FUNDAMENTA
MATHEMATICAE
Online First version

Increasing sequences of principal left ideals of $\beta\mathbb{Z}$ are finite

by

Yevhen Zelenyuk (Johannesburg)

Abstract. We show that increasing sequences of principal left ideals of $\beta\mathbb{Z}$ are finite. As a consequence, $\beta\mathbb{Z}\setminus\mathbb{Z}$ is a disjoint union of maximal principal left ideals of $\beta\mathbb{Z}$. Another consequence is that increasing chains of idempotents ($p \le q \Leftrightarrow p+q = q+p = p$) in $\beta\mathbb{Z}$ are finite. All these are answers to long-standing open questions.

Addition of integers extends to the Stone–Čech compactification $\beta\mathbb{Z}$ of the discrete space $\mathbb{Z}$ so that for each $a \in \mathbb{Z}$, the left translation $\beta\mathbb{Z} \ni x \mapsto a + x \in \beta\mathbb{Z}$ is continuous, and for each $q \in \beta\mathbb{Z}$, the right translation $\beta\mathbb{Z} \ni x \mapsto x + q \in \beta\mathbb{Z}$ is continuous.

We take the points of $\beta\mathbb{Z}$ to be the ultrafilters on $\mathbb{Z}$, identifying the principal ultrafilters with the points of $\mathbb{Z}$. The topology of $\beta\mathbb{Z}$ has a base consisting of subsets of the form $\overline{A} = \{p \in \beta\mathbb{Z} : A \in p\}$, where $A \subseteq \mathbb{Z}$, and $\overline{A}$ is the closure of A. For $p, q \in \beta\mathbb{Z}$, the ultrafilter $p + q$ has a base consisting of subsets of the form $\bigcup_{x\in A}(x + B_x)$, where $A \in p$ and $B_x \in q$.

The semigroup $\beta\mathbb{Z}$ is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. The first application to Ramsey theory was the proof of Hindman's theorem: whenever the set $\mathbb{N}$ of positive integers is finitely colored, there is an infinite sequence all of whose sums are monochrome. Such a sequence was constructed using an idempotent of the semigroup $\mathbb{Z}^* = \beta\mathbb{Z} \setminus \mathbb{Z}$. An elementary introduction to $\beta\mathbb{Z}$ can be found in [4].

Sometime in the 1970's or 1980's M. Rudin was asked by some, now anonymous, analysts whether every point of $\mathbb{Z}^*$ is a member of a maximal orbit closure under the continuous extension of the shift function $n \mapsto 1+n$,

---

2020 *Mathematics Subject Classification*: Primary 22A15, 54H20; Secondary 05D10, 54D80.
Key words and phrases: Stone–Čech compactification, orbit closure, principal left ideal, chain of idempotents, ultrafilter, Ramsey theory.
Received 17 January 2021; revised 27 July 2021.
Published online 10 January 2022.

DOI: 10.4064/fm17-8-2021

equivalently under the action $\mathbb{Z} \times \mathbb{Z}^* \ni (x,p) \mapsto x+p \in \mathbb{Z}^*$. A simpler question is whether there is any infinite (strictly) increasing sequence of orbit closures of $\mathbb{Z}^*$. Clearly, a negative answer to the latter question implies an affirmative to the former. Since $\overline{\mathbb{Z}+p} = \beta\mathbb{Z}+p$, the orbit closures of $\mathbb{Z}^*$ are the (proper) principal left ideals of $\beta\mathbb{Z}$. Thus the second question in algebraic terms asks whether there is any infinite increasing sequence of principal left ideals of $\beta\mathbb{Z}$ (see [4, p. 135]). There are infinite increasing sequences of principal right ideals of $\beta\mathbb{Z}$ [2]. Since any two distinct maximal principal left ideals of $\beta\mathbb{Z}$ are disjoint [4, Corollary 6.20], the first question is equivalent to asking whether $\mathbb{Z}^*$ is a disjoint union of maximal principal left ideals of $\beta\mathbb{Z}$ (see [1]).

The inclusion relation on principal left and right ideals of $\beta\mathbb{Z}$ induces the left and right preorderings on idempotents of $\mathbb{Z}^*$:

$$p \leq_L q \Leftrightarrow \beta\mathbb{Z}+p \subseteq \beta\mathbb{Z}+q \Leftrightarrow p+q=p,$$
$$p \leq_R q \Leftrightarrow p+\beta\mathbb{Z} \subseteq q+\beta\mathbb{Z} \Leftrightarrow q+p=p,$$

and the standard order on idempotents is defined by

$$p \leq q \Leftrightarrow p+q=q+p=p.$$

There are infinite decreasing chains of idempotents in $\beta\mathbb{Z}$ [3]. In [5] an infinite increasing right chain of idempotents in $\beta\mathbb{Z}$ was constructed. However, the question whether there is any infinite increasing chain (left chain) of idempotents in $\beta\mathbb{Z}$ remained open [4, Question 9.27]. Notice that as distinguished from left (right) chains of idempotents, chains of idempotents are subsemigroups.

In this paper we show

Theorem 1. *Increasing sequences of principal left ideals of $\beta\mathbb{Z}$ are finite.*

As a consequence we obtain

Corollary 2. *$\mathbb{Z}^*$ is a disjoint union of maximal principal left ideals of $\beta\mathbb{Z}$.*

Another consequence is

Corollary 3. *Increasing chains (left chains) of idempotents in $\beta\mathbb{Z}$ are finite.*

In the rest of the paper we prove Theorem 1.

Let W denote the set of finite words over $\mathbb{N}$. For every sequence $w_1, w_2, \ldots$ in W, we write $w_1 < w_2 < \cdots$ if $|w_j| \geq j$ and $w_j(j) < w_{j+1}(j) < \cdots$ for each j. Here $|w|$ denotes the length of w and $w(j)$ the jth letter of w. For every finite sequence $w_1 < \cdots < w_n$, define $[w_1, \ldots, w_n] \in W$ to be the word obtained from w_n by replacing, for each $j < n$, $w_n(j)$ with $w_j(j)$.

PROPOSITION 4. *Assume that there is an infinite increasing sequence of principal left ideals of $\beta\mathbb{Z}$. Then for every finite coloring of W, there is an infinite sequence $w_1 < w_2 < \cdots$ such that the set*

$$\{[w_{j_1},\dots,w_{j_k}] : k \geq 1, 1 \leq j_1 < \cdots < j_k\}$$

is monochrome.

Proof. Let $(p_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{Z}^*$ such that $(\beta\mathbb{Z}+p_n)_{n=0}^{\infty}$ is increasing. Passing to a subsequence, one may suppose that $\{p_n : n = 0,1,\dots\}$ is discrete. For every n, pick $A_n \in p_n$ such that all A_n are pairwise disjoint and $\overline{A_{n+1}} \cap \overline{\mathbb{Z}+p_n} = \emptyset$ and so $x+p_n \notin \overline{A_{n+1}}$ for all $x \in \mathbb{Z}$. For all $n \geq 1$ and $\delta \in \{0,1\}$, let $X_{n,n-\delta} = \{x \in \mathbb{Z} : x+p_n \in \overline{A_{n-\delta}}\}$. Notice that $p_{n-1} \in \overline{\{x+p_n : x \in X_{n,n-1}\}}$ (in particular, $X_{n,n-1}$ is infinite).

We shall construct, for every $n \geq 1$, a decreasing sequence $(A_{n,j})_{j=1}^{\infty}$ of members of p_n with $A_{n,1} \subseteq A_n$ and a sequence $(x_{n,j})_{j=1}^{\infty}$ in $X_{n,n-1} \cup X_{n,n}$ such that the following three conditions are satisfied.

(1) All the subsets $x_{n,j} + A_{n,j}$, where $n,j \geq 1$, are pairwise disjoint, and each $x_{n,j}+A_{n,j}$ is disjoint from $A_{n,j}$; moreover, if $x_{n,j} \in X_{n,n-\delta}$, then $x_{n,j}+A_{n,j} \subseteq A_{n-\delta}$, and if $n-\delta \geq 1$, then $x_{n,j}+A_{n,j} \subseteq A_{n-\delta,k-1} \setminus A_{n-\delta,k}$ for some k (we put $A_{n,0} = A_n$ for all $n \geq 0$).

Thus, for every $n \geq 0$, A_n contains the subsets $x_{n+1,j} + A_{n+1,j}$ where $x_{n+1,j} \in X_{n+1,n}$, and the subsets $x_{n,j}+A_{n,j}$ where $x_{n,j} \in X_{n,n}$ (the second family may be empty; it is so in particular if $n = 0$). Let

$$C_n = \bigcup_{\delta=0}^{1} \bigcup \{x_{n+\delta,j} + A_{n+\delta,j} : x_{n+\delta,j} \in X_{n+\delta,n}\}$$

and $Z_n = A_n \setminus C_n$.

Notice that if $x_{n,j}+A_{n,j} \subseteq A_{n,k}$, then $j > k$ (otherwise $A_{n,j} \supseteq A_{n,k}$, and $x_{n,j}+A_{n,j}$ is disjoint from $A_{n,j}$).

(2) For every $n \geq 0$ and every $x \in X_{n+1,n}$, there are finite $J_0, J_1 \subseteq \mathbb{N}$ with $J_1 \neq \emptyset$ such that for each $k \in J_0$,

$$-\sum_{J_0 \ni j<k} x_{n,j} + x + p_{n+1} \in \overline{x_{n,k}+A_{n,k}},$$

and for each $k \in J_1$,

$$-\Big(\sum_{j\in J_0} x_{n,j} + \sum_{J_1 \ni j<k} x_{n+1,j}\Big) + x + p_{n+1} \in \overline{x_{n+1,k}+A_{n+1,k}},$$

and

$$x = \sum_{j\in J_0} x_{n,j} + \sum_{j \in J_1} x_{n+1,j}.$$

We call the latter an *X-decomposition* of x.

From the above relations we deduce that the subsets J_0, J_1 are determined inductively as follows.

Find $x_{n+\delta,k} + A_{n+\delta,k} \subseteq A_n$ such that $x + p_{n+1} \in \overline{x_{n+\delta,k} + A_{n+\delta,k}}$. Then k is the first element of J_0 if $\delta = 0$, while $J_0 = \emptyset$ and k is the first element of J_1 otherwise. If a nonempty initial segment J_0' of J_0 is already determined, then $-\sum_{J_0' \ni j < s} x_{n,j} + x + p_{n+1} \in \overline{x_{n,s} + A_{n,s}}$, where $s = \max J_0'$, whence $-\sum_{j \in J_0'} x_{n,j} + x + p_{n+1} \in \overline{A_{n,s}}$, so find $x_{n+\delta,k} + A_{n+\delta,k} \subseteq A_{n,s}$ such that $-\sum_{j \in J_0'} x_{n,j} + x + p_{n+1} \in \overline{x_{n+\delta,k} + A_{n+\delta,k}}$. Then k is the next element of J_0 if $\delta = 0$, while $J_0 = J_0'$ and k is the first element of J_1 otherwise. If J_0 and a nonempty initial segment J_1' of J_1 are already determined, then $-(\sum_{j \in J_0} x_{n,j} + \sum_{j \in J_1'} x_{n+1,j}) + x + p_{n+1} \in \overline{A_{n+1,s}}$, where $s = \max J_1'$, so find $x_{n+1+\delta,k} + A_{n+1+\delta,k} \subseteq A_{n+1,s}$ such that $-(\sum_{j \in J_0} x_{n,j} + \sum_{j \in J_1'} x_{n+1,j}) + x + p_{n+1} \in \overline{x_{n+1+\delta,k} + A_{n+1+\delta,k}}$. Then $\delta = 0$ (because $y + p_{n+1} \notin \overline{A_{n+2}}$ for all $y \in \mathbb{Z}$) and k is the next element of J_1. Determining J_1 terminates when $x = \sum_{j \in J_0} x_{n,j} + \sum_{j \in J_1} x_{n+1,j}$. (Notice that for each $j \in J_0$, $x_{n,j} \in X_{n,n}$, and for each $j \in J_1$, $x_{n+1,j} \in X_{n+1,n}$ if $j = \min J_1$, while $x_{n+1,j} \in X_{n+1,n+1}$ otherwise.)

This also shows that the X-decomposition of x is unique.

Let $\theta(x) = \min J_1$ and $\phi(x) = \max J_1$. There is $A(x) \in p_{n+1}$ (contained in $A_{n+1,\phi(x)}$) such that for each $k \in J_0$,

$$-\sum_{J_0 \ni j < k} x_{n,j} + x + A(x) \subseteq x_{n,k} + A_{n,k},$$

and for each $k \in J_1$,

$$-\Big(\sum_{j \in J_0} x_{n,j} + \sum_{J_1 \ni j < k} x_{n+1,j}\Big) + x + A(x) \subseteq x_{n+1,k} + A_{n+1,k}.$$

(3) For every $n \geq 0$ and every $a \in A_n$, there are $l = l(a) \geq 0$ and finite $J_0, \dots, J_l \subseteq \mathbb{N}$ with $J_1, \dots, J_l \neq \emptyset$ such that for all $i \leq l$ and $k \in J_i$,

$$-\Big(\sum_{j \in J_0} x_{n,j} + \dots + \sum_{j \in J_{i-1}} x_{n+i-1,j} + \sum_{J_i \ni j < k} x_{n+i,j}\Big) + a \in x_{n+i,k} + A_{n+i,k},$$

$$-\Big(\sum_{j \in J_0} x_{n,j} + \dots + \sum_{j \in J_l} x_{n+l,j}\Big) + a \in Z_{n+l},$$

and so

$$a = \sum_{j \in J_0} x_{n,j} + \dots + \sum_{j \in J_l} x_{n+l,j} + z$$

for some $z \in Z_{n+l}$. We call the latter an *A-decomposition* of a.

From the above relations we deduce that the subsets $J_0, \dots, J_l$ are determined inductively as follows.

If $a \in Z_n$, then $l = 0$ and $J_0 = \emptyset$. Otherwise find $x_{n+\delta,k} + A_{n+\delta,k} \subseteq A_n$ such that $a \in x_{n+\delta,k} + A_{n+\delta,k}$. Then k is the first element of J_0 if $\delta = 0$, while $J_0 = \emptyset$ and k is the first element of J_1 otherwise. If a nonempty initial segment J_0' of J_0 is already determined, then $-\sum_{j\in J_0'} x_{n,j} + a \in A_{n,s}$, where $s = \max J_0'$, so find $x_{n+\delta,k} + A_{n+\delta,k} \subseteq A_{n,s}$ such that $-\sum_{j\in J_0'} x_{n,j} + a \in x_{n+\delta,k} + A_{n+\delta,k}$. Then k is the next element of J_0 if $\delta = 0$, while $J_0 = J_0'$ and k is the first element of J_1 otherwise. If J_0 and a nonempty initial segment J_1' of J_1 are already determined, then $-(\sum_{j\in J_0} x_{n,j} + \sum_{j\in J_1'} x_{n+1,j}) + a \in A_{n+1,s}$, where $s = \max J_1'$, so find $x_{n+1+\delta,k} + A_{n+1+\delta,k} \subseteq A_{n+1,s}$ such that $-(\sum_{j\in J_0} x_{n,j} + \sum_{j\in J_1'} x_{n+1,j}) + a \in x_{n+1+\delta,k} + A_{n+1+\delta,k}$. Then k is the next element of J_1 if $\delta = 0$, while $J_1 = J_1'$ and k is the first element of J_2 otherwise, and so on. Determining $J_0, \dots, J_l$ terminates when $-(\sum_{j\in J_0} x_{n,j} + \dots + \sum_{j\in J_l} x_{n+l,j}) + a \in Z_{n+l}$. (Notice that for each $j \in J_0$, $x_{n,j} \in X_{n,n}$, and for all $i \in \{1, \dots, l\}$ and $j \in J_i$, $x_{n+i,j} \in X_{n+i,n+i-1}$ if $j = \min J_i$, while $x_{n+i,j} \in X_{n+i,n+i}$ otherwise.)

This also shows that the A-decomposition of a is unique.

The X-decomposition of x and the A-decomposition of a are unrelated in general. However, if $x \in X_{n+1,n}$ and $a \in A(x)$, and

$$\begin{aligned} x &= \sum_{j\in I_0} x_{n,j} + \sum_{j\in I_1} x_{n+1,j}, \\ a &= \sum_{j\in J_1} x_{n+1,j} + \sum_{j\in J_2} x_{n+2,j} + \dots + \sum_{j\in J_{l+1}} x_{n+l+1,j} + z \end{aligned}$$

are their X-decomposition and A-decomposition respectively, then

$$x + a = \sum_{j\in I_0} x_{n,j} + \sum_{j\in I_1\cup J_1} x_{n+1,j} + \sum_{j\in J_2} x_{n+2,j} + \dots + \sum_{j\in J_{l+1}} x_{n+l+1,j} + z$$

is the A-decomposition of $x + a$ and $\max I_1 < \min J_1$ if $J_1 \neq \emptyset$. Indeed, since $a \in A(x)$ and $A(x) \subseteq A_{n+1,\phi(x)}$, we have $\min J_1 > \phi(x) = \max I_1$ if $J_1 \neq \emptyset$. And since $a \in A(x)$, for each $k \in I_0$ we have

$$-\sum_{I_0\ni j<k} x_{n,j} + x + a \in x_{n,k} + A_{n,k},$$

and for each $k \in I_1$,

$$-\Big(\sum_{j\in I_0} x_{n,j} + \sum_{I_1\ni j<k} x_{n+1,j}\Big) + x + a \in x_{n+1,k} + A_{n+1,k}.$$

Next, for each $k \in J_1$,

$$-\Big(\sum_{j\in I_0} x_{n,j} + \sum_{j\in I_1} x_{n+1,j} + \sum_{J_1\ni j<k} x_{n+1,j}\Big) + x + a = -\sum_{J_1\ni j<k} x_{n+1,j} + a \in x_{n+1,k} + A_{n+1,k},$$

and for each $2 \le i \le l+1$ and each $k \in J_i$,

$$-\Big(\sum_{j\in I_0} x_{n,j} + \sum_{j\in I_1\cup J_1} x_{n+1,j} + \dots + \sum_{j\in J_{i-1}} x_{n+i-1,j} + \sum_{J_i\ni j<k} x_{n+i,j}\Big) + x + a$$
$$= -\Big(\sum_{j\in J_1} x_{n+1,j} + \dots + \sum_{j\in J_{i-1}} x_{n+i-1,j} + \sum_{J_i\ni j<k} x_{n+i,j}\Big) + a \in x_{n+i,k} + A_{n+i,k},$$

and

$$-\Big(\sum_{j\in I_0} x_{n,j} + \sum_{j\in I_1\cup J_1} x_{n+1,j} + \dots + \sum_{j\in J_{l+1}} x_{n+l+1,j}\Big) + x + a = -\Big(\sum_{j\in J_1} x_{n+1,j} + \dots + \sum_{j\in J_{l+1}} x_{n+l+1,j}\Big) + a = z \in Z_{n+l+1}.$$

Now to construct the sequences, enumerate the set of pairs $e = (x, n)$ with $n \ge 0$ and $x \in X_{n+1,n}$ as $\{e_m : m = 0, 1, \dots\}$, and the set $\bigcup_{n=0}^{\infty} A_n$ as $\{a_m : m = 0, 1, \dots\}$. The construction goes by induction on m.

For $m = 0$, let $e_0 = (x, r)$ and $a_0 = a \in A_s$. Pick $B \in p_{r+1}$ such that $B \subseteq A_{r+1}$, $x + B \subseteq A_r$, and $a \notin x + B$ if $s = r$. Put $A_{r+1,1} = B$, $x_{r+1,1} = x$, and define k_n^0 for all $n \ge 1$ and Z_n^0 for all $n \ge 0$ by

$$k_n^0 = \begin{cases} 1 & \text{if } n = r+1, \\ 0 & \text{otherwise,} \end{cases} \qquad Z_n^0 = \begin{cases} \{a\} & \text{if } n = s, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $x = x_{r+1,1}$ will be an X-decomposition with $J_0 = \emptyset$ and $J_1 = \{1\}$, and $a = a$ an A-decomposition with $l(a) = 0$ and $J_0 = \emptyset$.

Fix $m \ge 0$ and suppose that we have defined k_n^m, $(A_{n,j})_{j=1}^{k_n^m}$, $(x_{n,j})_{j=1}^{k_n^m}$ for $n \ge 1$, and $Z_n^m \subseteq A_n$ for $n \ge 0$ so that

(i) $\sum_{n=1}^{\infty} k_n^m \le m+1$ and $\sum_{n=0}^{\infty} |Z_n^m| \le m+1$,

(ii) the subsets $x_{n,j} + A_{n,j}$ with $n \ge 1$ and $1 \le j \le k_n^m$ are pairwise disjoint, and $x_{n,j} + A_{n,j}$ is disjoint from $A_{n,j}$; moreover, if $x_{n,j} \in X_{n,n-\delta}$, then $x_{n,j} + A_{n,j} \subseteq A_{n-\delta}$, and if $n-\delta \ge 1$, then $x_{n,j} + A_{n,j} \subseteq A_{n-\delta,k-1} \setminus A_{n-\delta,k}$ for some $1 \le k \le k_{n-\delta}^m$ or $x_{n,j} + A_{n,j} \subseteq A_{n-\delta,k_{n-\delta}^m}$,

(iii) Z_n^m is disjoint from

$$C_n^m = \bigcup_{\delta=0}^{1} \bigcup \{x_{n+\delta,j} + A_{n+\delta,j} : j \le k_{n+\delta}^m \text{ and } x_{n+\delta,j} \in X_{n+\delta,n}\}.$$

Notice that $C_n^m \notin p_n$.

Let $a_{m+1} = a \in A_t$. Similarly to the description in (3), find $l \geq 0$ and finite $J_0, \dots, J_l \subseteq \mathbb{N}$ with $J_1, \dots, J_l \neq \emptyset$ such that for all $i \leq l$ and $k \in J_i$,

$$-\Big(\sum_{j\in J_0} x_{t,j} + \cdots + \sum_{j\in J_{i-1}} x_{t+i-1,j} + \sum_{J_i\ni j<k} x_{t+i,j}\Big) + a \in x_{t+i,k} + A_{t+i,k},$$

$$z = -\Big(\sum_{j\in J_0} x_{t,j} + \cdots + \sum_{j\in J_l} x_{t+l,j}\Big) + a \in A_{t+l,u} \setminus C^m_{t+l},$$

where $u = \max J_l$ ($\max\emptyset = 0$; if $t = 0$, then $J_0 = \emptyset$, so if $t = l = 0$, then $u = 0$). Put

$$Z_n^{m+1} = \begin{cases} Z^m_{t+l} \cup \{z\} & \text{if } n = t+l, \\ Z^m_n & \text{otherwise.} \end{cases}$$

Then

$$a = \sum_{j\in J_0} x_{t,j} + \cdots + \sum_{j\in J_l} x_{t+l,j} + z$$

will be an A-decomposition.

Let $e_{m+1} = (x, i)$. Similarly to the description in (2), find finite $J_0, J_1 \subseteq \mathbb{N}$ such that for each $k \in J_0$,

$$-\sum_{J_0\ni j<k} x_{i,j} + x + p_{i+1} \in \overline{x_{i,k} + A_{i,k}},$$

and for each $k \in J_1$,

$$-\Big(\sum_{j\in J_0} x_{i,j} + \sum_{J_1\ni j<k} x_{i+1,j}\Big) + x + p_{i+1} \in \overline{x_{i+1,k} + A_{i+1,k}},$$

and either

$$x = \sum_{j\in J_0} x_{i,j} + \sum_{j\in J_1} x_{i+1,j}$$

or

$$x' = -\Big(\sum_{j\in J_0} x_{i,j} + \sum_{j\in J_1} x_{i+1,j}\Big) + x \neq 0 \quad \text{and} \quad x' + p_{i+1} \in \overline{A_{i+\delta,v} \setminus C^m_{i+\delta}},$$

where

$$v = \begin{cases} \max J_1 & \text{if } J_1 \neq \emptyset, \\ \max J_0 & \text{otherwise} \end{cases}$$

($\delta = 0$ if and only if $J_1 = \emptyset$; if $i = 0$, then $J_0 = \emptyset$, so if $i = \delta = 0$, then $v = 0$).

In the first case, $J_1 \neq \emptyset$, and so

$$x = \sum_{j\in J_0} x_{i,j} + \sum_{j\in J_1} x_{i+1,j}$$

will be an X-decomposition. Indeed, $-\sum_{j\in J_0} x_{i,j}+x+p_{i+1}\in \overline{A_{i,k}}$, where $k=\max J_0$, and consequently if $J_1=\emptyset$, then $p_{i+1}\in\overline{A_{i,k}}\subseteq\overline{A_i}$. So put $k_n^{m+1}=k_n^m$.

In the second case, choose $B\in p_{i+1}$ such that $B\subseteq A_{i+1,k_{i+1}^m}$, $x'+B\subseteq A_{i+\delta}$, and if $i+\delta\geq 1$, then $x'+B\subseteq A_{i+\delta,j-1}\setminus A_{i+\delta,j}$ for some $1\leq j\leq k_{i+\delta}^m$ or $x'+B\subseteq A_{i+\delta,k_{i+\delta}^m}$, and the subsets

$$B, x'+B, C_{i+1}^m\cup Z_{i+1}^{m+1}, \text{ and } C_i^m\cup Z_i^{m+1}$$

are pairwise disjoint. Put $A_{i+1,k_{i+1}^m+1}=B$, $x_{i+1,k_{i+1}^m+1}=x'$, and

$$k_n^{m+1}=\begin{cases} k_{i+1}^m+1 & \text{if } n=i+1,\\ k_n^m & \text{otherwise.}\end{cases}$$

Then

$$x=\sum_{j\in J_0} x_{i,j}+\sum_{j\in J_1\cup\{k_{i+1}^{m+1}\}} x_{i+1,j}$$

will be an X-decomposition.

As a result, the sequences $(A_{n,j})_{j=1}^\infty$ and $(x_{n,j})_{j=1}^\infty$, $n\geq 1$, so constructed satisfy conditions (1)–(3): we need only check that $\lim_{m\to\infty} k_n^m=\infty$ for all $n\geq 1$. (Then (1) is obvious, and so are (2) and (3).) In fact, we show that $\{j : x_{n,j}\in X_{n,n-1}\}$ is infinite.

We proceed by induction on n. Let any m_0 be given. Pick $x\in X_{1,0}$ such that $x+p_1\notin\overline{x_{1,j}+A_{1,j}}$ for each $1\leq j\leq k_1^{m_0}$ (using the fact that $p_0\in\overline{\{x+p_1 : x\in X_{1,0}\}}$). Write a presumed X-decomposition that x has got at some stage m, say $x=\sum_{j\in J_0} x_{0,j}+\sum_{j\in J_1} x_{1,j}$, and let $t=\theta(x)=\min J_1$. Then $J_0=\emptyset$ and $x+p_1\in\overline{x_{1,t}+A_{1,t}}$, so $t>k_1^{m_0}$ and $x_{1,t}\in X_{1,0}$. Fix $n\geq 2$ and suppose that $\lim_{m\to\infty} k_{n-1}^m=\infty$. Let any m_0 be given. Pick s such that $A_{n-1,s}$ is disjoint from $x_{n,j}+A_{n,j}$ for each $1\leq j\leq k_n^{m_0}$, and pick $x\in X_{n,n-1}$ such that $x+p_n\in\overline{A_{n-1,s}}$. Let $t=\theta(x)$. Then $t>k_n^{m_0}$ and $x_{n,t}\in X_{n,n-1}$.

Having finished the construction, for every $n\geq 0$ let

$$D_n=\{a\in A_n : l(a)\geq 1\}.$$

Notice that $C_n\supseteq D_n\in p_n$. Indeed, for every $a\in Z_n$, $l(a)=0$ and $J_0=\emptyset$, so $D_n\subseteq C_n$. And since $p_n\in\overline{\{x+p_{n+1} : x\in X_{n+1,n}\}}$, for every $B\in p_n$ one can choose $x\in X_{n+1,n}$ and $B'\in p_{n+1}$ such that $x+B'\subseteq B$; then for any $a\in B'\cap A(x)$, $x+a\in B$ and $l(x+a)\geq 1$, so $D_n\in p_n$.

Define $f : D_0\to W$ as follows. For every $a\in D_0$, if

$$a=\sum_{j\in J_0} x_{0,j}+\sum_{j\in J_1} x_{1,j}+\cdots+\sum_{j\in J_n} x_{n,j}+z$$

is an A-decomposition and $\alpha_s=\min J_s$ for each $s\in\{1,\ldots,n\}$, put

$$f(a)=\alpha_1\ldots\alpha_n.$$

Now suppose that W is finitely colored and pick $B_0 \in p_0$ such that $B_0 \subseteq D_0$ and $f(B_0)$ is monochrome. Construct inductively a sequence of elements $y_{1,j} \in X_{1,0}$ and a decreasing sequence of members $B_{1,j} \in p_1$, $j \geq 1$, such that $y_{1,j} + B_{1,j} \subseteq B_0$, $B_{1,j} \subseteq A(y_{1,j})$, and $\theta(y_{1,j}) < \theta(y_{1,j+1})$. To satisfy the inequality, pick $y_{1,j+1}$ such that $y_{1,j+1} + p_1 \notin \overline{x_{1,k} + A_{1,k}}$ for each $1 \leq k \leq \theta(y_{1,j})$. Then inductively, for each $i \geq 2$, construct a sequence of elements $y_{i,j} \in X_{i,i-1}$ and a decreasing sequence of members $B_{i,j} \in p_i$, $j \geq i$, such that $y_{i,j} + B_{i,j} \subseteq B_{i-1,j}$, $B_{i,j} \subseteq A(y_{i,j})$, and $\theta(y_{i,j}) < \theta(y_{i,j+1})$ (using the fact that $p_{i-1} \in \overline{\{x + p_i : x \in X_{i,i-1}\}}$). To satisfy the inequality, pick t such that $A_{i-1,t}$ is disjoint from $x_{i,k} + A_{i,k}$ for each $1 \leq k \leq \theta(y_{1,j})$, and pick $y_{i,j+1}$ with $y_{i,j+1} + p_i \in \overline{A_{i-1,t}}$.

It then follows that for all $n \geq 1$ and $j_1 \leq \cdots \leq j_n$ with $j_i \geq i$,

$$y_{1,j_1} + \cdots + y_{n,j_n} + B_{n,j_n} \subseteq B_0,$$

and for every $b \in B_{n,j_n}$, if

$$y_{1,j_1} = \sum_{j \in I_0} x_{0,j} + \sum_{j \in J_1} x_{1,j}, \ \ldots \ , y_{n,j_n} = \sum_{j \in I_{n-1}} x_{n-1,j} + \sum_{j \in J_n} x_{n,j}$$

are X-decompositions and

$$b = \sum_{j \in I_n} x_{n,j} + \sum_{j \in J_{n+1}} x_{n+1,j} + \cdots + \sum_{j \in J_{n+l}} x_{n+l,j} + z$$

is an A-decomposition, then

$$\begin{aligned} y_{1,j_1} + \cdots + y_{n,j_n} + b = &\sum_{j \in I_0} x_{0,j} + \sum_{j \in J_1 \cup I_1} x_{1,j} + \cdots + \sum_{j \in J_n \cup I_n} x_{n,j} \\ &+ \sum_{j \in J_{n+1}} x_{n+1,j} + \cdots + \sum_{j \in J_{n+l}} x_{n+l,j} + z \end{aligned}$$

is an A-decomposition (by induction, starting from $y_{n,j_n} + b$), and so

$$f(y_{1,j_1} + \cdots + y_{n,j_n} + b) = \theta(y_{1,j_1}) \ldots \theta(y_{n,j_n}) \alpha_{n+1} \ldots \alpha_{n+l},$$

where $\alpha_{n+s} = \min J_{n+s}$.

For every n, pick $b_n \in B_{n,n}$ and let

$$a_n = y_{1,n} + \cdots + y_{n,n} + b_n \text{ and } w_n = f(a_n).$$

If $J_{n,n}, J_{n,n+1}, \ldots, J_{n,n+l_n}$ are finite sets from the A-decomposition of b_n and $\alpha_{n,n+s} = \min J_{n,n+s}$, then

$$w_n = \theta(y_{1,n}) \ldots \theta(y_{n,n}) \alpha_{n,n+1} \ldots \alpha_{n,n+l_n}.$$

Clearly, $w_1 < w_2 < \cdots$.

Let $1 \le j_1 < \cdots < j_k = n$ be given and let $w = [w_{j_1}, \dots, w_{j_k}]$ and $a = y_{1,j_1} + \cdots + y_{k,j_k} + y_{k+1,n} + \cdots + y_{n,n} + b_n$. Then

$$w = \theta(y_{1,j_1}) \dots \theta(y_{k,j_k})\theta(y_{k+1,n}) \dots \theta(y_{n,n})\alpha_{n,n+1} \dots \alpha_{n,n+l_n},$$
$$f(a) = \theta(y_{1,j_1}) \dots \theta(y_{k,j_k})\theta(y_{k+1,n}) \dots \theta(y_{n,n})\alpha_{n,n+1} \dots \alpha_{n,n+l_n},$$

so $w = f(a)$, and since $a \in B_0$, we have $w \in f(B_0)$. ∎

The next lemma tells us that the conclusion of Proposition 4 is false, and so is the hypothesis, which finishes the proof of Theorem 1.

Lemma 5. *There is a 2-coloring of W such that for every infinite sequence $w_1 < w_2 < \cdots$, the set*

$$\{[w_{j_1}, \dots, w_{j_k}] : k \ge 1, 1 \le j_1 < \cdots < j_k\}$$

is not monochrome.

Proof. For every word $w = \alpha_1 \dots \alpha_m$ with $|w| > 1$, define inductively $r(w) \in \mathbb{N}$ and a decreasing sequence $s(w) : m = i_0 > i_1 > \cdots > i_{r(w)} = 1$ by

$$i_{t+1} = \min\big\{i \in \{1, \dots, i_t - 1\} : i_t - j \le \alpha_j \text{ for each } j \in \{i, i+1, \dots, i_t - 1\}\big\},$$

and let $d_1(w) = i_{r(w)-1} - 1$ and $d_2(w) = i_{r(w)-2} - 1$. Since $i_{r(w)-1} = d_1(w) + 1$ and $i_0 = |w|$, $d_2(w)$ is defined if $|w| > d_1(w) + 1$. Now define a coloring $\chi : W \to \mathbb{Z}_2$ by

$$\chi(w) = \begin{cases} 1 & \text{if } r(w) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

To show that χ is as required, let $w_1 < w_2 < \cdots$ be given.

Suppose first that $\{d_1(w_j) : j \in \mathbb{N}\}$ is infinite. Pick j such that $d_1(w_j) > w_1(1) = \alpha_1$ and let $w = [w_1, w_j]$. If $w_j = \beta_1 \dots \beta_m$ and $s(w_j) : m = i_0 > i_1 > \cdots > i_{r-1} > i_r = 1$, then $w = \alpha_1\beta_2 \dots \beta_m$ and $s(w) : m = i_0 > i_1 > \cdots > i_{r-1} > 2 > 1$. Hence, $r(w) = r(w_j) + 1$.

Suppose now that $\{d_1(w_j) : j \in \mathbb{N}\}$ is finite. Then for every $j > \max\{d_1(w_i) : i \in \mathbb{N}\} + 1$, one has $|w_j| > d_1(w_j) + 1$, so $d_2(w_j)$ is defined. We claim that $\{d_2(w_j) : j \in \mathbb{N}\}$ is infinite.

To see this, assume the contrary. Pick $j \ge \max\{d_2(w_i) : i \in \mathbb{N}\} + 1$ and let $w = w_j$. If $s(w) : i_0 > i_1 > \cdots > i_{r-1} > i_r = 1$, then $i_{r-2} - 1 \le j - 1$, so $i_{r-2} \le j$, and since $w(1) \ge j > i_{r-2} - 1$, $w(2) \ge j - 1 > i_{r-2} - 2, \dots,$ one has $i_{r-1} = 1$, a contradiction.

Now pick j_1 such that $\alpha_1 = w_{j_1}(1) \ge \max\{d_1(w_j) : j \in \mathbb{N}\}$ and let $k = \alpha_1 + 1$, so $(k+1) - 1 > \alpha_1$. Pick $j_2 < \cdots < j_k < j_{k+1}$ with $j_1 < j_2$ such that $\alpha_2 = w_{j_2}(2) \ge (k+1) - 2 = k - 1$, $\alpha_3 = w_{j_3}(3) \ge k - 2, \dots,$ $\alpha_k = w_{j_k}(k) \ge 1$, and $d_2(w_{j_{k+1}}) > \alpha_k + k - 1$. Let $w_{j_{k+1}} = \beta_1 \dots \beta_m$ and $s(w_{j_{k+1}}) : m = i_0 > i_1 > \cdots > i_{r-1} > i_r = 1$. Since $i_{r-1} - 1 \le \alpha_1 = k - 1$, one has $i_{r-1} \le k$, and $i_{r-2} - 1 > \alpha_k + k - 1$ gives $i_{r-2} > k + 1$ and $i_{r-2} - k > \alpha_k$.

Let $w = [w_{j_1}, \ldots, w_{j_k}, w_{j_{k+1}}]$. Then $w = \alpha_1 \ldots \alpha_k \beta_{k+1} \ldots \beta_m$ and $s(w) : m = i_0 > i_1 > \cdots > i_{r-2} > k+1 > 2 > 1$. Hence, $r(w) = r(w_{j_{k+1}}) + 1$. ∎

Remark 6. In Theorem 1 and Corollary 2, $\mathbb{Z}$ cannot be replaced with $\mathbb{N}$. To see this, pick $p \in \mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)$, for example any $p \in \{2^n : n \geq 0\}^*$ (see [6, Lemma 1]). For every $n \geq 0$, let $p_n = -n + p$, so $p_n \in \mathbb{N}^*$ and $1 + p_{n+1} = p_n$. Then the sequence of principal left ideals $\{p_n\} \cup (\beta\mathbb{N} + p_n)$ of $\beta\mathbb{N}$ generated by p_n, $n \geq 0$, is increasing, and every principal left ideal of $\beta\mathbb{N}$ generated by a point of $\mathbb{N}^*$ and containing p is a member of this sequence.

Remark 7. In the proof of Proposition 4 commutativity of $\mathbb{Z}$ was not used, so the proof also works for $\mathbb{Z}$ replaced with any countably infinite discrete group G, and consequently, Theorem 1, Corollary 2, and Corollary 3 also hold for $\mathbb{Z}$ replaced with G.

References

[1] G. Botha, Y. Zelenyuk, and Yu. Zelenyuk, *Finer closed left ideal decompositions of G^**, Topology Appl. 165 (2014), 58–63.

[2] N. Hindman, J. van Mill, and P. Simon, *Increasing chains of ideals and orbit closures in $\beta\mathbb{Z}$*, Proc. Amer. Math. Soc. 114 (1992), 1167–1172.

[3] N. Hindman and D. Strauss, *Chains of idemptotents in $\beta\mathbb{N}$*, Proc. Amer. Math. Soc. 123 (1995), 3881–3888.

[4] N. Hindman and D. Strauss, *Algebra in the Stone–Čech Compactification*, De Gruyter, Berlin, 1998.

[5] N. Hindman, D. Strauss, and Y. Zelenyuk, *Longer chains of idempotents in βG*, Fund. Math. 220 (2013), 243–261.

[6] Y. Zelenyuk, *Principal left ideals of βG may be both minimal and maximal*, Bull. London Math. Soc. 45 (2013), 613–617.

Yevhen Zelenyuk
School of Mathematics
University of the Witwatersrand
Private Bag 3, Wits 2050
Johannesburg, South Africa
E-mail: yevhen.zelenyuk@wits.ac.za