Divisibility of some binomial sums

by

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1. Introduction. In [10], Ramanujan listed 17 interesting convergent series involving $1/\pi$. For example, he found that

$$\sum_{k=0}^{\infty} \frac{6k + 1}{256^k} \cdot \binom{2k}{k}^3 = \frac{4}{\pi}.$$  \hfill (1.1)

Nowadays, the theory of Ramanujan-type series has been greatly developed. In [4], Guillera gave a summary of the methods used to deal with Ramanujan-type series. Recently, a $q$-analogue of (1.1) was given in [7].

In recent years, the arithmetic properties of truncated Ramanujan-type series were also investigated. In [13], Van Hamme proposed 13 conjectured congruences concerning truncated Ramanujan-type series. For example,

$$\sum_{k=0}^{(p-1)/2} \frac{6k + 1}{256^k} \cdot \binom{2k}{k}^3 \equiv (-1)^{(p-1)/2} p \pmod{p^4},$$  \hfill (1.2)

where $p > 3$ is a prime. Now all conjectures of Van Hamme have been confirmed. The reader may refer to [15, 20] for the history of the proofs of Van Hamme’s conjectures.

On the other hand, Sun discovered that the convergent series involving $\pi$ often correspond to divisibility of some binomial sums. For example, Sun [19, (4)] proved that for each integer $n \geq 2$,

$$\sum_{k=0}^{n-1} (4k + 1) \binom{2k}{k}^3 \cdot (-64)^{n-1-k} \equiv 0 \pmod{2n \binom{2n}{n}}.$$  \hfill (1.3)

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Congruence (1.3) corresponds to the following identity proved by G. Brauer in 1859, which was also rediscovered by Ramanujan [16]:

\[(1.4) \sum_{k=0}^{\infty} \frac{4k + 1}{(-64)^k} \cdot \binom{2k}{k}^3 = \frac{2}{\pi}.\]

Sun also conjectured many similar congruences. For example, a conjectured congruence [18] corresponding to (1.1) is

\[(1.5) \sum_{k=0}^{n-1} (6k + 1) \binom{2k}{k}^3 \cdot 256^{n-1-k} \equiv 0 \pmod{2n \binom{2n}{n}}.\]

Later, (1.5) was confirmed by He [12] with the help of the WZ method, and its \(q\)-analogue was also established by Guo [6] by using the \(q\)-WZ method.

In this paper, we shall consider divisibility of some binomial sums similar to (1.5). For \(\alpha \in \mathbb{Q} \setminus \mathbb{Z}\) and \(n \in \mathbb{Z}^+\), define

\[N_{\alpha,n} := \text{the numerator of } n \left| \binom{-\alpha}{n} \right|.\]

It is easy to see that \(N_{1/2,n}\) coincides with the odd part of \(n \binom{2n}{n}\).

**Theorem 1.1.** Suppose that \(\rho\) is a positive integer and \(\alpha\) is a non-integral rational number. Then for each integer \(n \geq 1,

\[(1.6) \sum_{k=0}^{n-1} (2k + \alpha) \binom{-\alpha}{k}^\rho \equiv 0 \pmod{N_{\alpha,n}}.\]

For example, applying the Dougall formula ([1] Corollary 3.5.2] with \(a = c = d = e = \alpha\), we find that for any non-integral \(\alpha < 1/2,

\[(1.7) \sum_{k=0}^{\infty} (2k + \alpha) \binom{-\alpha}{k}^4 = \frac{\sin(\alpha \pi)}{\pi} \cdot \frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)^2},\]

where \(\Gamma\) denotes the gamma function. In particular, setting \(\alpha = -1/2\), we get a convergent series involving \(1/\pi^2\):

\[(1.8) 1 - \sum_{k=0}^{\infty} (4k + 3) \cdot \frac{C_k^4}{16^{2k+1}} = \frac{8}{\pi^2},\]

where \(C_k = \binom{2k}{k}/(k+1)\) denotes the \(k\)th Catalan number. Evidently when \(\rho = 4\), (1.6) can be viewed as a congruence corresponding to (1.7).

On the other hand, since

\[n \binom{2n}{n} = \binom{2k}{k} \cdot \frac{2^{n-k} \cdot (2n-1)(2n-3) \cdots (2k+1)}{(n-1)(n-2) \cdots (k+1)},\]

we have

\[\text{ord}_2 \left( n \binom{2n}{n} \right) \leq n - k + \text{ord}_2 \left( \binom{2k}{k} \right),\]
where \( \text{ord}_2(n) = \max \{ k : 2^k | n \} \) denotes the 2-adic order of \( n \). Thus substituting \( \alpha = 1/2 \) in (1.6) and noting that \( \binom{2k}{k} \) is always even for \( k \geq 1 \), we obtain

\[
(1.9) \quad \sum_{k=0}^{n-1} (4k + 1) \binom{2k}{k}^\rho \cdot (-4)^{\rho(n-1-k)} \equiv 0 \pmod{2^\rho 2^n \binom{2n}{n}}
\]

for each \( \rho \geq 1 \) and \( n \geq 2 \). Furthermore, Guo [5] proposed a conjectured \( q \)-analogue of (1.9). We shall discuss Guo’s conjecture in the last section of this paper.

We shall use \( q \)-congruences to prove Theorem 1.1. In recent years, the \( q \)-analogues of the Ramanujan-type supercongruences were widely investigated, and a variety of techniques, such as asymptotic estimates, basic hypergeometric transformation, creative microscoping, \( q \)-WZ pairs and \( q \)-Zeilberger algorithm etc., were involved. For example, Guo and Zudilin [11] introduced a new method called creative microscoping, and used it to prove several new Ramanujan-type \( q \)-congruences in a unified way. For more related results and the latest progress, the reader may refer to [5, 3, 8, 9, 10, 17].

In Section 2, we shall introduce the notion of \( q \)-congruences and give a general result concerning divisibility of \( q \)-binomial sums, whose proof will be given in Section 3. In Section 4, with the help of the above general result, we shall complete the proof of Theorem 1.1. Finally, in Section 5, we shall consider Guo’s \( q \)-analogue of (1.9) and confirm his conjecture.

### 2. \( q \)-congruence.

First, let us introduce the notion of \( q \)-congruence. For any \( x \in \mathbb{Q} \), define

\[
[x]_q := \frac{1 - qx}{1 - q}.
\]

Clearly if \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \), then \([n]_q = 1 + q + \cdots + q^{n-1}\) is a polynomial in \( q \). For \( a, b \in \mathbb{N} \) and \( n \in \mathbb{Z}^+ \), if \( a \equiv b \pmod{n} \), then putting \( m = (a - b)/n \), we have

\[
[a]_q - [b]_q = \frac{q^b - q^a}{1 - q} = q^b \cdot \frac{1 - q^{nm}}{1 - q} = q^b [m]_{q^n} \cdot [n]_q \equiv 0 \pmod{[n]_q},
\]

where the above congruence is considered over the polynomial ring \( \mathbb{Z}[q] \).

Furthermore,

\[
(2.1) \quad [nm]_q \cdot [n]_q = \frac{1 - q^{nm}}{1 - q^n} = 1 + q^n + q^{2n} + \cdots + q^{(m-1)n} \equiv 1 + 1 + \cdots + 1 = m \pmod{[n]_q}.
\]

Note that (2.1) is still valid when \( m \) is a negative integer, since \([nm]_q = -q^{nm} [n]_q\).
For $d \geq 2$, let $\Phi_d(q)$ denote the $d$th cyclotomic polynomial, i.e.,

$$\Phi_d(q) := \prod_{1 \leq k \leq d \atop (d,k)=1} (q - e^{2\pi \sqrt{-1} \cdot k/d}).$$

It is well known that $\Phi_d(q)$ is an irreducible polynomial with integral coefficients. Also, we have

$$[n]_q = \prod_{d \geq 2 \atop d
mid n} \Phi_d(q).$$

So $\Phi_d(q)$ divides $[n]_q$ if and only if $d$ divides $n$. Furthermore,

$$\Phi_d(1) = \begin{cases} p & \text{if } d = p^k \text{ for some prime } p, \\ 1 & \text{otherwise}. \end{cases}$$

For $n \in \mathbb{N}$, define

$$(x ; q)_n := \begin{cases} (1 - x)(1 - xq) \cdots (1 - xq^{n-1}) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Also, define the $q$-binomial coefficient

$$\left[ \begin{array}{c} x \\ n \end{array} \right]_q := \frac{(q^{x-n+1} ; q)_n}{(q ; q)_n}.$$ 

Clearly,

$$\lim_{q \to 1} \left[ \begin{array}{c} x \\ n \end{array} \right]_q = \left( \begin{array}{c} x \\ n \end{array} \right).$$

Furthermore, it is easy to check that

$$\left[ \begin{array}{c} -r/m \\ n \end{array} \right]_q = (-1)^n q^{-nr-m(n)} \cdot \left( \begin{array}{c} q^r \cdot q^m \\ (q^m ; q^m)_n \end{array} \right).$$

Suppose that $r \in \mathbb{Z}$, $m \in \mathbb{Z}^+$ and $(r,m) = 1$. For each positive integer $d$ with $(d,m) = 1$, let $\lambda_{r,m}(d)$ be the integer lying in \{0,1,\ldots,d-1\} such that

$$r + \lambda_{r,m}(d)m \equiv 0 \pmod{d}. \quad (2.3)$$

Thus, $\lambda_{r,m}(d)$ can be viewed as the least non-negative residue of $-r/m$ modulo $d$. Let

$$S_{r,m}(n) := \left\{ d \geq 2 : \left\lfloor \frac{n-1 - \lambda_{r,m}(d)}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \right\},$$

where $\lfloor x \rfloor = \max\{k \in \mathbb{N} : k \leq x\}$ for every $x \in \mathbb{R}$. Evidently for each $d > \max_{0 \leq j \leq n-1} |r+jm|$, we must have $\lambda_{r,m}(d) > n-1$, whence $d \notin S_{r,m}(n)$. 
So \( S_{r,m}(n) \) is always a finite set. Let

\[
A_{r,m,n}(q) := \prod_{d \in S_{r,m}(n)} \Phi_d(q),
\]

\[
C_{m,n}(q) := \prod_{d \mid n, d > 1 \atop (d,m) = 1} \Phi_d(q).
\]

Clearly, if \( d \mid n \), then we cannot have \( d \in S_{r,m}(n) \). So \( A_{r,m,n}(q) \) and \( C_{m,n}(q) \) are co-prime. Furthermore, as we shall see in the next section,

\[
A_{r,m,n}(1)C_{m,n}(1) = N_{r/m,n}.
\]

The following theorem is the key ingredient of this paper.

**Theorem 2.1.** Suppose that \( r \in \mathbb{Z} \) and \( m \in \mathbb{Z}^+ \). Assume that \( \nu_0(q), \nu_1(q), \ldots \) is a sequence of rational functions in \( q \) such that for any integer \( d \geq 2 \) with \( (m,d) = 1 \),

(i) \( \nu_k(q) \) is \( \Phi_d(q) \)-integral for each \( k \geq 0 \), i.e., the denominator of \( \nu_k(q) \) is not divisible by \( \Phi_d(q) \);

(ii) for any \( s,t \in \mathbb{N} \) with \( 0 \leq t \leq d - 1 \),

\[
\nu_{sd+t}(q) \equiv \mu_s(q)\nu_t(q) \pmod{\Phi_d(q)},
\]

where \( \mu_s(q) \) is a \( \Phi_d(q) \)-integral rational function only depending on \( s \);

(iii) we have

\[
\sum_{k=0}^{d-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv 0 \pmod{\Phi_d(q)}.
\]

Then for each positive integer \( n \),

\[
\sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv 0 \pmod{A_{r,m,n}(q)C_{m,n}(q)).
\]

Clearly Theorem 2.1 gives a tool to prove the divisibility of \( q \)-binomial sums. Before we give the proof of Theorem 2.1 which will occupy the subsequent section, let us explain how to deduce divisibility of binomial sums from Theorem 2.1.

**Corollary 2.1.** Under the assumptions of Theorem 2.1, additionally assume that for each positive integer \( n \), there exists a polynomial \( B_n(q) \) with integral coefficients such that

(1) \( B_n(q) \sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \) is a polynomial with integral coefficients;

(2) \( B_n(1) \) is not divisible by any prime \( p \) with \( p \nmid m \).
Then for any integer \( n \geq 1 \),

\[
\sum_{k=0}^{n-1} (-1)^k \binom{-r/m}{k} \cdot \nu_k(1) \equiv 0 \pmod{N_{r/m,n}}.
\]

**Proof.** By Theorem 2.1, we have

\[
B_n(q) \sum_{k=0}^{n-1} \binom{q^r; q^m}{k} \cdot \nu_k(q) = A_{r,m,n}(q)C_{m,n}(q) \cdot H(q),
\]

where \( H(q) \) is a polynomial in \( q \) with rational coefficients. Notice that the greatest common divisor of all coefficients of \( A_{r,m,n}(q)C_{m,n}(q) \) is just 1, i.e., \( A_{r,m,n}(q)C_{m,n}(q) \) is a primitive polynomial. According to a well known result of Gauss (cf. [14, Lemma 6.11]), we know that the coefficients of \( H(q) \) must all be integers. Hence substituting \( q = 1 \) in (2.7), we get

\[
B_n(1) \sum_{k=0}^{n-1} (-1)^k \binom{-r/m}{k} \cdot \nu_k(1) = N_{r/m,n} \cdot H(1) \equiv 0 \pmod{N_{r/m,n}}.
\]

Since \( N_{r/m,n} \) is prime to \( B_n(1) \), (2.8) is proved. □

Furthermore, substituting \( r = m = 1 \) in Theorem 2.1 we also have

**Corollary 2.2.** Assume that \( \nu_0(q), \nu_1(q), \ldots \) is a sequence of polynomials in \( q \) such that for any integer \( d \geq 2 \),

(i) for any \( s, t \in \mathbb{N} \) with \( 0 \leq t \leq d - 1 \),

\[
\nu_{sd+t}(q) \equiv \mu_s(q)\nu_t(q) \pmod{\Phi_d(q)}
\]

for some polynomial \( \mu_s(q) \) only depending on \( s \);

(ii) \( \sum_{k=0}^{d-1} \nu_k(q) \equiv 0 \pmod{\Phi_d(q)} \).

Then for any integer \( n \geq 1 \),

\[
\sum_{k=0}^{n-1} \nu_k(q) \equiv 0 \pmod{[n]_q}.
\]

In particular, if \( \nu_0(q), \nu_1(q), \ldots \) are polynomials with integral coefficients, then

\[
\sum_{k=0}^{n-1} \nu_k(1) \equiv 0 \pmod{n}.
\]

**Proof.** Clearly \( \lambda_{1,1}(d) = d - 1 \) for each \( d \geq 2 \). So we have \( S_{1,1,n} = \emptyset \) for any \( n \geq 1 \), i.e., \( A_{1,1,n}(q) = 1 \). On the other hand, we have \( C_{1,n}(q) = [n]_q \). Thus (2.9) follows immediately from Theorem 2.1 by setting \( r = m = 1 \). □
3. Proof of Theorem 2.1. In this section, we shall complete the proof of Theorem 2.1. First, we need several auxiliary lemmas.

**Lemma 3.1.** Let \( r \in \mathbb{Z} \) and let \( m, d \in \mathbb{Z}^+ \) with \( (m, d) = 1 \). Then

\[
\frac{(q^r; q^m)_d}{1 - q^d} \equiv r + \lambda_{r,m}(d)m \pmod{\Phi_d(q)},
\]

where \( \lambda_{r,m}(d) \) is defined by (2.3).

**Proof.** Clearly by (2.1),

\[
\frac{(q^r; q^m)_d}{1 - q^d} = \frac{1 - q^{r+\lambda_{r,m}(d)m}}{1 - q^d} \prod_{0 \leq j \leq d-1} (1 - q^{r+jm})
\]

\[
\equiv \frac{1 - q^{d \cdot r + \lambda_{r,m}(d)m}}{1 - q^d} \prod_{j=1}^{d-1} (1 - q^{r+j}) \equiv \frac{r + \lambda_{r,m}(d)m}{d} \cdot (q; q)_{d-1} \pmod{\Phi_d(q)}.
\]

Now for every primitive \( d \)th root of unity \( \xi \), we have

\[
(q; q)_{d-1}|_{q=\xi} = \prod_{j=1}^{d-1} (1 - \xi^j) = \lim_{x \to 1} \prod_{j=1}^{d-1} (x - \xi^j) = \lim_{x \to 1} \frac{x^d - 1}{x - 1} = d.
\]

Consequently,

\[
(q; q)_{d-1} \equiv d \pmod{\Phi_d(q)}.
\]

**Lemma 3.2.** Under the assumptions of Lemma 3.1, for any \( s, t \in \mathbb{N} \) with \( 0 \leq t \leq d - 1 \),

\[
\frac{(q^r; q^m)_{sd+t}}{(q^m; q^m)_{sd+t}} \equiv \left(\frac{r + \lambda_{r,m}(d)m}{md}\right)_s \cdot \frac{(q^r; q^m)_t}{(q^m; q^m)_t} \pmod{\Phi_d(q)}.
\]

**Proof.** By Lemma 3.1 we have

\[
\frac{(q^r; q^m)_{sd+t}}{(1 - q^d)^s} = (q^{r+smd}; q^m)_t \prod_{j=0}^{s-1} \frac{(q^{r+jmd}; q^m)_d}{1 - q^d}
\]

\[
\equiv (q^r; q^m)_t \prod_{j=0}^{s-1} (r + \lambda_{r,m}(d)m + jmd) \pmod{\Phi_d(q)}.
\]

Similarly,

\[
\frac{(q^m; q^m)_{sd+t}}{(1 - q^d)^s} \equiv (q^m; q^m)_t \prod_{j=0}^{s-1} (m + (d - 1)m + jmd) \pmod{\Phi_d(q)}.
\]

Clearly,

\[
\prod_{j=0}^{s-1} \frac{r + \lambda_{r,m}(d)m + jmd}{md + jmd} = \left(\frac{r + \lambda_{r,m}(d)m}{md}\right)_s.
\]

Since \((q^m; q^m)_t\) is prime to \(\Phi_d(q)\) for each \(0 \leq t \leq d - 1\), we get (3.2).
Lemma 3.3. Suppose that $r \in \mathbb{Z}$, $m \in \mathbb{N}$ and $(r, m) = 1$. Then

$$
(3.3) \quad \frac{(q^r; q^m)_n}{(q^m; q^m)_n} \prod_{(d, m) > 1} \Phi_d(q)^{\left[\frac{n(d, m)}{d}\right]} = (-1)^\delta q^{\Delta} \prod_{d \in \mathcal{S}_{r, m}(n)} \Phi_d(q),
$$

where $\delta = |\{0 \leq j \leq n-1 : r + jm < 0\}|$ and

$$
\Delta = \sum_{\substack{0 \leq j \leq n-1 \\text{mod } \mod d}} (r + jm).
$$

Proof. Note that for any $h \in \mathbb{N}$,

$$
1 - q^h = \prod_{d | h} \Phi_d(q).
$$

Hence

$$
(q^r; q^m)_n = (-1)^\delta q^{\Delta} \prod_{(d, m) = 1} \Phi_d(q)^{|\{0 \leq j \leq n-1: r+jm \equiv 0 \mod d\}|}.
$$

It is easy to check that

$$
|\{0 \leq j \leq n-1: r + jm \equiv 0 \mod d\}| = 1 + \left\lfloor \frac{n-1 - \lambda_{r, m}(d)}{d} \right\rfloor.
$$

Similarly,

$$
(q^m; q^m)_n = \prod_{d \geq 1} \Phi_d(q)^{|\{1 \leq j \leq n: jm \equiv 0 \mod d\}|},
$$

and

$$
|\{1 \leq j \leq n: jm \equiv 0 \mod d\}| = \left\lfloor \frac{n (m, d)}{d} \right\rfloor.
$$

Hence $d \in \mathcal{S}_{r, m}(n)$ if and only if $(d, m) = 1$ and

$$
|\{0 \leq j \leq n-1: r+jm \equiv 0 \mod d\}| = |\{1 \leq j \leq n: jm \equiv 0 \mod d\}| + 1.
$$

We immediately get (3.3). \hfill \blacksquare

Let

$$
(3.4) \quad B_{r, m, n}(q) = \prod_{l | m, l \geq 2, (d, m) = l} \Phi_d(q)^{\left[\frac{nl}{d}\right]}.
$$

Then (3.3) is equivalent to

$$
\frac{(q^r; q^m)_n}{(q^m; q^m)_n} = (-1)^\delta q^{\Delta} \cdot \frac{A_{r, m, n}(q)}{B_{r, m, n}(q)}.
$$

By (2.4), (2.5) and (3.4), clearly $B_{r, m, n}(q)$ is prime to $A_{r, m, n}(q)C_{m, n}(q)$. Also, $A_{r, m, n}(1)C_{m, n}(1)$ and $B_{r, m, n}(1)$ are co-prime integers by (2.2). Moreover,
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$B_{r,m,n}(q)$ is divisible by

$$\frac{[n]_q}{C_{m,n}(q)} = \prod_{d|n, (d,m)>1} \Phi_d(q).$$

So $A_{r,m,n}(1)C_{m,n}(1)$ must coincide with the numerator of $n \cdot \left| \binom{-r/m}{n} \right|$, i.e., (2.6) is valid.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. It suffices to show that the left side of (2.7) is divisible by $\Phi_d(q)$ for $d \in S_{r,m}(n)$ and for those $d \mid n$ with $(d, m) = 1$.

Suppose that $d \in S_{r,m}(n)$. Write $n = ud + v$ where $0 \leq v \leq d - 1$. Let $h = \lambda_{r,m}(d)$, $w = \frac{r + \lambda_{r,m}(d)m}{d}$.

Note that $d \in S_{r,m}(n)$ implies that $v \geq 1 + h$. Hence for any $v \leq t \leq d - 1$,

$$(q^r; q^m)_t = (1 - q^{r+hm}) \prod_{0 \leq j \leq t-1, j \neq h} (1 - q^{r+jm}) \equiv 0 \pmod{\Phi_d(q)}.$$ 

In view of (3.2),

$$\frac{(q^r; q^m)_{ud+t}}{(q^m; q^m)_{ud+t}} \equiv 0 \pmod{\Phi_d(q)}.$$ 

Thus applying Lemma 3.2, we get

$$\sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv \sum_{k=0}^{ud+d-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv \sum_{s=0}^{u} \sum_{t=0}^{d-1} \frac{(q^r; q^m)_{sd+t}}{(q^m; q^m)_{sd+t}} \cdot \nu_{sd+t}(q)$$

$$\equiv \sum_{s=0}^{u} \frac{(\frac{w}{m})_s}{(1)_s} \cdot \mu_s(q) \sum_{t=0}^{d-1} \frac{(q^r; q^m)_t}{(q^m; q^m)_t} \cdot \nu_t(q) \equiv 0 \pmod{\Phi_d(q)}.$$ 

Furthermore, assume that $d \mid n$ and $(d, m) = 1$. Let $u = n/d$. Then in view of (3.2), we also have

$$\sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv \sum_{s=0}^{u-1} \frac{(\frac{w}{m})_s}{(1)_s} \cdot \mu_s(q) \sum_{t=0}^{d-1} \frac{(q^r; q^m)_t}{(q^m; q^m)_t} \cdot \nu_t(q)$$

$$\equiv 0 \pmod{\Phi_d(q)}.$$ 

4. Proof of Theorem 1.1. In this section, we shall complete the proof of Theorem 1.1. In fact, we have the following $q$-analogue of (1.6).
Hence

\[
\sum_{k=0}^{n-1} (-1)^k q^{c(rk+mk^2)-mk+\rho m(k+1)} [2mk + r]_q \cdot \frac{(q^r; q^m)_k^\rho}{(q^m; q^m)_k^\rho} \equiv 0 \pmod{A_{r,m,n}(q)C_{m,n}(q)}.
\]

**Proof.** Write \( \alpha = r/m \), where \( r \in \mathbb{Z} \), \( m \in \mathbb{Z}^+ \) and \( (r, m) = 1 \). It is easy to verify that

\[
\frac{(q^r; q^m)_k}{(q^m; q^m)_k} = (-1)^k q^{-rk+mk(c-1)/2} \left[ \frac{-r/m}{k} \right]_{q^m}.
\]

Assume \( d \geq 1 \) and \( (m, d) = 1 \). Let \( h = \lambda_{r,m}(d) \). Clearly \( r \equiv -hm \pmod{d} \). Then

\[
\left[ \frac{-r/m}{k} \right]_{q^m} = (-1)^k q^{-rk-m(c-1)/2} \cdot \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \equiv (-1)^k q^m h^{-rk-m(c-1)/2} \cdot \frac{(q^{-hm}; q^m)_k}{(q^m; q^m)_k} = \left[ \frac{h}{k} \right]_{q^m} \pmod{\Phi_d(q)}.
\]

Let \( c_* = c + \rho \). Note that

\[
\sum_{k=0}^{d-1} q^{m(h-k)-c_* mk(h-k)} [2mk - hm]_q \left[ \frac{h}{k} \right]_{q^m}^\rho
\]

\[
= \sum_{k=0}^{h} q^{mk-c_* mk(h-k)} [2m(h-k) - hm]_q \left[ \frac{h}{k} \right]_{q^m}^\rho
\]

\[
= - \sum_{k=0}^{h} q^{m(h-k)-c_* mk(h-k)} [2mk - hm]_q \left[ \frac{h}{k} \right]_{q^m}^\rho.
\]

We must have

\[
\sum_{k=0}^{d-1} q^{m(h-k)-c_* mk(h-k)} [2mk - hm]_q \cdot \left[ \frac{h}{k} \right]_{q^m}^\rho = 0.
\]

Hence

\[
\sum_{k=0}^{d-1} (-1)^k q^{c(rk+mk^2)-mk+\rho m(k+1)} [2mk + r]_q \cdot \frac{(q^r; q^m)_k^\rho}{(q^m; q^m)_k^\rho}
\]

\[
= \sum_{k=0}^{d-1} q^{c(rk+mk^2)-mk} [2mk + r]_q \cdot \left[ \frac{-r/m}{k} \right]_{q^m}^\rho
\]

\[
\equiv q^r \sum_{k=0}^{d-1} q^{m(h-k)-c_* mk(h-k)} [2mk - hm]_q \cdot \left[ \frac{h}{k} \right]_{q^m}^\rho \equiv 0 \pmod{\Phi_d(q)}.
\]

Thus the requirement (iii) of Theorem 2.1 is satisfied.
Next, we verify (ii). By Lemma 3.2, for any \( s, t \in \mathbb{N} \) with \( 0 \leq t \leq d - 1 \),

\[
q^{c(r(sd+t)+m(sd+t)^2)-m(sd+t)}[2m(sd + t) + r]q \cdot \frac{(q^r; q^m)_{sd+t}^{\rho-1}}{(q^m; q^m)_{sd+t}^{\rho-1}} \equiv \frac{(r + \lambda_{r,m}(d)m)_{md}^{\rho-1}}{(1)^{s-1}} \cdot q^{c(rt+mt^2)-mt}[2mt + r]q \cdot \frac{(q^r; q^m)_{t}^{\rho-1}}{(q^m; q^m)_{t}^{\rho-1}} \pmod{\Phi_d(q)},
\]

and

\[
(-1)^{sd+t}q^{m \left( \frac{sd+t+1}{2} \right)} \equiv (-1)^{sd+t}q^{msd(t+1)+m \left( \frac{sd}{2} \right) + m \left( \frac{t+1}{2} \right)} \equiv (-1)^{sd}q^{m \left( \frac{sd}{2} \right)} \cdot (-1)^{t}q^{m \left( \frac{t+1}{2} \right)} \pmod{\Phi_d(q)}.
\]

If \( d \) is odd, then clearly

\[
(-1)^{sd}q^{m \left( \frac{sd}{2} \right)} = (-1)^{s}q^{md \cdot s \left( \frac{sd-1}{2} \right)} \equiv (-1)^{s} \pmod{\Phi_d(q)}.
\]

If \( d \) is even, then

\[
1 + q^{d/2} = \frac{1 - q^{d}}{1 - q^{d/2}} \equiv 0 \pmod{\Phi_d(q)},
\]

i.e., \( q^{d/2} \equiv -1 \pmod{\Phi_d(q)} \). So

\[
(-1)^{sd}q^{m \left( \frac{sd}{2} \right)} = (q^{d/2})^{msd} \equiv (-1)^{s} \pmod{\Phi_d(q)},
\]

by noting that \( m \) is odd since \( (m,d) = 1 \). Hence, we always have

\[
(-1)^{\rho(sd+t)}q^{\rho m \left( \frac{sd+t+1}{2} \right)} \equiv (-1)^{\rho s} \cdot (-1)^{\rho t}q^{\rho m \left( \frac{t+1}{2} \right)} \pmod{\Phi_d(q)}.
\]

Thus (ii) is also valid, whence (4.1) follows.

Of course, we still need to explain how to deduce Theorem 1.1 from Theorem 4.1 and Corollary 2.1. In view of (3.4), clearly \( B_{r,m,n}(q) \) is divisible by \( B_{r,m,k}(q) \) provided \( 0 \leq k \leq n - 1 \). It follows from Lemma 3.3 that

\[
B_{r,m,n}(q)^\rho \sum_{k=0}^{n} q^{m(h-k)}[2mk + r]q \cdot \frac{(q^r; q^m)_k^{\rho}}{(q^m; q^m)_k^{\rho}}
\]

is a polynomial with integral coefficients. And by (2.2), each prime factor of \( B_{r,m,n}(1) \) must divide \( m \). In view of Corollary 2.1 we have

\[
\sum_{k=0}^{n-1} (2mk + r) \cdot \binom{-r/m}{k}^\rho \equiv 0 \pmod{N_{r/m,n}}.
\]

So (1.6) is valid since \( N_{r/m,n} \) and \( m \) are co-prime.

5. Guo’s \( q \)-analogue of (1.9). In [5, Conjecture 5.4], Guo conjectured the following.
Conjecture 5.1. For each \( n \geq 2 \) and \( r \geq 2 \),

\[
\sum_{k=0}^{n-1} (-1)^k q^{k^2 + (r-2)k}[4k + 1]_q \cdot \left[ \frac{2k}{k} \right]_{q^{2r-1}} (-q^{k+1}; q)_{n-1-k}^{4r-2} \equiv 0 \pmod{(1 + q^{n-1})^{2r-2} [n]_q [2n-1]_q},
\]

and

\[
\sum_{k=0}^{n-1} q^{-(r-2)k}[4k + 1]_q \cdot \left[ \frac{2k}{k} \right]_{q^{2r}} (-q^{k+1}; q)_{n-1-k}^{4r} \equiv 0 \pmod{(1 + q^{n-1})^{2r-1} [n]_q [2n-1]_q}.
\]

Evidently Conjecture 5.1 is a \( q \)-analogue of (1.9). In fact, for \( r = 2 \), (5.1) and (5.2) have been respectively confirmed by Guo [5] and Guo–Wang [9] with the help of the \( q \)-WZ method. However, their discussion is not suitable for \( r \geq 3 \), unless the corresponding \( q \)-WZ pairs are found.

Here we shall completely confirm Guo’s conjecture.

Theorem 5.1. Conjecture 5.1 is true.

Proof. It is well known (cf. [2, Lemma 1]) that the \( q \)-binomial coefficient \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) can be factorized into

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \prod_{d \in \mathcal{D}_{n,k}} \Phi_d(q),
\]

where

\[
\mathcal{D}_{n,k} := \left\{ d \geq 2 : \left\lfloor \frac{n}{d} \right\rfloor > \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{n-k}{d} \right\rfloor \right\}.
\]

Thus

\[
[n]_q [2n-1]_q = \prod_{d \mid n \atop d \geq 2} \Phi_d(q) \cdot \prod_{d \in \mathcal{D}_{2n-1,n-1}} \Phi_d(q).
\]

Since \( d \not\in \mathcal{D}_{2n-1,n-1} \) if \( d \mid n \), \( [n]_q [2n-1]_q \) is the product of some distinct cyclotomic polynomials.

Clearly for each \( 0 \leq k \leq n-2 \),

\[
(-q^{k+1}; q)_{n-1-k} = (1 + q^{n-1}) \cdot (-q^{k+1}; q)_{n-2-k} \equiv 0 \pmod{1 + q^{n-1}},
\]

and

\[
\left[ \begin{array}{c} 2n-2 \\ n-1 \end{array} \right]_q = (1 + q^{n-1}) \left[ \begin{array}{c} 2n-3 \\ n-2 \end{array} \right]_q \equiv 0 \pmod{1 + q^{n-1}}.
\]
So for each \(0 \leq k \leq n - 1\), we always have

\[
\left[\begin{array}{c} 2k \\ k \end{array}\right]_q (-q^{k+1}; q)_{n-1-k} \equiv 0 \pmod{1 + q^{n-1}}.
\]

Thus the left sides of (5.1) and (5.2) are divisible by \((1 + q^{n-1})^{2r-1}\) and \((1 + q^{n-1})^{2r}\) respectively. So we only need to show that the left sides of (5.1) and (5.2) are divisible by \([n]_q[2n-1]_q\).

Since \(\lambda_{1,2}(d) = (d - 1)/2\), it is easy to check that if \(1 < d \in D_{2n-1,n-1}\) is odd if and only if \(d \in S_{1,2}(n)\). Thus

\[
\text{If } k \text{ is even.}
\]

Note that

\[
\text{If } k \text{ is even.}
\]

Substituting \(c = 1 - r\) and \(\rho = 2r - 1\) in Theorem 4.1 we get

\[
\sum_{k=0}^{d-1} (-1)^k q^{k^2+(r-2)k}[4k+1]_q \cdot \frac{(q; q^2)_k}{(q^2; q^2)_k} \equiv 0 \pmod{A_{1,2,n}(q)C_{2,n}(q)}.
\]

Similarly, setting \(c = -r\) and \(\rho = 2r\) in Theorem 4.1 we also have

\[
\sum_{k=0}^{d-1} q^{(r-2)k}[4k+1]_q \cdot \frac{(q; q^2)_k}{(q^2; q^2)_k} \equiv 0 \pmod{A_{1,2,n}(q)C_{2,n}(q)}.
\]

Let us turn to \(d\) even. Suppose that \(d \in D_{2n-1,n-1}\) is even. By (4.3), \(1 + q^{d/2}\) is divisible by \(\Phi_d(q)\). Write \(n = ud + v\) where \(0 \leq v \leq d - 1\). Since \(d \in D_{2n-1,n-1}\), we must have \(v > d/2\). So if \(0 \leq k < ud + d/2\), then

\[
(-q^{k+1}; q)_{n-1-k} = (1 + q^{ud+d/2}) \prod_{k+1 \leq j \leq n-1 \atop j \neq ud+d/2} (1 + q^j) \equiv 0 \pmod{\Phi_d(q)}.
\]

And if \(ud + d/2 \leq k \leq n - 1\), then evidently also \(d \in D_{2k,k}\), i.e.,

\[
\left[\begin{array}{c} 2k \\ k \end{array}\right]_q \equiv 0 \pmod{\Phi_d(q)}.
\]

So for each \(0 \leq k \leq n - 1\),

\[
\left[\begin{array}{c} 2k \\ k \end{array}\right]_q (-q^{k+1}; q)_{n-1-k} \equiv 0 \pmod{\Phi_d(q)}.
\]

Similarly, if \(d\) is an even divisor of \(n\), then

\[
(-q^{k+1}; q)_{n-1-k} = (1 + q^{n-d/2}) \prod_{k+1 \leq j \leq n-1 \atop j \neq n-d/2} (1 + q^j) \equiv 0 \pmod{\Phi_d(q)}.
\]
provided that $0 \leq k < n - d/2$. And if $n - d/2 \leq k \leq n - 1$, we also have $d \in \mathcal{D}_{2k,k}$. Thus for each $0 \leq k \leq n - 1$, $\Phi_d(q)$ always divides $\genfrac{[}{]}{0pt}{}{2k}{k}_q(-q^{k+1};q)_{n-1-k}$. Hence the left sides of (5.1) and (5.2) are divisible by $\Phi_d(q)$ provided $d \in \mathcal{D}_{2n-1,n-1}$ is even or $d \mid n$ is even.

Thus in view of (5.4), we find that $[n]_q\genfrac{[}{]}{0pt}{}{2n-1}{n-1}_q$ divides the left sides of (5.1) and (5.2). So (5.1) and (5.2) are proved.

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