suitable Lyapunov function is sufficient. Note, however, that such a Lyapunov function has to satisfy more restrictive assumptions than in Hypothesis 6.2. Indeed, if $\Omega = \mathbb{R}^d$ and $\mathscr{A} = \Delta$, the Laplace operator, then $V(x) = |x|^2$ can be used as a Lyapunov function in the sense of Hypothesis 6.2. However, there is no invariant probability measure for the heat semigroup on $\mathbb{R}^d$.

In [28], and also other references, using the Krylov–Bogolyubov theorem, invariant measures are constructed as certain weak accumulation points of Cesàro means of the semigroup. In our situation, it is more convenient to work with Abel means.

LEMMA 8.2. *Suppose that $\lambda_n \subset (0,\infty)$ is such that $\lambda_n \downarrow 0$ and there is a probability measure ν such that $\lambda_n R(\lambda_n, A_\mu)'\nu$ converges to $\nu^\star$ in the $\sigma(\mathscr{M}(\overline{\Omega}), C_b(\overline{\Omega}))$-topology. Then $\nu^\star$ is an invariant measure for T'_μ.*

Proof. As $R(\lambda, A_\mu)$ is a strong Feller operator, we may view $R(\lambda, A_\mu)'$ as an operator which is $\sigma(\mathscr{M}(\overline{\Omega}), C_b(\overline{\Omega}))$-continuous. Note that $R(\lambda, A_\mu)'$ is not necessarily injective, whence it may not be the resolvent of an operator. We may, however, view it as the resolvent of a multivalued and $\sigma(\mathscr{M}(\overline{\Omega}), C_b(\overline{\Omega}))$-closed operator which we may view as a multivalued generator of T'_μ. With a slight abuse of notation, we denote this operator by A'_μ.

Let $\nu_n := \lambda_n R(\lambda_n, A_\mu)'\nu$. Then $\nu_n \rightharpoonup \nu^\star$. Here and in what follows, $\rightharpoonup$ denotes $\sigma(\mathscr{M}(\overline{\Omega}), C_b(\overline{\Omega}))$-convergence. From the identity $(\lambda_n - A'_\mu)R(\lambda_n, A_\mu)' = I$, we obtain

$$A'_\mu \nu_n = \lambda_n \nu_n - \lambda_n \nu \rightharpoonup 0.$$

By the closedness of A'_μ, we find $\nu^\star \in D(A'_\mu)$ and $A'_\mu \nu^\star = 0$. From [26, Proposition 5.7] it follows that $\nu^\star$ is invariant. ∎

We can now prove a Lyapunov criterion that ensures the existence of an invariant probability measure.

THEOREM 8.3. *Assume that* $\ker A_\mu = \operatorname{span}\{\mathbb{1}\}$. *Suppose furthermore that there is a function $V \in C(\overline{\Omega}) \cap W(\Omega)$ such that*

(i) *$V \geq 0$ and $V(x) \to \infty$ as $|x| \to \infty$;*
(ii) *$\mathscr{A}V$ coincides almost everywhere on Ω with a continuous function that is bounded on bounded subsets and $\mathscr{A}V(x) \to -\infty$ as $|x| \to \infty$;*
(iii) *for every $z \in \partial\Omega$ the function V is $\mu(z)$-integrable and if $v_0(z) := \int_\Omega V(x)\,\mu(z,dx)$ then $v_0 \leq V$ on $\partial\Omega$.*

Then T_μ has a unique invariant probability measure.

Proof. In view of Lemma 8.2, it suffices to prove that for some $x_0 \in \Omega$ the set $\{\lambda R(\lambda, A_\mu)'\delta_{x_0} : 0 < \lambda \leq 1\}$ is tight.

As a first step, let us prove that the function $-\mathscr{A}V$ is integrable with respect to the measure $R(\lambda, A_\mu)'\delta_{x_0}$ whenever $\lambda \in (0,1]$. To that end, fix n_0 so large that $x_0 \in \Omega_{n_0}$. For $n \geq n_0$, put $\tilde{f}_n := R(\lambda, A_n)(\lambda - \mathscr{A})V$. Then $\tilde{f}_n \in D(A_n)$. In particular, $\tilde{f}_n$ satisfies $\tilde{f}_n(z) = \langle \tilde{f}_n, \mu_n(z)\rangle$ for all $z \in \partial\Omega_n$. Now put $f_n := \tilde{f}_n - \mathbb{1}_{\overline{\Omega_n}}V$. Then $(\lambda - \mathscr{A})f_n = 0$ on Ω_n. Since

$$\langle \mathbb{1}_{\overline{\Omega_n}}V, \mu_n(z)\rangle \leq \langle V, \mu(z)\rangle = v_0(z) \leq V(z)$$

for all $z \in \partial\Omega_n$, we infer that $f_n(z) \leq \langle f_n, \mu_n(z)\rangle$ for all $z \in \partial\Omega_n$. It follows from Lemma 4.8 that $f_n \leq 0$ on Ω_n and thus

$$-R(\lambda, A_n)\mathscr{A}V \leq R(\lambda, A_n)(\lambda - \mathscr{A})V \leq V$$

on Ω_n, as $\lambda R(\lambda, A_n)V \geq 0$.

Now pick $c > 0$ such that $c - \mathscr{A}V \geq 0$, which is possible in view of (ii). Note that $R(\lambda, A_n)c \leq c\lambda^{-1}$. By monotone convergence,

$$\int_{\overline{\Omega}} (c - \mathscr{A}V)\, dR(\lambda, A_\mu)'\delta_{x_0} = \sup_{n\in\mathbb{N}} \big(R(\lambda, A_n)c - R(\lambda, A_n)\mathscr{A}V(x_0)\big) \leq c/\lambda + V(x_0).$$

We can now prove the claimed tightness. To that end, let $\varepsilon > 0$. Since $\mathscr{A}V(x) \to -\infty$ as $|x| \to \infty$, we can find a radius $r > 0$ such that $\mathscr{A}V(x) \leq c - \varepsilon^{-1}$ for all $|x| > r$. Consequently, $\mathbb{1}_{B_r(0)^c} \leq \varepsilon(c - \mathscr{A}V)$ and hence

$$(\lambda R(\lambda, A_\mu)'\delta_{x_0})(B_r(0)^c) \leq \varepsilon\lambda \int_{\overline{\Omega}} (c - \mathscr{A}V) d(R(\lambda, A_\mu)'\delta_{x_0}) \leq \lambda\varepsilon(V(x_0) + c/\lambda) \leq \varepsilon(V(x_0) + c)$$

for all $0 < \lambda \leq 1$. ∎

Remark 8.4. If V satisfies the assumptions of Theorem 8.3, then it also satisfies Hypothesis 6.2.

9. Examples. In this section, we show how the assumptions of Theorem 8.3 can be verified in concrete situations. We assume that

$$\Omega = \mathbb{R}^d \setminus \overline{B}(0,1) = \{x \in \mathbb{R}^d : \|x\| > 1\}.$$

We note that $\partial\Omega$ is compact, so that whenever $\mu : \partial\Omega \to \mathscr{M}(\Omega)$ satisfies Hypothesis 1.2, the set $\{\mu(z) : z \in \partial\Omega\}$ is tight. In particular, the concentration condition from Theorem 6.4 is automatically fulfilled.

We assume that the coefficients a_{ij} and b_j belong to $C(\overline{\Omega})$ for $i, j = 1, \ldots, d$ and satisfy

$$\lim_{|x|\to\infty} \sum_{j=1}^{d} (a_{jj}(x) + b_j(x)x) = -\infty. \tag{9.1}$$

Then the function $V(x) = |x|^2$ satisfies $V(x) \geq 0$, $\lim_{|x|\to\infty} V(x) = \infty$ and $\lim_{|x|\to\infty} \mathscr{A}V(x) = -\infty$ (cf. [28, Corollary 6.4]).

Example 9.1. Condition (9.1) is for example satisfied in the following situations:

(a) if $a_{ij}(x) = \delta_{ij}$ and $b_j(x) = -x_j$, i.e. when $\mathscr{A}$ is the *Ornstein–Uhlenbeck operator*

$$\mathscr{A}u(x) = \Delta u(x) - \langle x, \nabla u(x)\rangle;$$

(b) for operators of the form

$$\mathscr{A}u(x) = \frac{1}{|x|^\alpha} \Delta u(x) - \langle x, \nabla u(x)\rangle,$$

where $\alpha > 0$ (recall that $|x| > 1$ for $x \in \Omega$),

(c) for operators of the form

$$\mathscr{A}u(x) = |x|^{\alpha}\Delta u(x) - |x|^{\beta-1}\langle x, \nabla u(x)\rangle,$$

where $\alpha > 0$ and $\beta > \alpha - 1$.

Corollary 9.2. *Let Ω be as above and assume that the continuous coefficients a_{ij} and b_j satisfy besides Hypothesis* 1.1 *also condition* (9.1). *Then $D(A_\mu) = D_{\max}$ and the semigroup T_μ is Markovian and enjoys the strong Feller property.*

Proof. It follows from (9.1) that $V(x) = |x|^2$ satisfies Hypothesis 6.2. Since $\partial\Omega$ is compact and thus $\{\mu(z) : z \in \partial\Omega\}$ is tight, the other condition of Theorem 6.4 is satisfied and $D(A_\mu) = D_{\max}$ follows from Corollary 6.6. The assertions concerning T_μ now follow from Theorem 7.5. ∎

Let us now turn to the existence of an invariant measure. If (9.1) is satisfied, then $V(x) = |x|^2$ satisfies conditions (i) and (ii) in Theorem 8.3. Condition (iii), however, is in general not satisfied by this function. Indeed, V need not be $\mu(z)$-integrable, for example if $d = 1$ and $\mu(z)$ has a density of the form $c|x|^{-2}$ with respect to Lebesgue measure. Even if $V(x)$ is $\mu(z)$-integrable for all $z \in \partial\Omega$, we cannot expect that for $z \in \partial\Omega$ we have $\int V(x)\,\mu(z, dx) \le 1 = V(z)$. However, sometimes we may modify V so that this is the case.

Corollary 9.3. *Let Ω be as above and assume that the continuous coefficients a_{ij} and b_j satisfy besides Hypothesis* 1.1 *also condition* (9.1). *Moreover, assume that*

$$\sup_{|z|=1} \int_{\Omega} |x|^2\,\mu(z, dx) < \infty.$$

Then the semigroup T_μ has a unique invariant measure.

Proof. We note that as $\mathbb{1} \in D(A_\mu)$ by Corollary 9.2, the semigroup T_μ can have at most one invariant probability measure. To prove that one exists, we show that we can modify the function $V(x) = |x|^2$ in such a way that the assumptions of Theorem 8.3 are satisfied.

We set $M := \sup_{|z|=1} \int_{\Omega} |x|^2\,\mu(z, dx)$. We claim that we can find $\varepsilon \in (0, 1)$ such that for the set $S_\varepsilon := B_{1+\varepsilon}(0) \setminus B_1(0)$ we have $\mu(z, S_\varepsilon) \le (1 + 2M)^{-1}$ for all $z \in \partial\Omega$. To see this, pick a continuous function $f_n : \overline{\Omega} \to [0, 1]$ such that $f_n(x) = 1$ for $1 \le |x| \le 1 + n^{-1}$ and $f_n(x) = 0$ for $|x| \ge 1 + 2n^{-1}$. Then $f_n \downarrow 0$ pointwise on Ω. By dominated convergence, $\langle f_n, \mu(z)\rangle \downarrow 0$ for every $z \in \partial\Omega$. Since $z \mapsto \langle f_n, \mu(z)\rangle$ is continuous, by Dini's theorem this convergence is uniform on the compact set $\partial\Omega$. Consequently, we can find an n such that $\langle f_n, \mu(z)\rangle \le (1 + M)^{-1}$ for *all* $z \in \partial\Omega$ and we may put $\varepsilon = 2n^{-1}$.

We now pick $\varphi \in C^2([1,\infty))$ such that

$$\varphi(t) \begin{cases} = M+1 & \text{for } t = 1, \\ \in [0, M+1] & \text{for } t \in (0,\varepsilon), \\ \in [0,t] & \text{for } t \in [\varepsilon, 1], \\ = t & \text{for } t > 1, \end{cases}$$

and set $\tilde{V}(x) = \varphi(|x|^2)$. Then $\tilde{V}$ is a C^2-function such that for $|x| > 1$ we have $\tilde{V}(x) = V(x)$ and $\mathscr{A}\tilde{V}(x) = \mathscr{A}V(x)$. In particular, $\tilde{V}(x) \to \infty$ for $|x| \to \infty$ and $\mathscr{A}\tilde{V}(x) \to -\infty$ for $|x| \to \infty$, so that conditions (i) and (ii) in Theorem 8.3 are fulfilled. Moreover,

$$\begin{aligned} \int_\Omega \tilde{V}(x)\,\mu(z,dx) &\le \int_{S_\varepsilon} (M+1)\,\mu(z,dx) + \int_{\Omega\setminus S_\varepsilon} |x|^2\,\mu(z,dx) \\ &\le \frac{1+M}{1+2M} + M \le M+1 = \tilde{V}(z). \end{aligned}$$

This proves that $\tilde{V}$ also satisfies condition (iii) in Theorem 8.3, so the existence of an invariant measure follows. ∎

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