

Mean dimension and an embedding theorem for real flows

by

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Abstract. We develop mean dimension theory for \mathbb{R} -flows. We obtain fundamental properties and examples and prove an embedding theorem: Any real flow (X, \mathbb{R}) of mean dimension strictly less than r admits an extension (Y, \mathbb{R}) whose mean dimension is equal to that of (X, \mathbb{R}) and such that (Y, \mathbb{R}) can be embedded in the \mathbb{R} -shift on the compact function space $\{f \in C(\mathbb{R}, [-1, 1]) : \text{supp}(\hat{f}) \subset [-r, r]\}$, where \hat{f} is the Fourier transform of f considered as a tempered distribution. These canonical embedding spaces appeared previously as a tool in embedding results for \mathbb{Z} -actions.

1. Introduction. Mean dimension was introduced by Gromov [Gro99] in 1999, and was systematically studied by Lindenstrauss and Weiss [LW00] as an invariant of topological dynamical systems. In recent years it has been extensively investigated in relation to the so-called embedding problem, mainly for \mathbb{Z}^k -actions ($k \in \mathbb{N}$). For \mathbb{Z} -actions, the problem is which \mathbb{Z} -actions (X, T) can be embedded in the shifts on the Hilbert cubes $(([0, 1]^N)^{\mathbb{Z}}, \sigma)$, where N is a natural number and the shift σ acts on $([0, 1]^N)^{\mathbb{Z}}$ by $\sigma((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$ for $x_n \in [0, 1]^N$. Under the conditions that X has finite Lebesgue covering dimension and the system (X, T) is aperiodic, Jaworski [Jaw74] proved in 1974 that (X, T) can be embedded in the shift on $[0, 1]^{\mathbb{Z}}$. Using Fourier and complex analysis, Gutman and Tsukamoto showed that if (X, T) is minimal and has mean dimension strictly less than $N/2$ then it can be embedded in $(([0, 1]^N)^{\mathbb{Z}}, \sigma)$ (see a more general result in [GQT19]). We note that the value $N/2$ is optimal since a minimal system of mean dimension $N/2$ which cannot be embedded in $(([0, 1]^N)^{\mathbb{Z}}, \sigma)$ was constructed in [LT14, Theorem 1.3]. More references for the embedding problem are given in [Aus88, Kak68, Lin99, Gut11, Gut15, GT14, GLT16, Gut16, Gut17, GQS18].

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In this paper, we develop the mean dimension theory for \mathbb{R} -actions and investigate the embedding problem in this context. Throughout this paper, by a *flow* we mean a pair (X, \mathbb{R}) , where X is a compact metric space and $\Gamma : \mathbb{R} \times X \rightarrow X$, $(r, x) \mapsto rx$, is a continuous map such that $\Gamma(0, x) = x$ and $\Gamma(r_1, \Gamma(r_2, x)) = \Gamma(r_1 + r_2, x)$ for all $r_1, r_2 \in \mathbb{R}$ and $x \in X$. Let $(X, \mathbb{R}) = (X, (\varphi_r)_{r \in \mathbb{R}})$ and $(Y, \mathbb{R}) = (Y, (\phi_r)_{r \in \mathbb{R}})$ be flows. We say that (Y, \mathbb{R}) can be *embedded* in (X, \mathbb{R}) if there is an \mathbb{R} -equivariant homeomorphism of Y onto a subspace of X ; that is, there is a homeomorphism $f : Y \rightarrow f(Y) \subset X$ such that $f \circ \phi_r = \varphi_r \circ f$ for all $r \in \mathbb{R}$.

This paper is organized as follows: In Section 2, we present basic notions and properties of mean dimension theory for flows. In Section 3 we construct minimal real flows with arbitrary mean dimension. In Section 4, we propose an embedding conjecture for flows and discuss its relation to the Lindenstrauss–Tsukamoto embedding conjecture for \mathbb{Z} -systems. In Section 5, we state the main embedding theorem and prove it using a key proposition. In Section 6, we prove the key proposition.

2. Mean dimension for real flows. We first introduce the definition of mean dimension for \mathbb{R} -actions. Let (X, d) be a compact metric space. Let $\epsilon > 0$ and let Y be a topological space. A continuous map $f : X \rightarrow Y$ is called a (d, ϵ) -*embedding* if for any $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$ we have $d(x_1, x_2) < \epsilon$. Define

$$\text{Widim}_\epsilon(X, d) = \min_{K \in \mathcal{K}} \dim(K),$$

where $\dim(K)$ is the Lebesgue covering dimension of the space K and \mathcal{K} denotes the collection of compact metrizable spaces K such that there is a (d, ϵ) -embedding $f : X \rightarrow K$. Note that \mathcal{K} is always nonempty since we can take $K = X$ which is a compact metric space and $f = \text{id}$ which is the identity map from X to itself.

Let (X, \mathbb{R}) be a flow. For $x, y \in X$ and a subset A of \mathbb{R} let

$$d_A(x, y) = \sup_{r \in A} d(rx, ry).$$

For $R > 0$ denote by d_R the metric $d_{[0, R]}$ on X . Clearly, the metric d_R is compatible with the topology on X .

PROPOSITION 2.1. *For any $\epsilon > 0$, we have*

- (1) $\text{Widim}_\epsilon(X, d) \leq \dim(X)$;
- (2) if $0 < \epsilon_1 < \epsilon_2$ then $\text{Widim}_{\epsilon_1}(X, d) \geq \text{Widim}_{\epsilon_2}(X, d)$;
- (3) if $0 \leq R_1 < R_2$ then $\text{Widim}_\epsilon(X, d_{R_1}) \leq \text{Widim}_\epsilon(X, d_{R_2})$;
- (4) $\text{Widim}_\epsilon(X, d_{[r_1, r_2]}) = \text{Widim}_\epsilon(X, d_{[r_0+r_1, r_0+r_2]})$ for any $r_0, r_1, r_2 \in \mathbb{R}$;
- (5) $\text{Widim}_\epsilon(X, d_{N+M}) \leq \text{Widim}_\epsilon(X, d_N) + \text{Widim}_\epsilon(X, d_M)$ for any $N, M \geq 0$.

Proof. Since (X, d) is a compact metric space that belongs to \mathcal{K} , we have (1). Points (2) and (3) follow from the definition.

Let $\epsilon > 0$. If K is a compact metrizable space and $f : X \rightarrow K$ is a continuous map such that $d_{[r_1, r_2]}(x_1, x_2) < \epsilon$ for any $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, then $f \circ r_0 : X \rightarrow K$ is a continuous map such that $d_{[r_1, r_2]}(r_0x_1, r_0x_2) < \epsilon$ for any $x_1, x_2 \in X$ with $f \circ r_0(x_1) = f \circ r_0(x_2)$ which implies that $d_{[r_0+r_1, r_0+r_2]}(x_1, x_2) < \epsilon$. This shows (4).

To see (5), let $\epsilon > 0$, K (resp. L) be a compact metrizable space and $f : X \rightarrow K$ (resp. $g : X \rightarrow L$) be a continuous map such that $d_N(x_1, x_2) < \epsilon$ (resp. $d_M(x_1, x_2) < \epsilon$) for any $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$ (resp. $g(x_1) = g(x_2)$). Define $F : X \rightarrow K \times L$ by $F(x) = (f(x), g(Nx))$ for every $x \in X$. Clearly, $K \times L$ is a compact metrizable space and the map F is continuous. For $x, y \in X$, if $F(x) = F(y)$ then $f(x) = f(y)$ and $g(Nx) = g(Ny)$, thus $d_N(x, y) < \epsilon$ and $d_M(Nx, Ny) < \epsilon$, and hence $d_{N+M}(x, y) < \epsilon$. It follows that $\text{Widim}_\epsilon(X, d_{N+M}) \leq \dim(K \times L) \leq \dim(K) + \dim(L)$. Thus, $\text{Widim}_\epsilon(X, d_{N+M}) \leq \text{Widim}_\epsilon(X, d_N) + \text{Widim}_\epsilon(X, d_M)$. ■

We define the *mean dimension* of a flow (X, \mathbb{R}) by

$$\text{mdim}(X, \mathbb{R}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{\text{Widim}_\epsilon(X, d_N)}{N}.$$

The limit exists by the Ornstein–Weiss lemma [LW00, Theorem 6.1], in view of subadditivity.

Next we recall the definition of mean dimension for \mathbb{Z} -actions [LW00, Definition 2.6]. Let (X, T) be a \mathbb{Z} -action. For $x, y \in X$ and $N \in \mathbb{N}$, denote

$$d_N^{\mathbb{Z}}(x, y) = \max_{n \in \mathbb{Z} \cap [0, N-1]} d(T^n(x), T^n(y)).$$

Define the *mean dimension* of (X, T) by

$$\text{mdim}(X, \mathbb{Z}) = \text{mdim}(X, T) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty (N \in \mathbb{N})} \frac{\text{Widim}_\epsilon(X, d_N^{\mathbb{Z}})}{N}.$$

PROPOSITION 2.2. *Let (X, \mathbb{R}) be a flow. If X is finite-dimensional then $\text{mdim}(X, \mathbb{R}) = 0$.*

Proof. We have $\text{Widim}_\epsilon(X, d_N) \leq \dim(X) < +\infty$. ■

Although the definition of mean dimension for \mathbb{R} -actions depends on the metric d , the next proposition shows that the mean dimension of a flow has the same value for all metrics compatible with the topology. Therefore mean dimension is an invariant of \mathbb{R} -actions.

PROPOSITION 2.3. *Let (X, \mathbb{R}) be a flow. Suppose that d and d' are compatible metrics on X . Then $\text{mdim}(X, \mathbb{R}; d) = \text{mdim}(X, \mathbb{R}; d')$.*

Proof. Since d and d' are equivalent, the identity map $\text{id} : (X, d') \rightarrow (X, d)$ is uniformly continuous. Thus, for every $\epsilon > 0$ there is $\delta > 0$ with $\delta < \epsilon$ such

that for any $x, y \in X$ with $d'(x, y) < \delta$ we have $d(x, y) < \epsilon$, which implies that $\text{Widim}_\epsilon(X, d_N) \leq \text{Widim}_\delta(X, d'_N)$ for every $N \in \mathbb{N}$. Noting that $\epsilon \rightarrow 0$ yields $\delta \rightarrow 0$ we obtain $\text{mdim}(X, \mathbb{R}; d) \leq \text{mdim}(X, \mathbb{R}; d')$. In the same way, also $\text{mdim}(X, \mathbb{R}; d') \leq \text{mdim}(X, \mathbb{R}; d)$. ■

PROPOSITION 2.4 ([LW00, Def. 2.6]). *Let (X, \mathbb{Z}) be a topological dynamical system. If d and d' are compatible metrics on X then we have $\text{mdim}(X, \mathbb{Z}; d) = \text{mdim}(X, \mathbb{Z}; d')$.*

Note that a flow $(X, (\varphi_r)_{r \in \mathbb{R}})$ naturally induces a “sub- \mathbb{Z} -action” (X, φ_1) .

PROPOSITION 2.5. *Let $(X, (\varphi_r)_{r \in \mathbb{R}})$ be a flow. Then $\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}(X, \varphi_1)$.*

Proof. Recall that for any compatible metric D on X and $R > 0$, we denote $D_R = D|_{[0, R]}$. For a flow $(X, d; \mathbb{R})$ and $N \in \mathbb{N}$, we have

$$(d_1)_N^{\mathbb{Z}} = (d_N^{\mathbb{Z}})_1 = d_N.$$

Thus,

$$\text{mdim}(X, \mathbb{R}; d) = \text{mdim}(X, \mathbb{Z}; d_1).$$

Since d_1 and d are compatible metrics on X , by Proposition 2.4 we have

$$\text{mdim}(X, \mathbb{Z}; d_1) = \text{mdim}(X, \mathbb{Z}; d).$$

Combining the two equalities we have as desired

$$\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}(X, \varphi_1). \quad \blacksquare$$

Thus if the space is not metrizable then we may take $\text{mdim}(X, \varphi_1)$ as the definition of mean dimension.

PROPOSITION 2.6. *Let $(X, (\varphi_r)_{r \in \mathbb{R}})$ be a flow. If the topological entropy of $(X, (\varphi_r)_{r \in \mathbb{R}})$ is finite then the mean dimension of $(X, (\varphi_r)_{r \in \mathbb{R}})$ is zero.*

Proof. By [HK03, Proposition 8.3.6], $h_{\text{top}}(X, \varphi_1) = h_{\text{top}}(X, (\varphi_r)_{r \in \mathbb{R}})$, which is finite. By [LW00, Theorem 4.2], $\text{mdim}(X, \varphi_1) = 0$. By Proposition 2.5, $\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = 0$. ■

The following proposition follows directly from the definition.

PROPOSITION 2.7. *For any flow $(X, (\varphi_r)_{r \in \mathbb{R}})$ and $c \in \mathbb{R}$,*

$$\text{mdim}(X, (\varphi_{cr})_{r \in \mathbb{R}}) = |c| \cdot \text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}).$$

3. Construction of minimal real flows with arbitrary mean dimension. By definition $\text{mdim}(X, \mathbb{R})$ belongs to $[0, +\infty]$. In this section we will show that for every $r \in [0, +\infty]$, there is a minimal flow (X, \mathbb{R}) with $\text{mdim}(X, \mathbb{R}) = r$.

Recall that there are natural constructions for passing from a \mathbb{Z} -action to a flow, and vice versa [BS02, Section 1.11]. Let (X, T) be a \mathbb{Z} -action

and $f : X \rightarrow (0, \infty)$ be a continuous function (in particular bounded away from 0). Consider the quotient space (equipped with the quotient topology)

$$S_f X = \{(x, t) \in X \times \mathbb{R}^+ : 0 \leq t \leq f(x)\} / \sim,$$

where \sim is the equivalence relation $(x, f(x)) \sim (Tx, 0)$. The *suspension* over (X, T) generated by the *roof function* f is the flow $(S_f X, (\psi_t)_{t \in \mathbb{R}})$ given by

$$\psi_t(x, s) = (T^n x, s') \quad \text{for } t \in \mathbb{R} \text{ and } (x, s) \in S_f X,$$

where n and s' satisfy

$$\sum_{i=0}^{n-1} f(T^i x) + s' = t + s, \quad 0 \leq s' \leq f(T^n x).$$

In other words, flow along $\{x\} \times \mathbb{R}^+$ to $(x, f(x))$, then continue from $(Tx, 0)$ (which is the same as $(x, f(x))$) along $\{Tx\} \times \mathbb{R}^+$ and so on. If $f \equiv 1$, then $S_f X$ is called the *mapping torus* over X .

Let d be a compatible metric on X . Bowen and Walters introduced a compatible metric \tilde{d} on $S_f X$ [BW72, Section 4] known today as the *Bowen–Walters metric* ⁽¹⁾. Let us recall the construction. First assume $f \equiv 1$. We will introduce $\tilde{d}_{S_1 X}$ on the space $S_1 X$. First, for $x, y \in X$ and $0 \leq t \leq 1$ define the *length* of the horizontal segment $((x, t), (y, t))$ by

$$d_h((x, t), (y, t)) = (1 - t)d(x, y) + td(Tx, Ty).$$

Clearly, $d_h((x, 0), (y, 0)) = d(x, y)$ and $d_h((x, 1), (y, 1)) = d(Tx, Ty)$. Secondly, for $(x, t), (y, s) \in S_1 X$ which are on the same orbit define the *length* of the vertical segment $((x, t), (y, t))$ by

$$d_v((x, t), (y, s)) = \inf\{|r| : \psi_r(x, t) = (y, s)\}.$$

Finally, for any $(x, t), (y, s) \in S_1 X$ define the *distance* $\tilde{d}_{S_1 X}((x, t), (y, s))$ to be the infimum of the lengths of paths between (x, t) and (y, s) consisting of a finite number of horizontal and vertical segments. Bowen and Walters showed that this construction gives rise to a compatible metric on $S_1 X$. Now assume a continuous function $f : X \rightarrow (0, \infty)$ is given. There is a natural homeomorphism $i_f : S_1 X \rightarrow S_f X$ given by $(x, t) \mapsto (x, tf(x))$. Define $\tilde{d}_{S_f X} = (i_f)_*(\tilde{d}_{S_1 X})$.

Recall from [LW00, Definition 4.1] that for a \mathbb{Z} -action (X, T) , the *metric mean dimension* $\text{mdim}_M(X, d)$ of X with respect to a metric d compatible with the topology on X is defined as follows. Let $\epsilon > 0$ and $n \in \mathbb{N}$. A subset S of X is called (ϵ, d, n) -*spanning* if for every $x \in X$ there is $y \in S$ such that $d_n^{\mathbb{Z}}(x, y) \leq \epsilon$. Set

$$A(X, \epsilon, d, n) = \min\{\#S : S \subset X \text{ is } (\epsilon, d, n)\text{-spanning}\}$$

⁽¹⁾ Note that in [BW72] it is assumed that $\text{diam}(X) < 1$ but this is unnecessary.

and define

$$\text{mdim}_M(X, T, d) = \liminf_{\epsilon \rightarrow 0} \frac{1}{|\log \epsilon|} \limsup_{n \rightarrow \infty} \frac{1}{n} \log A(X, \epsilon, d, n).$$

Similarly one may define metric mean dimension for flows but we will not pursue this direction.

THEOREM 3.1 (Lindenstrauss–Weiss [LW00, Theorem 4.2]). *For any \mathbb{Z} -action (X, T) and any metric d compatible with the topology on X ,*

$$\text{mdim}(X, T) \leq \text{mdim}_M(X, T, d).$$

THEOREM 3.2 (Lindenstrauss [Lin99, Theorem 4.3]). *If a \mathbb{Z} -action (X, T) is an extension of an aperiodic minimal system then there is a compatible metric d on X such that $\text{mdim}(X, T) = \text{mdim}_M(X, T, d)$.*

For related results we refer to [Gut17, Appendix A].

PROPOSITION 3.3. *Let $(Y, (\varphi_r)_{r \in \mathbb{R}})$ be the mapping torus over (X, T) (the suspension generated by the roof function 1). Assume that there is a compatible metric d on X with $\text{mdim}_M(X, T, d) = \text{mdim}(X, T)$. Then*

$$\text{mdim}(X, T) = \text{mdim}(Y, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}_M(Y, T, \tilde{d}).$$

Proof. By Proposition 2.5 $\text{mdim}(Y, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}(Y, \varphi_1)$. Since (X, T) is a subsystem of $(Y, T) = (Y, \varphi_1)$, we have $\text{mdim}(X, T) \leq \text{mdim}(Y, \varphi_1)$. Note that for every $r \in [0, 1)$, $\varphi_r(X)$ is a φ_1 -invariant closed subset of Y , and $(\varphi_r(X), \varphi_1)$ can be regarded as a copy of (X, T) . Let $\epsilon > 0$ and $n \in \mathbb{N}$. If $d_{n+1}^{\mathbb{Z}}(x, y) \leq \epsilon/2$ and $|t - t'| \leq \epsilon/2$ for $0 \leq t, t' < 1$ then $\tilde{d}_n^{\mathbb{Z}}((x, t), (y, t')) \leq \epsilon$. Thus it is easy to see that $A(Y, \epsilon, \tilde{d}, n) \leq ([1/\epsilon] + 1) \cdot A(X, \epsilon/2, d, n + 1)$. In particular

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log A(Y, \epsilon, \tilde{d}, n) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log A(X, \epsilon/2, d, n)$$

and we obtain $\text{mdim}_M(Y, \tilde{d}) \leq \text{mdim}_M(X, d)$. By Theorem 3.1 we know that $\text{mdim}(Y, \varphi_1) \leq \text{mdim}_M(Y, \tilde{d})$. Summarizing, we have

$$\begin{aligned} \text{mdim}(X, T) &\leq \text{mdim}(Y, \varphi_1) \leq \text{mdim}_M(Y, \varphi_1, \tilde{d}) \\ &\leq \text{mdim}_M(X, T, d) = \text{mdim}(X, T). \blacksquare \end{aligned}$$

We note that for general roof functions Proposition 3.3 does not hold. Indeed, Masaki Tsukamoto has informed us that he has constructed an example of a minimal topological dynamical system (X, T) with compatible metric d and $f : X \rightarrow (0, \infty)$, $f \not\equiv 1$ such that $\text{mdim}(X, T) = \text{mdim}_M(X, d) = 0$ but $\text{mdim}_M(S_f X, \varphi_1, \tilde{d}) > 0$ [Tsu].

PROBLEM 3.4. *Is Proposition 3.3 always true without assuming that there is a compatible metric d on X with $\text{mdim}_M(X, d) = \text{mdim}(X, T)$?*

PROBLEM 3.5. *Is it possible to find a topological dynamical system (X, T) with compatible metric d and $f : X \rightarrow (0, \infty)$ such that $\text{mdim}(X, T) = 0$ and $\text{mdim}(S_f X, (\varphi_r)_{r \in \mathbb{R}}) \neq 0$?*

In Proposition 3.3, if (X, T) is minimal then $(Y, (\varphi_r)_{r \in \mathbb{R}})$ is minimal. In particular, by Theorem 3.2 we have the following:

PROPOSITION 3.6. *Suppose that (X, T) is minimal and (Y, \mathbb{R}) is the mapping torus over (X, T) (the suspension generated by the roof function 1). Then (Y, \mathbb{R}) is also minimal and $\text{mdim}(X, T) = \text{mdim}(Y, \mathbb{R})$.*

PROPOSITION 3.7. *For every $c \in [0, +\infty]$ there is a minimal flow $(X, (\varphi_r)_{r \in \mathbb{R}})$ such that $\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = c$.*

Proof. By the \mathbb{Z} -version result due to Lindenstrauss and Weiss [LW00, Proposition 3.5] there is a minimal \mathbb{Z} -action (Y, \mathbb{Z}) such that $\text{mdim}(Y, \mathbb{Z}) = c$. By Proposition 3.6 we obtain a minimal flow (X, \mathbb{R}) with $\text{mdim}(X, \mathbb{R}) = c$. ■

4. An embedding conjecture. We now state the main embedding theorem of this paper. We recall some necessary notions and results in Fourier analysis. A C^∞ function $f : \mathbb{R} \rightarrow \mathbb{C}$ is said to be *rapidly decreasing* if there are constants $M_{n,m} > 0$ such that $|f^{(m)}(x)| < M_{n,m}|x|^{-n}$ as $x \rightarrow \infty$, for all $n, m \in \mathbb{N}$. The space of such functions is called the *Schwartz space* and is denoted by \mathcal{S} . For $f \in \mathcal{S}$ the definitions of the Fourier transform and its inverse are given by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i t \xi} f(t) dt, \quad \bar{\mathcal{F}}(f)(t) = \int_{-\infty}^{\infty} e^{2\pi i t \xi} f(\xi) d\xi.$$

Then $\mathcal{F}(\mathcal{S}) = \mathcal{S}$, $\bar{\mathcal{F}}(\mathcal{S}) = \mathcal{S}$ and for all $f \in \mathcal{S}$, $\bar{\mathcal{F}}(\mathcal{F}(f)) = \mathcal{F}(\bar{\mathcal{F}}(f)) = f$. The operators \mathcal{F} and $\bar{\mathcal{F}}$ can be extended to tempered distributions in a standard way (for details see [Sch66, Chapter 7] and [Str03, Chapters 3 & 4]). The tempered distributions include in particular all bounded continuous functions.

Let $a < b$ be real numbers. We define $V[a, b]$ as the space of bounded continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$ satisfying the condition $\text{supp}(\mathcal{F}(f)) \subset [a, b]$. We denote $B_1(V[a, b]) = \{f \in V[a, b] : \|f\|_\infty \leq 1\}$ and $B_1(V^{\mathbb{R}}[-a, a]) = \{f \in B_1(V[-a, a]) : f(\mathbb{R}) \subset \mathbb{R}\}$. One may show that $B_1(V[a, b])$ is a compact metric space with respect to the distance

$$d(f_1, f_2) = \sum_{n=1}^{\infty} \frac{\|f_1 - f_2\|_{L^\infty([-n, n])}}{2^n}.$$

This metric coincides with the standard topology of tempered distributions (for details see [Sch66, Chapter 7, Section 4]). Let $\mathbb{R} = (\tau_r)_{r \in \mathbb{R}}$ act on $B_1(V[a, b])$ by the shift: for every $r \in \mathbb{R}$ and $f \in B_1(V[a, b])$, $(\tau_r f)(t) = f(t + r)$ for all $t \in \mathbb{R}$. Thus we obtain a flow $(B_1(V[a, b]), \mathbb{R})$.

Lindenstrauss and Tsukamoto posed the following conjecture:

CONJECTURE 4.1 ([LT14, Conjecture 1.2]). *Let (X, T) be a \mathbb{Z} -dynamical system and D an integer. For $r \in \mathbb{N}$, define $P_r(X, T) = \{x \in X : rx = x\}$. Suppose that $\dim P_r(X, T) < rD/2$ for every $r \in \mathbb{N}$, and $\text{mdim}(X, T) < D/2$. Then (X, T) can be embedded in $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$.*

By [LW00, Proposition 3.3], $\text{mdim}(([0, 1]^D)^{\mathbb{Z}}, \sigma) = D$. It is not hard to see that for $r \in \mathbb{N}$,

$$\dim P_r((([0, 1]^D)^{\mathbb{Z}}, \sigma)) = rD.$$

Thus the above conjecture may be rephrased as follows: if

$$\dim P_r(X, T) < \frac{\dim P_r((([0, 1]^D)^{\mathbb{Z}}, \sigma))}{2} \quad \text{for all } r \in \mathbb{N}$$

and

$$\text{mdim}(X, T) < \frac{\text{mdim}((([0, 1]^D)^{\mathbb{Z}}, \sigma))}{2}$$

then $(X, T) \hookrightarrow (([0, 1]^D)^{\mathbb{Z}}, \sigma)$. We expect that a similar phenomenon holds for flows where the role of $(([0, 1]^D)^{\mathbb{Z}}, \sigma)$ is played by $(B_1(V^{\mathbb{R}}[-a, a]), \mathbb{R})$. By [GQT19, footnote 4], $\text{mdim}(B_1(V^{\mathbb{R}}[-a, a]), \mathbb{R}) = 2a$. For $r \in \mathbb{R}_{>0}$ denote

$$P_r(X, \mathbb{R}) = \{x \in X : rx = x\}.$$

We now calculate $\dim P_r(B_1(V^{\mathbb{R}}[-a, a]), \mathbb{R})$.

PROPOSITION 4.2. *Let $r > 0$. Then $\dim P_r(B_1(V^{\mathbb{R}}[-a, a])) = 2\lfloor ar \rfloor + 1$.*

Proof. Let $f \in B_1(V^{\mathbb{R}}[-a, a])$ with $f(x) = f(x+r)$ for all $x \in \mathbb{R}$. In particular we have a periodic $f \in C^\infty(\mathbb{R}, \mathbb{R})$ that is a restriction of a holomorphic function, and hence the Fourier series representation of f , $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x / r}$, converges uniformly to f and $c_{-k} = \overline{c_k}$ for all k . Since $\mathcal{F}(f) = c_0 \mathcal{F}(1) + \sum_{k=1}^{\infty} c_k \mathcal{F}(e^{2\pi i k t / r}) + \overline{c_k} \mathcal{F}(e^{-2\pi i k t / r})$ is supported in $[-a, a]$, we have $c_k = 0$ for $|k| > ar$. Let $N = \lfloor ar \rfloor$. Choose $x_0 < x_1 < x_2 < \dots < x_N$ such that $e^{2\pi i x_i / r} \neq e^{2\pi i x_j / r}$ for $i \neq j$. The Vandermonde determinant formula indicates that $\det(e^{2\pi i k x_l / r})_{l,k=0}^N \neq 0$. This implies that the functions $e^{2\pi i k x / r}$, $0 \leq k \leq N$, are linearly independent. Thus, we conclude that $\dim P_r(B_1(V^{\mathbb{R}}[-a, a])) = 2\lfloor ar \rfloor + 1$. ■

We now conjecture:

CONJECTURE 4.3. *Let (X, \mathbb{R}) be a flow and $a > 0$ a real number. Suppose that $\text{mdim}(X, \mathbb{R}) < a$ and for every $r \in \mathbb{R}$, $\dim P_r(X, \mathbb{R}) < \lfloor ar \rfloor + 1/2$. Then (X, \mathbb{R}) can be embedded in the flow $(B_1(V^{\mathbb{R}}[-a, a]), \mathbb{R})$.*

PROBLEM 4.4. *Does Conjecture 4.3 imply Conjecture 4.1? Does Conjecture 4.1 imply Conjecture 4.3?*

We give a partial answer:

PROPOSITION 4.5. *Assume Conjecture 4.3 holds. Let (X, T) be a topological dynamical system such that:*

- (i) *there exists $D \in \mathbb{N}$ such that $\text{mdim}(X, T) < D/2$,*
- (ii) *there exists $b \in \mathbb{R}$ such that $b < D/2$ and for all $r > 3/(D - 2b)$, $\dim P_r(X, T) < br$,*
- (iii) *for all $r \leq 1/(D - 2b)$, $P_r(X, T) = \emptyset$,*
- (iv) $\text{mdim}(S_1X, \mathbb{R}) = \text{mdim}(X, T)$.

Then (X, T) can be embedded in the system $(([0, 1]^D)^\mathbb{Z}, \sigma)$.

Proof. Note that the periodic orbits of the suspension (S_1X, \mathbb{R}) have positive integer lengths, and orbits of length $r \in \mathbb{N}$ in S_1X correspond to r -periodic points of (X, T) , so that $P_r(X, T) = \emptyset$ implies $P_r(S_1X, \mathbb{R}) = \emptyset$ and $P_r(X, T) \neq \emptyset$ implies

$$\dim P_r(S_1X, \mathbb{R}) = \dim P_r(X, T) + 1.$$

Consider the following sequence of embeddings:

$$(X, T) \xrightarrow{(1)} (S_1X, \psi_1) \xrightarrow{(2)} (B_1(V^\mathbb{R}[-c, c]), \sigma) \xrightarrow{(3)} (([-1, 1]^D)^\mathbb{Z}, \sigma).$$

Embedding (1) is the trivial embedding from (X, T) into (S_1X, ψ_1) where ψ_1 is the time-1 map. Embedding (3) is a consequence of [GQT19, Lemma 2.4] as long as $c < D/2$. We now justify (2). This \mathbb{Z} -embedding is induced from an \mathbb{R} -embedding $(S_1X, \mathbb{R}) \hookrightarrow (B_1(V^\mathbb{R}[-c, c]), \mathbb{R})$ whose existence follows from Conjecture 4.3 which we assume to hold. We need to verify the conditions appearing in Conjecture 4.3. Let c be a real number such that $\text{mdim}(X, T) < c < D/2$. Thus $\text{mdim}(S_1X, \mathbb{R}) = \text{mdim}(X, T) < c$. Let r be an integer such that $r > 3/(D - 2b)$. Then $\dim P_r(X, \mathbb{R}) < br + 1$, whereas $1/2 \dim P_r(B_1(V^\mathbb{R}[-c, c]), \text{shift}) = \lfloor rc \rfloor + 1/2 = cr - t_r + 1/2$, where $0 \leq t_r < 1$. Note $cr - t_r + 1/2 \geq br + 1$ if $(c - b)r \geq 3/2 > t_r + 1/2$, i.e. if $r \geq \frac{3}{2(c-b)}$. Thus it is enough to check it for the minimal integer r_0 such that $r_0 > \frac{3}{D-2b} = \frac{3}{2(D/2-b)}$. We thus choose $b < c < D/2$ such that $r_0 \geq \frac{3}{2(c-2b)} > \frac{3}{2(D/2-2b)}$, and this ends the proof. ■

5. An embedding theorem. For every $n \in \mathbb{N}$ denote by S_n the circle of circumference $n!$ (identified with $[0, n!]$). Let \mathbb{R} act on $\prod_{n \in \mathbb{N}} S_n$ as follows: $(x_i)_i \mapsto (x_i + r \pmod{i!})_i$, $r \in \mathbb{R}$. Define the *solenoid* [NS60, V.8.15]

$$S = \left\{ (x_n)_n \in \prod_{n \in \mathbb{N}} S_n : x_n = x_{n+1} \pmod{n!} \right\}.$$

It is easy to see that (S, \mathbb{R}) is a (minimal) flow.

The following definitions are standard: A continuous surjective map $\psi : (X, \mathbb{Z}) \rightarrow (Y, \mathbb{Z})$ is called an *extension* (of topological dynamical system) if $\psi(n.x) = n.\psi(x)$ for all $n \in \mathbb{Z}$ and $x \in X$. A continuous surjective map

$\psi : (X, \mathbb{R}) \rightarrow (Y, \mathbb{R})$ is called an *extension* (of flows) if $\psi(r.x) = r.\psi(x)$ for all $r \in \mathbb{R}$ and $x \in X$.

The following embedding result, which is the main result of this paper, provides a partial positive answer to Conjecture 4.3. This result may be understood as an analog for flows of [GT14, Corollary 1.8] which states that Conjecture 4.1 is true for any \mathbb{Z} -system which is an extension of an aperiodic subshift, i.e. an aperiodic subsystem of a symbolic shift $(\{1, 2, \dots, l\}^{\mathbb{Z}}, \sigma)$ for some $l \in \mathbb{N}$.

THEOREM 5.1. *Let $a < b$ be two real numbers. If (X, \mathbb{R}) is an extension of (S, \mathbb{R}) and $\text{mdim}(X, \mathbb{R}) < b - a$, then (X, \mathbb{R}) can be embedded in $(B_1(V[a, b]), \mathbb{R})$.*

COROLLARY 5.2. *Conjecture 4.3 holds for (X, \mathbb{R}) which is an extension of (S, \mathbb{R}) .*

Proof. Suppose $\text{mdim}(X, \mathbb{R}) < a$ for some $a > 0$. As (X, \mathbb{R}) is an extension of an aperiodic system, it is aperiodic and in particular for every $r \in \mathbb{R}$, $\dim P_r(X, \mathbb{R}) = 0$. We have to show that (X, \mathbb{R}) may be embedded in the flow $(B_1(V^{\mathbb{R}}[-a, a]), \mathbb{R})$. Indeed, by Theorem 5.1, (X, \mathbb{R}) may be embedded in $(B_1(V[0, a]), \mathbb{R})$. It is now enough to notice that one has the embedding

$$B_1(V[0, a]) \rightarrow B_1(V^{\mathbb{R}}[-a, a]), \quad \varphi \mapsto \frac{1}{2}(\varphi + \bar{\varphi}). \quad \blacksquare$$

Since for any flow (X, \mathbb{R}) , the product flow $(X \times S, \mathbb{R} \times \mathbb{R})$ is an extension of the flow (S, \mathbb{R}) , the following result is a direct corollary of Theorem 5.1.

THEOREM 5.3. *For every flow (X, \mathbb{R}) with $\text{mdim}(X, \mathbb{R}) < b - a$ (where $a < b$ are real numbers) there is an extension (Y, \mathbb{R}) with $\text{mdim}(X, \mathbb{R}) = \text{mdim}(Y, \mathbb{R})$ that can be embedded in $(B_1(V[a, b]), \mathbb{R})$.*

In our proof of Theorem 5.1, the key step is to embed (X, \mathbb{R}) in a product flow (Theorem 5.4):

THEOREM 5.4. *Suppose that $a < b$, $\text{mdim}(X, \mathbb{R}) < b - a$ and $\Phi : (X, \mathbb{R}) \rightarrow (S, \mathbb{R})$ is an extension. Then for a dense G_δ subset of $f \in C_{\mathbb{R}}(X, B_1(V[a, b]))$ the map*

$$(f, \Phi) : X \rightarrow B_1(V[a, b]) \times S, \quad x \mapsto (f(x), \Phi(x)),$$

is an embedding.

REMARK 5.5. *It is possible to prove a similar theorem where (S, \mathbb{R}) is replaced by a solenoid defined by circles of circumference $r_n \rightarrow \infty$ as $n \rightarrow \infty$ but we will not pursue this direction.*

The proof is given in the next section. We start with an auxiliary result:

PROPOSITION 5.6. *There is an embedding of (S, \mathbb{R}) in $(B_1(V[0, c]), \mathbb{R})$ for any $c > 0$.*

Proof. Define a continuous and \mathbb{R} -equivariant map

$$\phi : (S, \mathbb{R}) \rightarrow (B_1(V[0, c]), \mathbb{R})$$

by

$$S \ni x = (x_n)_n \mapsto f_x(t) = \sum_{n \geq m(c)} \frac{1}{2^n} \cdot e^{2\pi i(t+x_n)/n!} = \sum_{n \geq m(c)} \left(\frac{1}{2^n} \cdot e^{\frac{2\pi i}{n!} x_n} \right) \cdot e^{\frac{2\pi i}{n!} t}$$

where $m(c) \in \mathbb{N}$ is sufficiently large so that the right-hand side belongs to $B_1(V[0, c])$.

Assume $f_x(t) = f_y(t)$ for some $x = (x_n)_n, y = (y_n)_n \in S$. We claim $x = y$. This implies that the map is an embedding. Indeed, it is enough to show that $\frac{1}{2^n} \cdot e^{\frac{2\pi i}{n!} x_n} = \frac{1}{2^n} \cdot e^{\frac{2\pi i}{n!} y_n}$ for all n . This is a consequence of the following more general lemma:

LEMMA 5.7. *Let a_n be an absolutely summable series ($\sum |a_n| < \infty$). Let λ_n be a pairwise distinct sequence of real numbers such that value $|\lambda_n| \leq M$ for some $M > 0$. Then $f(z) = \sum a_n e^{i\lambda_n z}$, $z \in \mathbb{C}$, defines an entire function such that $f \equiv 0$ iff $a_n = 0$ for all n .*

Proof (compare with the proof of [Man72, Theorem I.3.1]). We claim that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) e^{-i\lambda_m t} dt = a_m \quad \text{for all } m.$$

Thus $f \equiv 0$ implies $a_m = 0$ for all m . Indeed,

$$\frac{1}{T} \int_0^T f(t) e^{-i\lambda_m t} dt = \frac{1}{T} \int_0^T \sum_{n \neq m} a_n e^{i(\lambda_n - \lambda_m)t} dt + \frac{1}{T} \int_0^T a_m dt.$$

For $n \neq m$, since $\lambda_n - \lambda_m \neq 0$, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i(\lambda_n - \lambda_m)t} dt = 0.$$

As absolute summability implies that one may interchange the limiting operations,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{n \neq m} a_n e^{i(\lambda_n - \lambda_m)t} dt = \sum_{n \neq m} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a_n e^{i(\lambda_n - \lambda_m)t} dt = 0,$$

This completes the proof. ■

Now we show Theorem 5.1 assuming Theorem 5.4.

Proof of Theorem 5.1 assuming Theorem 5.4. We take $a < c_1 < c_2 < b$ with $\text{mdim}(X, \mathbb{R}) < c_1 - a$. By Theorem 5.4, (X, \mathbb{R}) can be embedded in $(B_1(V[a, c_1]) \times S, \mathbb{R} \times \mathbb{R})$, which, by Proposition 5.6, can be embedded in

$(B_1(V[a, c_1]) \times B_1(V[c_2, b]), \mathbb{R} \times \mathbb{R})$, and finally embedded in $(B_1(V[a, b]), \mathbb{R})$ by the following embedding:

$$B_1(V[a, c_1]) \times B_1(V[c_2, b]) \rightarrow B_1(V[a, b]), \quad (\varphi_1, \varphi_2) \mapsto \frac{1}{2}(\varphi_1 + \varphi_2).$$

This ends the proof. ■

6. Embedding in a product. Let $C_{\mathbb{R}}(X, B_1(V[a, b]))$ be the space of \mathbb{R} -equivariant continuous maps $f : X \rightarrow B_1(V[a, b])$. This space is nonempty because it contains the constant 0. The metric on $C_{\mathbb{R}}(X, B_1(V[a, b]))$ is chosen to be the uniform distance $\sup_{x \in X} \mathbf{d}(f(x), g(x))$. This space is completely metrizable and hence is a Baire space (see [Mun00, Theorem 48.2]).

We denote by d the metric on X . To prove Theorem 5.4, it suffices to show that the set

$$\bigcap_{n=1}^{\infty} \{f \in C_{\mathbb{R}}(X, B_1(V[a, b])) : (f, \Phi) \text{ is a } \frac{1}{n}\text{-embedding with respect to } d\}$$

is a dense G_{δ} subset of $C_{\mathbb{R}}(X, B_1(V[a, b]))$. It is obviously a G_{δ} subset of $C_{\mathbb{R}}(X, B_1(V[a, b]))$. Therefore it remains to prove the following:

PROPOSITION 6.1. *For any $\delta > 0$ and $f \in C_{\mathbb{R}}(X, B_1(V[a, b]))$, there is $g \in C_{\mathbb{R}}(X, B_1(V[a, b]))$ such that:*

- (1) $|f(x)(t) - g(x)(t)| < \delta$ for all $x \in X$ and $t \in \mathbb{R}$;
- (2) $(g, \Phi) : X \rightarrow B_1(V[a, b]) \times S$ is a δ -embedding with respect to d .

To show Proposition 6.1, we prove several auxiliary results. We start by quoting [GT14, Lemma 2.1]:

LEMMA 6.2. *Let (X, d') be a compact metric space, and let $F : X \rightarrow [-1, 1]^M$ be a continuous map. Suppose that $\delta', \epsilon > 0$ satisfy*

$$(6.1) \quad d'(x, y) < \epsilon \implies \|F(x) - F(y)\|_{\infty} < \delta'.$$

Then if $\text{Widim}_{\epsilon}(X, d') < M/2$ then there is an ϵ -embedding $G : X \rightarrow [-1, 1]^M$ satisfying

$$\sup_{x \in X} \|F(x) - G(x)\|_{\infty} < \delta'.$$

We say that a holomorphic function g in $S \subset \mathbb{C}$ is of *exponential type* if for all $z \in S$, $|g(z)| \leq Ce^{T|z|}$ for some $C, T > 0$. The following classical theorem is proven in [DM72, Section 3.1.7].

THEOREM 6.3 (Phragmén–Lindelöf principle). *Let g be a function of exponential type that is holomorphic in the sector*

$$S = \{z \in \mathbb{C} : \alpha < \arg z < \beta\}$$

of angle $\beta - \alpha < \pi$, and continuous on its boundary. If $|g(z)| \leq 1$ for $z \in \partial S$ then $|g(z)| \leq 1$ for $z \in S$.

According to the classical Paley–Wiener theorem [Rud87, Theorem 19.3], if $f \in L^2(\mathbb{R})$ extends to an entire function F for which there exist $A, C > 0$ such that for all $z = x + iy \in \mathbb{C}$, $|f(x + iy)| \leq Ce^{2\pi A|y|}$, then $\mathcal{F}(f) \in L^2(\mathbb{R})$ is supported in $[-A, A]$. We will need a generalized version:

THEOREM 6.4. *Let $f \in L^\infty(\mathbb{R})$ be a function which extends to an entire function $F : \mathbb{C} \rightarrow \mathbb{C}$ ($F|_{\mathbb{R}} = f$) such that there exist $A, C > 0$ and $M \in \mathbb{N}$ such that*

$$|F(z)| \leq C(1 + |z|)^M \cdot e^{2\pi A|y|} \quad \text{for all } z = x + iy \in \mathbb{C}.$$

Then $f \in V[-A, A]$.

Proof. See [Str03, Theorem 7.2.3] ⁽²⁾. ■

Let $\rho > 0$ and $N \in \mathbb{N}$ be such that $\rho N! \in \mathbb{N}$. Define

$$L(\rho) = \left\{ \frac{k}{\rho} \right\}_{k \in \mathbb{Z}}, \quad L^*(\rho) = L(\rho) \setminus \{0\}.$$

In the next lemma we write $x \lesssim y$ for two real numbers x and y if there exists a constant $C > 0$ which depends only on ρ and N such that $x \leq Cy$.

LEMMA 6.5. *Let*

$$f(z) = \lim_{A \rightarrow \infty} \prod_{\lambda \in L(\rho), 0 < |\lambda| < A} \left(1 - \frac{z}{\lambda} \right).$$

Then f defines a holomorphic function in \mathbb{C} satisfying

$$f(0) = 1, \quad f(\lambda) = 0, \quad \forall \lambda \in L^*(\rho).$$

Moreover, for all $z \in \mathbb{C}$ we have

$$|f(z)| \lesssim (1 + |z|)^{5\rho N!} \cdot e^{\pi\rho|y|},$$

where y is the imaginary part of z .

Proof. We first show the convergence of $f(z)$. Notice that

$$f(z) = \lim_{A \rightarrow \infty} \prod_{\lambda \in L(\rho), 0 < \lambda < A} \left(1 - \frac{z^2}{\lambda^2} \right).$$

As $\sum_{\lambda \in L(\rho), 0 < \lambda} 1/\lambda^2$ converges, the limit above is locally uniform (see [Kno51, §29, Theorems 6 & 7]). Thus, $f(z)$ is a holomorphic function which satisfies

$$f(0) = 1, \quad f(\lambda) = 0, \quad \forall \lambda \in L^*(\rho).$$

Next we shall estimate the growth of f on the real line. Suppose $x > 0$ and let k be the integer with $kN! \leq x < (k+1)N!$. We may assume $k > 0$, as

⁽²⁾ Reading the proof in that reference one should note that in [Str03] the Fourier transform is defined as $\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} e^{it\xi} f(t) dt$.

the case $k = 0$ is easier and can be dealt with in a similar way. For $n \in \mathbb{Z}$, set

$$L_n = L(\rho) \cap [nN!, (n+1)N!).$$

For $\lambda \in L_n$ with $n \leq -2$ or $n \geq k+1$ we have

$$|1 - x/\lambda| \leq 1 - x/((n+1)N!)$$

and hence

$$\prod_{\lambda \in L_n} \left| 1 - \frac{x}{\lambda} \right| \leq \left| 1 - \frac{x}{(n+1)N!} \right|^{\rho N!}.$$

For $\lambda \in L_n$ with $1 \leq n < k$ we have

$$|1 - x/\lambda| \leq x/(nN!) - 1$$

and hence

$$\prod_{\lambda \in L_n} \left| 1 - \frac{x}{\lambda} \right| \leq \left| 1 - \frac{x}{nN!} \right|^{\rho N!}.$$

The factors for $n = -1, 0, k$ need to be treated separately. Recall Euler's sine product formula [Cia15]:

$$\frac{\sin z}{z} = \lim_{A \rightarrow \infty} \prod_{0 < |n| < A} \left(1 - \frac{z}{n\pi} \right).$$

Using this it is easy to see that $|f(x)|$ is bounded by

$$\begin{aligned} & \prod_{0 \neq \lambda \in L_{-1} \cup L_0 \cup L_k} \left| 1 - \frac{x}{\lambda} \right| \cdot \lim_{A \rightarrow \infty} \prod_{|n| < A, n \neq 0, k, k+1} \left| 1 - \frac{x}{nN!} \right|^{\rho N!} \\ &= \prod_{0 \neq \lambda \in L_{-1} \cup L_0 \cup L_k} \left| 1 - \frac{x}{\lambda} \right| \cdot \left| \frac{\sin \frac{\pi x}{N!}}{\frac{\pi x}{N!} \left(1 - \frac{x}{kN!} \right) \left(1 - \frac{x}{(k+1)N!} \right)} \right|^{\rho N!}. \end{aligned}$$

The first factor is easy to estimate:

$$\prod_{0 \neq \lambda \in L_{-1} \cup L_0 \cup L_k} \left| 1 - \frac{x}{\lambda} \right| \lesssim (1+x)^{3\rho N!}.$$

Set $t = x/N!$; then

$$\frac{\sin \frac{\pi x}{N!}}{\frac{\pi x}{N!} \left(1 - \frac{x}{kN!} \right) \left(1 - \frac{x}{(k+1)N!} \right)} = \frac{k(k+1) \sin \pi t}{\pi t(k-t)(k+1-t)}.$$

By the mean value theorem,

$$\left| \frac{\sin \pi t}{t} \right| \leq \pi, \quad \left| \frac{\sin \pi t}{k-t} \right| \leq \pi, \quad \left| \frac{\sin \pi t}{k+1-t} \right| \leq \pi.$$

Thus,

$$\left| \frac{k(k+1) \sin \pi t}{\pi t(k-t)(k+1-t)} \right| \lesssim k(k+1) \lesssim (1+x)^2.$$

Therefore

$$|f(x)| \lesssim (1+x)^{5\rho N!}.$$

The case $x < 0$ is similar so we get

$$|f(x)| \lesssim (1+|x|)^{5\rho N!}.$$

We now turn to estimating $|f(yi)|$ for $y \in \mathbb{R} \setminus \{0\}$. For $r > 0$ we set

$$n(r) = \#(L^*(\rho) \cap (-r, r)).$$

We have

$$n(r) < 2\rho r.$$

Note that for $0 < r \leq 1/\rho$, one has $n(r) = 0$. Since

$$|f(yi)|^2 = \prod_{\lambda \in L^*(\rho)} (1 + y^2/\lambda^2),$$

as $n(r)$ is increasing, we may use the Riemann–Stieltjes integral to write

$$\log |f(yi)| = \frac{1}{2} \sum_{\lambda \in L^*(\rho)} \log \left(1 + \frac{y^2}{\lambda^2} \right) = \frac{1}{2} \int_{1/\rho}^{\infty} \log \left(1 + \frac{y^2}{r^2} \right) dn(r).$$

Using integration by parts for the Riemann–Stieltjes integral [Gor94, Theorem 12.14], we see that for all $R \geq 1/\rho$,

$$\frac{1}{2} \int_{1/\rho}^R \log \left(1 + \frac{y^2}{r^2} \right) dn(r) = \frac{1}{2} \left(\log \left(1 + \frac{y^2}{r^2} \right) n(r) \Big|_{1/\rho}^R - \int_{1/\rho}^R n(r) d \log \left(1 + \frac{y^2}{r^2} \right) \right).$$

Letting $R \rightarrow \infty$, we conclude that

$$\log |f(yi)| = y^2 \int_{1/\rho}^{\infty} \frac{n(r)}{r(r^2 + y^2)} dr.$$

Since $n(r) \leq 2\rho r$, we deduce that

$$\log |f(yi)| \leq 2\rho y^2 \int_{1/\rho}^{\infty} \frac{dr}{r^2 + y^2}.$$

It is a standard exercise to show that

$$\int_0^{\infty} \frac{dr}{r^2 + y^2} = \frac{1}{|y|} \int_0^{\infty} \frac{dr}{1 + r^2} = \frac{\pi}{2|y|}.$$

It follows that

$$|f(yi)| \leq e^{\pi\rho|y|}.$$

Finally, we show that $|f(z)|$ grows at most exponentially. Let $z = x + yi$. We may assume $x, y > 0$, as all the other cases are similar. Let k be the integer with $kN! \leq x < (k+1)N!$. Set

$$L' = L(\rho) \setminus (L_{k-1} \cup L_k \cup L_{k+1}).$$

We estimate

$$\prod_{0 \neq \lambda \in L_{k-1} \cup L_k \cup L_{k+1}} \left| 1 - \frac{z}{\lambda} \right| \lesssim (1 + |z|)^{3\rho N!}.$$

We have

$$\begin{aligned} \lim_{A \rightarrow \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left| 1 - \frac{z}{\lambda} \right|^2 &= \lim_{A \rightarrow \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left\{ \left(1 - \frac{x}{\lambda} \right)^2 + \frac{y^2}{\lambda^2} \right\} \\ &= \left\{ \lim_{A \rightarrow \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left(1 - \frac{x}{\lambda} \right)^2 \right\} \cdot \prod_{0 \neq \lambda \in L'} \left\{ 1 + \frac{y^2}{(\lambda - x)^2} \right\}. \end{aligned}$$

As in the proof of $|f(x)| \lesssim (1 + |x|)^{5\rho N!}$ we estimate

$$\lim_{A \rightarrow \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left(1 - \frac{x}{\lambda} \right)^2 \lesssim (1 + x)^{12\rho N!}.$$

Replacing y by y^2 in $|f(yi)| \leq e^{\pi\rho|y|}$, we obtain

$$\prod_{0 \neq \lambda \in L'} \left\{ 1 + \frac{y^2}{(\lambda - x)^2} \right\} \leq e^{2\pi\rho|y|}.$$

Thus, we deduce that $|f(z)|$ grows at most exponentially.

We have thus shown that $f(z)$ has exponential type and satisfies $|f(x)| \lesssim (1 + |x|)^{5\rho N!}$ and $|f(yi)| \leq e^{\pi\rho|y|}$. By the Phragmén–Lindelöf principle of Theorem 6.3 (e.g. in the first quadrant $x, y \geq 0$) applied to $(1+z)^{-5\rho N!} e^{\pi\rho iz} f(z)$, the claim follows. ■

Next we proceed to constructing an interpolation function relying on [Beu89, pp. 351–365]:

PROPOSITION 6.6. *Let $a < b$. Let $\rho > 0$ be such that $\rho \in \mathbb{Q}$ and $\rho < b - a$. There exists $\varphi \in V[a, b]$ rapidly decreasing such that $\varphi(0) = 1$ and $\varphi(\lambda) = 0$ for all $\lambda \in L^*(\rho)$.*

Proof. Fix $\tau > 0$ with $\rho + \tau < b - a$. Let $\psi(\xi) \in \mathcal{S}$ be a nonnegative smooth function in \mathbb{R} satisfying

$$\text{supp}(\psi) \subset \left[-\frac{\tau}{2}, \frac{\tau}{2} \right], \quad \int_{-\infty}^{\infty} \psi(\xi) d\xi = 1.$$

Define the function $h : \mathbb{C} \rightarrow \mathbb{C}$ by

$$h(z) = \int_{-\tau/2}^{\tau/2} \psi(\xi) e^{2\pi i z \xi} d\xi.$$

It is easy to see that h is an entire function which satisfies

$$(6.2) \quad h|_{\mathbb{R}} = \bar{\mathcal{F}}(\psi) \in \mathcal{S}, \quad h(0) = 1, \quad |h(x + yi)| \leq e^{\pi\tau|y|}, \quad \forall x, y \in \mathbb{R}.$$

Let

$$g(z) = \lim_{A \rightarrow \infty} \prod_{\lambda \in L(\rho), 0 < |\lambda| < A} \left(1 - \frac{t}{\lambda}\right).$$

By Lemma 6.5, $g(z)$ is an entire function. Thus we may define the following entire functions:

$$\tilde{\varphi}(z) = h(z)g(z), \quad \varphi(z) = e^{\pi iz(a+b)} \tilde{\varphi}(z).$$

It is easy to see that $\varphi(0) = 1$ and $\varphi(\lambda) = 0$ for all $\lambda \in L^*(\rho)$. By Lemma 6.5, $g|_{\mathbb{R}}$ has polynomial growth. Therefore as $\bar{\mathcal{F}}(\psi)$ is rapidly decreasing, so are $\varphi|_{\mathbb{R}}$ and $\tilde{\varphi}|_{\mathbb{R}}$. By Lemma 6.5 and (6.2) (recall the convention $z = x + iy$):

$$|\tilde{\varphi}(z)| \lesssim (1 + |z|)^{5\rho N!} \cdot e^{\pi(\rho+\tau)|y|}.$$

As in addition $\tilde{\varphi}|_{\mathbb{R}}$ is bounded (as it is rapidly decreasing), it follows from Theorem 6.4 that $\tilde{\varphi} \in V\left[\frac{-\rho-\tau}{2}, \frac{\rho+\tau}{2}\right] \subset V\left[\frac{a-b}{2}, \frac{b-a}{2}\right]$. This immediately implies $\varphi \in V[a, b]$, which finishes the proof. ■

Now we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. We take $\delta > 0$ and $f \in C_{\mathbb{R}}(X, B_1(V[a, b]))$. Without loss of generality, we assume that $|f(x)(t)| \leq 1 - \delta$ for all $x \in X$ and $t \in \mathbb{R}$ (by replacing f with $(1 - \delta)f$ if necessary). Fix $\rho \in \mathbb{Q}$ with

$$\text{mdim}(X, \mathbb{R}) < \rho < b - a.$$

Let φ be the function constructed in Proposition 6.6. As φ is rapidly decreasing, we may find $K > 0$ such that

$$(6.3) \quad |\varphi(t)| \leq \frac{K}{1 + |t|^2}.$$

Let $\delta' > 0$ be such that

$$(6.4) \quad \delta' \cdot \sum_{\lambda \in L(\rho)} \frac{K}{1 + |t - \lambda|^2} < \delta \quad \text{for all } t \in \mathbb{R}.$$

Fix $\epsilon \in (0, \delta)$. Let $N \in \mathbb{N}$ satisfy $\rho N! \in \mathbb{N}$, $\text{Widim}_{\epsilon}(X, d_{N!}) < \rho N!$, and

$$(6.5) \quad d_{N!}(x, y) < \epsilon \quad \text{implies} \quad |f(x)(t) - f(y)(t)| < \delta'/2 \quad \text{for all } t \in [0, N!].$$

Define

$$F : X \rightarrow [0, 1]^{2\rho N!} = ([0, 1]^2)^{\rho N!}, \quad F(x) = (\operatorname{Re} f(x)|_{L(\rho, N)}, \operatorname{Im} f(x)|_{L(\rho, N)}).$$

$$F^{\mathbb{C}} : X \rightarrow \mathbb{C}^{\rho N!}, \quad F^{\mathbb{C}}(x) = f(x)|_{L(\rho, N)}.$$

Let $M = 2\rho N!$ and $d' = d_{N!}$. Equation (6.5) implies that (6.1) holds, so Lemma 6.2 implies that there is a $(d_{N!}, \epsilon)$ -embedding $G : X \rightarrow [-1, 1]^{2\rho N!}$ such that $\sup_{x \in X} \|F(x) - G(x)\|_{\infty} < \delta'/2$. Similarly to $F^{\mathbb{C}}(x)(k)$, we introduce the notation $G^{\mathbb{C}}(x)(k)$, $k = 0, \dots, \rho N! - 1$, in the natural way. Notice that

$$(6.6) \quad \sup_{x \in X} \|F^{\mathbb{C}}(x) - G^{\mathbb{C}}(x)\|_{\infty} < \delta'.$$

Take $x \in X$. Denote $\Phi(x) = (\Phi(x)_n)_{n \in \mathbb{N}}$, where $\Phi(x)_n \in S_{n!}$. For every $n \in \mathbb{Z}$ let

$$\Lambda(x, n) = nN! - \Phi(x)_N + L(\rho, N), \quad \Lambda(x) = \bigcup_{n \in \mathbb{Z}} \Lambda(x, n) \subset \mathbb{R}.$$

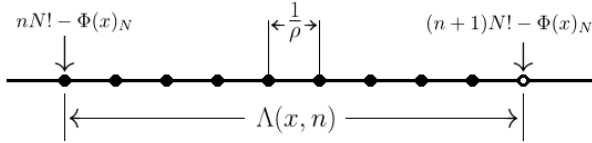


Fig. 6.1. The set $\Lambda(x, n)$

Next we construct a perturbation g of f :

$$g(x)(t) = f(x)(t) + h(x)(t),$$

where $h(x)(t)$ is defined by

$$\sum_{n \in \mathbb{Z}} \sum_{k=0}^{\rho N! - 1} (G^{\mathbb{C}}(T^{nN! - \Phi(x)_N} x)(k) - F^{\mathbb{C}}(T^{nN! - \Phi(x)_N} x)(k)) \times \varphi(t - (k/\rho + nN! - \Phi(x)_N)).$$

As φ is rapidly decreasing, the sum defining $g(x)$ for fixed x converges in the compact-open topology to a function in $V[a, b]$. Moreover the mapping $x \mapsto g(x)$ is continuous. To see that $g(x)$ is \mathbb{R} -equivariant, it suffices to deal with $h(x)$ (because f is already \mathbb{R} -equivariant). To see that $h(x)$ is \mathbb{R} -equivariant, note that for $0 \leq r < N! - \Phi(x)_N$ we have $\Phi(T^r x)_N = \Phi(x)_N + r$ and hence from the definition of h it follows that $h(T^r x)(t) = h(x)(t + r)$. Similarly, if $N! - \Phi(x)_N \leq r < N!$ then $\Phi(T^r x)_N = r - (N! - \Phi(x)_N)$ and hence $(T^{nN! - \Phi(x)_N} T^r x)(k) = (T^{(n+1)N! - \Phi(x)_N - r} T^r x)(k)$. Using that information in each summand in the sum over k 's appearing in the definition of $h(T^r x)(t)$, and then replacing $n + 1$ by n when summing over $n \in \mathbb{Z}$, we get, as desired,

$h(T^r x)(t) = h(x)(t + r)$ for r 's in this range. If $r = sN!$ where $s \in \mathbb{Z}$ then $\Phi(T^r x)_N = r - (sN! - \Phi(x)_N)$ and hence $(T^{nN! - \Phi(T^r x)_N} T^r x)(k) = (T^{(n+s)N! - \Phi(x)_N} x)(k)$. Using this information in each summand in the sum over k 's appearing in the definition of $h(T^r x)(t)$, and replacing $n + s$ by n when summing over $n \in \mathbb{Z}$, we obtain, as desired, $h(T^r x)(t) = h(x)(t + r)$ for r 's in this range. Finally, if $r = sN! + r'$ where $s \in \mathbb{Z}$ and $0 < r' < N!$, we use the additivity properties of the terms involved in order to combine the two cases and get the desired result. Note that by (6.3) and (6.4), for all $x \in X$ and $t \in \mathbb{R}$,

$$\sum_{n \in \mathbb{Z}} \sum_{k=0}^{\rho N! - 1} \varphi(t - (k/\rho + nN! - \Phi(x)_N)) < \delta/\delta'.$$

By (6.6), for all $x \in X$ and $k = 0, \dots, \rho N! - 1$,

$$|G^{\mathbb{C}}(T^{nN! - \Phi(x)_N} x)(k) - F^{\mathbb{C}}(T^{nN! - \Phi(x)_N} x)(k)| < \delta'.$$

Combining the last two inequalities we get $|g(x)(t) - f(x)(t)| < \delta$ for all $x \in X$ and $t \in \mathbb{R}$. Since $|f(x)(t)| \leq 1 - \delta$, we have $g(x) \in B_1(V[a, b])$. Thus, $g \in C_{\mathbb{R}}(X, B_1(V[a, b]))$. It remains to check that the map

$$(g, \Phi) : X \rightarrow B_1(V[a, b]) \times S, \quad x \mapsto (g(x), \Phi(x)),$$

is a δ -embedding with respect to d . We take $x, x' \in X$ with $(g(x), \Phi(x)) = (g(x'), \Phi(x'))$. We calculate, for $k = 0, \dots, \rho N! - 1$,

$$\begin{aligned} g(x)(-\Phi(x)_N + k/\rho) &= f(x)(-\Phi(x)_N + k/\rho) \\ &\quad + (G^{\mathbb{C}}(T^{-\Phi(x)_N} x)(k) - F^{\mathbb{C}}(T^{-\Phi(x)_N} x)(k)). \end{aligned}$$

As $F^{\mathbb{C}}(T^{-\Phi(x)_N} x)(k) = f(T^{-\Phi(x)_N} x)(k/\rho) = f(x)(-\Phi(x)_N + k/\rho)$, we conclude that $g(x)(-\Phi(x)_N + k/\rho) = G^{\mathbb{C}}(T^{-\Phi(x)_N} x)(k)$ for $k = 0, \dots, \rho N! - 1$.

Similarly $g(x')(-\Phi(x')_N + k/\rho) = G^{\mathbb{C}}(T^{-\Phi(x')_N} x')(k)$. Thus

$$g(x)(-\Phi(x)_N + k/\rho) = g(x')(-\Phi(x)_N + k/\rho) = g(x')(-\Phi(x')_N + k/\rho)$$

implies

$$G^{\mathbb{C}}(T^{-\Phi(x)_N} x)(k) = G^{\mathbb{C}}(T^{-\Phi(x')_N} x')(k) = G^{\mathbb{C}}(T^{-\Phi(x)_N} x')(k).$$

Since $G^{\mathbb{C}} : X \rightarrow [0, 1]^{\rho N!}$ is a $(d_{N!}, \epsilon)$ -embedding, we have

$$d_{N!}(T^{-\Phi(x)_N} x, T^{-\Phi(x)_N} x') < \epsilon < \delta,$$

which implies $d(x, x') < \epsilon < \delta$. This ends the proof. ■

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