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On the zeros of certain weakly holomorphic modular forms for $\mathit{\Gamma}_0^+(5)$ and $\mathit{\Gamma}_0^+(7)$

by

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1. Introduction. Let $k \geq 4$ be an even integer and E_k be the Eisenstein series of weight k for $\mathrm{SL}_2(\mathbb{Z})$. The standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1,\, -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0 \right\} \cup \left\{ z \in \mathbb{H} \;\middle|\; |z| > 1,\, 0 < \mathrm{Re}(z) < \frac{1}{2} \right\}.$$

In 1970, Rankin and Swinnerton-Dyer proved that all the zeros of E_k on the standard fundamental domain lies on the lower boundary arc [6]. Their method is based on a simple real estimation which can be applied to other holomorphic modular forms. In recent years the location of the zeros of holomorphic modular forms has been studied in several cases: in the case of the Eisenstein series for the Fricke group of level 2 and 3 by Miezaki, Nozaki, and Shigezumi [5], and for level 5 and 7 by Shigezumi [8].

In 2008, Duke and Jenkins [3] considered weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$. They constructed a natural basis of the space of weakly holomorphic modular forms and proved that the zeros of almost all elements in the natural basis on the standard fundamental domain lie on the lower boundary arc. By using their method, Choi and Im [1] studied the zeros of certain weakly holomorphic modular forms for the Fricke group of level 2; Hanamoto and the present author [4] investigated the case of level 3.

In this paper, we give similar results for levels 5 and 7. In §2, we define some notations and state the main theorem. In §3, we prove some lemmas which play an important role in the proof of the main theorem. In §4, we prove the main theorem.

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2. The main result. Let p be 1 or a prime, write

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{p} \right\},$$

$$\Gamma_0^+(p) = \Gamma_0(p) \cup \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix} \Gamma_0(p),$$

and set $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Let $k \in 2\mathbb{Z}$ and $q = e^{2\pi i z}$ for $z \in \mathbb{H}$. A holomorphic function f on $\mathbb{H}$ is a *weakly holomorphic modular form* of weight k for $\Gamma_0^+(p)$ if it satisfies the following two conditions:

- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^+(p)$ and $z \in \mathbb{H}$.
- f is "meromorphic at the cusp $i\infty$".

By the first condition, we have $f(z+1) = f(z)$ since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0^+(p)$. Hence f has an expansion of the form

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^n, \qquad q = e^{2\pi i z}.$$

The second condition above means that $a_n = 0$ for almost all $n < 0$. Let n_f be the smallest integer n such that $a_n \neq 0$. Then f is a *holomorphic modular form* if $n_f \geq 0$, and a *cusp form* if $n_f \geq 1$. We denote the space of weakly holomorphic modular forms of weight k for $\Gamma_0^+(p)$ by $M_k^!(\Gamma_0^+(p))$, the space of holomorphic modular forms by $M_k(\Gamma_0^+(p))$, and the space of cusp forms by $S_k(\Gamma_0^+(p))$.

Now we assume that $p = 5$ or 7. A fundamental domain for $\Gamma_0^+(p)$ is given by

$$\mathbb{F}^+(p) = \left\{ z \in \mathbb{H} \,\middle|\, |z| \geq \frac{1}{\sqrt{p}},\ \left|z + \frac{1}{2}\right| \geq \frac{1}{2\sqrt{p}},\ -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0 \right\}$$
$$\cup \left\{ z \in \mathbb{H} \,\middle|\, |z| > \frac{1}{\sqrt{p}},\ \left|z - \frac{1}{2}\right| > \frac{1}{2\sqrt{p}},\ 0 < \mathrm{Re}(z) < \frac{1}{2} \right\}.$$

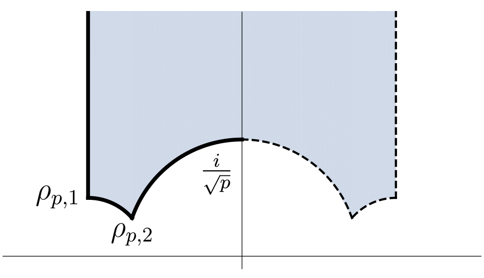

Fig. 1. A fundamental domain for level 5 and 7

We put $\rho_{5,1} = -\frac{1}{2} + \frac{i}{2\sqrt{5}}$, $\rho_{5,2} = -\frac{2}{5} + \frac{i}{5}$, $\rho_{7,1} = -\frac{1}{2} + \frac{i}{2\sqrt{7}}$, and $\rho_{7,2} = -\frac{5}{14} + \frac{\sqrt{3}}{14}i$.

Let

$$\delta = \delta_p = \begin{cases} 4 & \text{if } p = 5, \\ 12 & \text{if } p = 7, \end{cases}$$

and let $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ be the Dedekind eta function. Then $\Delta_p^+(z) = (\eta(z)\eta(pz))^{\delta}$ is a cusp form of weight δ for $\Gamma_0^+(p)$. The Hauptmodul for $\Gamma_0^+(p)$ is given by

$$j_p^+(z) = \left(\frac{\eta(z)}{\eta(pz)}\right)^{\frac{24}{p-1}} + \frac{24}{p-1} + p^{\frac{12}{p-1}}\left(\frac{\eta(pz)}{\eta(z)}\right)^{\frac{24}{p-1}}.$$

For $k \in 2\mathbb{Z}$, we write

$$k = \delta\ell_k + r_k \quad \text{where} \quad \ell_k \in \mathbb{Z} \text{ and } r_k = \begin{cases} \delta + 2 & \text{if } k \equiv 2 \pmod{\delta}, \\ k - [k/\delta]\delta & \text{otherwise.} \end{cases}$$

Put $m' = m_{p,k} = \frac{p+1}{24}\delta\ell_k + \dim S_{r_k}(\Gamma_0^+(p))$. Theorem 2.4 of [2] says that there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!(\Gamma_0^+(p))$ such that

$$f_{k,m}(z) = q^{-m} + O(q^{m'+1})$$

for each integer $m \geq -m'$. The functions $\{f_{k,m}\}_{m\geq -m'}$ form the *natural basis* of $M_k^!(\Gamma_0^+(p))$.

The function $f_{k,m}$ can be explicitly written by using Δ_p^+, j_p^+, and $\Delta_{p,r_k} = f_{r_k,-m_{p,r_k}}$ as follows:

$$f_{k,m} = (\Delta_p^+)^{\ell_k}\Delta_{p,r_k}F_{k,m+m'}(j_p^+) \tag{1}$$

where $F_{k,m+m'}$ is a monic polynomial of degree $m+m'$. We can check that $F_{k,m+m'}$ has rational coefficients, since the q-coefficients of Δ_p^+ and Δ_{p,r_k} are rational. Hence, the q-coefficients of $f_{k,m}$ are also rational.

The following is our main result.

THEOREM 2.1. *Let $p = 5$ or 7. For sufficiently large $m \in \mathbb{Z}$, all the zeros in $\mathbb{F}^+(p)$ of $f_{k,m}$ lie on the lower boundary arcs of the fundamental domain.*

3. Lemmas. First, we introduce the following lemma in order to use a real estimation.

LEMMA 3.1. *If $f(z) = \sum_{n\geq n_f}^{\infty} a(n)q^n \in M_k^!(\Gamma_0^+(p))$ has real coefficients, then $e^{ik\theta/2}f\left(\frac{1}{\sqrt{p}}e^{i\theta}\right)$ and $e^{ik\theta/2}f\left(-\frac{1}{2}+\frac{1}{2\sqrt{p}}e^{i\theta}\right)$ are real for all $\theta \in (0,\pi)$. In particular, $e^{ik\theta/2}f_{k,m}\left(\frac{1}{\sqrt{p}}e^{i\theta}\right)$ and $e^{ik\theta/2}f_{k,m}\left(-\frac{1}{2}+\frac{1}{2\sqrt{p}}e^{i\theta}\right)$ are real for all $\theta \in (0,\pi)$.*

Proof. For all $z, z' \in \mathbb{H}$, we note that

$$f\left(\frac{-1}{pz}\right) = f\left(\begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix} z\right) = (\sqrt{p}\, z)^k f(z),$$

$$f\left(\frac{\sqrt{p}\, z' + \frac{p-1}{2\sqrt{p}}}{2\sqrt{p}\, z' + \sqrt{p}}\right) = f\left(\begin{pmatrix} \sqrt{p} & \frac{p-1}{2\sqrt{p}} \\ 2\sqrt{p} & \sqrt{p} \end{pmatrix} z'\right) = (2\sqrt{p}\, z' + \sqrt{p})^k f(z'),$$

and

$$\overline{f(z)} = \overline{\sum_{n \geq n_f} a(n) e^{2\pi i n z}} = \sum_{n \geq n_f} a(n) \overline{e^{2\pi i n z}} = \sum_{n \geq n_f} a(n) e^{2\pi i n(-\overline{z})} = f(-\overline{z}).$$

Put $z = \frac{1}{\sqrt{p}} e^{i\theta}$ and $z' = -\frac{1}{2} + \frac{1}{2\sqrt{p}} e^{i\theta}$. Then $\frac{-1}{pz} = -\frac{1}{\sqrt{p}} e^{-i\theta} = -\overline{z}$ and

$$\frac{\sqrt{p}\, z' + \frac{p-1}{2\sqrt{p}}}{2\sqrt{p}\, z' + \sqrt{p}} = \frac{1}{2} - \frac{1}{2\sqrt{p}} e^{-i\theta} = -\overline{z'}.$$

Hence,

$$\begin{aligned} \overline{e^{ik\theta/2} f\left(\frac{1}{\sqrt{p}} e^{i\theta}\right)} &= e^{-ik\theta/2} \overline{f(z)} = e^{-ik\theta/2} f(-\overline{z}) \\ &= e^{-ik\theta/2} f\left(\frac{-1}{pz}\right) = e^{-ik\theta/2} (\sqrt{p}\, z)^k f(z) = e^{ik\theta/2} f\left(\frac{1}{\sqrt{p}} e^{i\theta}\right) \end{aligned}$$

and

$$\begin{aligned} &\overline{e^{ik\theta/2} f\left(-\frac{1}{2} + \frac{1}{2\sqrt{p}} e^{i\theta}\right)} \\ &\qquad = e^{-ik\theta/2} \overline{f(z')} = e^{-ik\theta/2} f(-\overline{z'}) = e^{-ik\theta/2} f\left(\frac{\sqrt{p}\, z' + \frac{p-1}{2\sqrt{p}}}{2\sqrt{p}\, z' + \sqrt{p}}\right) \\ &\qquad = e^{-ik\theta/2} (2\sqrt{p}\, z' + \sqrt{p})^k f(z') = e^{ik\theta/2} f\left(-\frac{1}{2} + \frac{1}{2\sqrt{p}} e^{i\theta}\right). \blacksquare \end{aligned}$$

Let $v_\rho(f)$ be the order of $f \in M_k^!(\Gamma_0^+(p))$ at $\rho \in \mathbb{H}$. Then the valence formulas for $\Gamma_0^+(5)$ and $\Gamma_0^+(7)$ are given as follows.

LEMMA 3.2. *Let $f \in M_k^!(\Gamma_0^+(5)) \setminus \{0\}$. Then*

$$v_\infty(f) + \tfrac{1}{2} v_{i/\sqrt{5}}(f) + \tfrac{1}{2} v_{\rho_{5,1}}(f) + \tfrac{1}{2} v_{\rho_{5,2}}(f) + \sum_{\substack{\rho \neq i/\sqrt{5},\, \rho_{5,1},\, \rho_{5,2} \\ \rho \in \mathbb{F}^+(5)}} v_\rho(f) = k/4.$$

LEMMA 3.3. *Let $f \in M_k^!(\Gamma_0^+(7)) \setminus \{0\}$. Then*

$$v_\infty(f) + \tfrac{1}{2} v_{i/\sqrt{7}}(f) + \tfrac{1}{2} v_{\rho_{7,1}}(f) + \tfrac{1}{3} v_{\rho_{7,2}}(f) + \sum_{\substack{\rho \neq i/\sqrt{7},\, \rho_{7,1},\, \rho_{7,2} \\ \rho \in \mathbb{F}^+(7)}} v_\rho(f) = k/3.$$

The proofs are very similar to that of the valence formula for $\mathrm{SL}_2(\mathbb{Z})$ [7]. Moreover, the following lemmas give us information about the orders of weakly holomorphic modular forms at $i/\sqrt{p}$, $\rho_{p,1}$, and $\rho_{p,2}$.

Lemma 3.4. *For $f \in M_k^!(\Gamma_0^+(5))$, we have*

$$v_{i/\sqrt{5}}(f) \geq s_k,\ v_{\rho_{5,1}}(f) \geq s_k,\ v_{\rho_{5,2}}(f) \geq s_k \quad (s_k = 0, 1 \text{ with } 2s_k \equiv k \pmod 4).$$

Lemma 3.5. *For $f \in M_k^!(\Gamma_0^+(7))$, we have*

$$v_{i/\sqrt{7}}(f) \geq s_k,\ v_{\rho_{7,1}}(f) \geq s_k \quad (s_k = 0, 1 \text{ with } 2s_k \equiv k \pmod 4),$$
$$v_{\rho_{7,2}}(f) \geq t_k \quad (t_k = 0, 1, 2 \text{ with } -2t_k \equiv k \pmod 6).$$

Proof. (1) implies that there exists a polynomial F of degree $m' - v_\infty(f)$ such that

$$f = (\Delta_p^+)^{\ell_k} \Delta_{p,r_k} F(j_p^+).$$

Hence the claims follow from [8, Propositions 2.5 and 2.6]. ∎

Now we consider two continuous real valued functions $F_{p,k,m}^1, F_{p,k,m}^2 : (0,\pi) \to \mathbb{R}$ defined by

$$F_{p,k,m}^1(\theta) = e^{-\frac{2}{\sqrt{p}}m\pi\sin\theta} e^{ik\theta/2} f_{k,m}\big(\tfrac{1}{\sqrt{p}}e^{i\theta}\big),$$
$$F_{p,k,m}^2(\theta) = e^{-\frac{1}{\sqrt{p}}m\pi\sin\theta} e^{ik\theta/2} f_{k,m}\big(-\tfrac{1}{2} + \tfrac{1}{2\sqrt{p}}e^{i\theta}\big).$$

By Lemma 3.1, these functions are well-defined. The following four lemmas play important roles in the proof of the main theorem.

Lemma 3.6. *Let $\alpha_5 \in [0,\pi]$ be the angle which satisfies $\tan\alpha_5 = 2$. If m is sufficiently large, we have the following bounds.*

(a) *For all $\theta \in \big(\frac{\pi}{2}, \frac{\pi}{2} + \alpha_5 - \frac{1}{2m}\big]$, we have*

$$\big|F_{5,k,m}^1(\theta) - 2\cos\big(\tfrac{k\theta}{2} - \tfrac{2}{\sqrt{5}}m\pi\cos\theta\big)\big| < 2.$$

(b) *For all $\theta \in \big[\alpha_5 + \frac{2}{3m}, \frac{\pi}{2}\big)$, we have*

$$\big|F_{5,k,m}^2(\theta) - 2\cos\big(\tfrac{k\theta}{2} - \tfrac{1}{\sqrt{5}}m\pi\cos\theta + m\pi\big)\big| < 2.$$

Lemma 3.7. *Let $u_5 \in [0, 2\pi)$ be the angle such that $\frac{k\pi}{4} + \frac{k\alpha_5}{2} + \frac{4m\pi}{5} \equiv u_5 \pmod{2\pi}$.*

(a) *There exists $0 < t_1 < 1/2$ such that for sufficiently large m,*

$$F_{5,k,m}^1\big(\tfrac{\pi}{2} + \alpha_5 - \tfrac{t_1}{m}\big)\cos u_5 > 0 \quad \textit{if } r_k = 0,$$
$$F_{5,k,m}^1\big(\tfrac{\pi}{2} + \alpha_5 - \tfrac{t_1}{m}\big)\sin(u_5 + \alpha_5) > 0 \quad \textit{if } r_k = 6.$$

(b) *There exists $0 < t_2 < 2/3$ such that for sufficiently large m,*

$$F_{5,k,m}^2\big(\alpha_5 + \tfrac{t_2}{m}\big)\cos(\ell_k\pi + u_5) > 0 \quad \textit{if } r_k = 0,$$
$$F_{5,k,m}^2\big(\alpha_5 + \tfrac{t_2}{m}\big)\sin((\ell_k + 1)\pi + u_5 + \alpha_5) > 0 \quad \textit{if } r_k = 6.$$

LEMMA 3.8. *Let $\alpha_7 \in [0, \pi]$ be the angle which satisfies $\tan\alpha_7 = 5/\sqrt{3}$. If m is sufficiently large, we have the following bounds.*

(a) *For all $\theta \in \left(\frac{\pi}{2}, \frac{\pi}{2} + \alpha_7 - \frac{70}{53m}\right]$, we have*

$$\left|F^1_{7,k,m}(\theta) - 2\cos\left(\frac{k\theta}{2} - \frac{2}{\sqrt{7}} m\pi\cos\theta\right)\right| < 2.$$

(b) *For all $\theta \in \left[\alpha_7 - \frac{\pi}{6} + \frac{70}{53m}, \frac{\pi}{2}\right)$, we have*

$$\left|F^2_{7,k,m}(\theta) - 2\cos\left(\frac{k\theta}{2} - \frac{1}{\sqrt{7}} m\pi\cos\theta + m\pi\right)\right| < 2.$$

LEMMA 3.9. *Let $u_7 \in [0, 2\pi)$ be the angle such that $\frac{k\pi}{4} + \frac{k\alpha_7}{2} + \frac{5m\pi}{7} \equiv u_7 \pmod{2\pi}$.*

(a) *There exists $0 < t_1 < \frac{70}{53}$ such that for sufficiently large m,*

$$\begin{aligned} F^1_{7,k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{t_1}{m}\right)\cos u_7 &> 0 \quad \text{if } r_k = 0, 6,\\ F^1_{7,k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{t_1}{m}\right)\sin(u_7 + \alpha_7) &> 0 \quad \text{if } r_k = 4, 10,\\ F^1_{7,k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{t_1}{m}\right)\sin\left(u_7 + 2\alpha_7 - \tfrac{\pi}{2}\right) &> 0 \quad \text{if } r_k = 8, 14. \end{aligned}$$

(b) *There exists $0 < t_2 < \frac{70}{53}$ such that for sufficiently large m,*

$$\begin{aligned} F^2_{7,k,m}\left(\alpha_7 - \tfrac{\pi}{6} + \tfrac{t_2}{m}\right)\cos u_7 &> 0 \quad \text{if } r_k = 0, 6,\\ F^2_{7,k,m}\left(\alpha_7 - \tfrac{\pi}{6} + \tfrac{t_2}{m}\right)\sin(u_7 + \pi + \alpha_7) &> 0 \quad \text{if } r_k = 4, 10,\\ F^2_{7,k,m}\left(\alpha_7 - \tfrac{\pi}{6} + \tfrac{t_2}{m}\right)\sin\left(u_7 + 2\alpha_7 - \tfrac{\pi}{2}\right) &> 0 \quad \text{if } r_k = 8, 14. \end{aligned}$$

Before the proof of the above four lemmas, we recall an integral representation of $f_{k,m}$ [2, p. 756]:

$$f_{k,m}(z) = \frac{1}{2\pi i}\oint_C \frac{f_k(z) f_{2-k}(\tau) {q'}^{-m-1}}{j_p^+(\tau) - j_p^+(z)}\, dq',$$

where $f_k = f_{k,-m_{p,k}} = (\Delta_p^+)^{\ell_k}\Delta_{p,r_k}$, $q' = e^{2\pi i\tau}$ and C is the circle centered at 0 in the q'-plane with a sufficiently small radius. Since $q' = e^{2\pi i\tau}$, $\frac{dq'}{d\tau} = 2\pi i q'$, $\ell_{2-k} = -\ell_k - 1$, $r_{2-k} = \delta + 2 - r_k$, and $\frac{dj_p^+}{d\tau} = -2\pi i\frac{\Delta_{p,\delta+2}}{\Delta_p^+}$, we can rearrange the integral formula of $f_{k,m}$ as follows:

(2) $$\begin{aligned} f_{k,m}(z) &= \int_{-1/2+iA}^{1/2+iA} \frac{f_k(z)f_{2-k}(\tau)e^{-2m\pi i\tau}}{j_p^+(\tau)-j_p^+(z)}\,d\tau \\ &= \int_{-1/2+iA}^{1/2+iA} \frac{\{\Delta_p^+(z)^{\ell_k}\Delta_{p,r_k}(z)\}\{\Delta_p^+(\tau)^{-\ell_k-1}\Delta_{p,\delta+2-r_k}(\tau)\}e^{-2m\pi i\tau}}{j_p^+(\tau)-j_p^+(z)}\,d\tau \\ &= \int_{-1/2+iA}^{1/2+iA} \frac{e^{-2m\pi i\tau}\Delta_p^+(z)^{\ell_k}\Delta_{p,r_k}(z)\Delta_{p,\delta+2-r_k}(\tau)}{\Delta_p^+(\tau)^{\ell_k+1}(j_p^+(\tau)-j_p^+(z))}\,d\tau \\ &= \int_{-1/2+iA}^{1/2+iA} e^{-2m\pi i\tau}\frac{\Delta_p^+(z)^{\ell_k}\Delta_{p,r_k}(z)}{\Delta_p^+(\tau)^{\ell_k}\Delta_{p,r_k}(\tau)}\cdot\frac{\Delta_{p,\delta+2}(\tau)}{\Delta_p^+(\tau)}\cdot\frac{1}{j_p^+(\tau)-j_p^+(z)}\,d\tau \\ &= \int_{-1/2+iA}^{1/2+iA} \frac{e^{-2m\pi i\tau}}{-2\pi i}\frac{\Delta_p^+(z)^{\ell_k}\Delta_{p,r_k}(z)}{\Delta_p^+(\tau)^{\ell_k}\Delta_{p,r_k}(\tau)}\frac{\frac{d}{d\tau}(j_p^+(\tau)-j_p^+(z))}{j_p^+(\tau)-j_p^+(z)}\,d\tau, \end{aligned}$$

where $A>0$ is sufficiently large. We write briefly

(3) $$\begin{aligned} G(\tau,z) &:= e^{-2m\pi i\tau}\frac{\Delta_p^+(z)^{\ell_k}\Delta_{p,r_k}(z)}{\Delta_p^+(\tau)^{\ell_k}\Delta_{p,r_k}(\tau)}\cdot\frac{\Delta_{p,\delta+2}(\tau)}{\Delta_p^+(\tau)}\cdot\frac{1}{j_p^+(\tau)-j_p^+(z)} \\ &= \frac{e^{-2m\pi i\tau}}{-2\pi i}\frac{\Delta_p^+(z)^{\ell_k}\Delta_{p,r_k}(z)}{\Delta_p^+(\tau)^{\ell_k}\Delta_{p,r_k}(\tau)}\frac{\frac{d}{d\tau}(j_p^+(\tau)-j_p^+(z))}{j_p^+(\tau)-j_p^+(z)}. \end{aligned}$$

Now, we move the contour of integration given in (2) downward to a height A' which has no pole of $G(\tau,z)$. As we do so, each pole τ_0 of $G(\tau,z)$ in the region D defined by

$$D=\{\tau\in\mathbb{H}\mid -1/2<\mathrm{Re}(\tau)<1/2,\ A'<\mathrm{Im}(\tau)<A\}$$

will contribute a term $2\pi i\cdot\mathrm{Res}_{\tau=\tau_0}G(\tau,z)$ to the equation.

When $z\not\equiv\frac{i}{\sqrt{p}},\rho_{p,1},\rho_{p,2}\pmod{\Gamma_0^+(p)}$, we have $\Delta_{p,r_k}(z)\neq 0$ and $\frac{dj_p^+}{d\tau}\big|_{\tau=z}\neq 0$ from Lemmas 3.2–3.5. Therefore, (3) implies that the simple poles of $G(\tau,z)$ occur only at $\tau\in D$ with $\tau\not\equiv z\pmod{\Gamma_0^+(p)}$. For any $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0^+(p)$ and $z'=\frac{az+b}{cz+d}$, we obtain

(4) $$\begin{aligned} &\mathrm{Res}_{z'}G(\tau,z) \\ &\quad=\lim_{\tau\to z'}(\tau-z')G(\tau,z) \\ &\quad=\frac{e^{-2m\pi iz'}}{-2\pi i}\frac{\Delta_p^+(z)^{\ell_k}\Delta_{p,r_k}(z)\frac{dj_p^+}{d\tau}\big|_{\tau=z'}}{\Delta_p^+(z')^{\ell_k}\Delta_{p,r_k}(z')\frac{dj_p^+}{d\tau}\big|_{\tau=z'}}=\frac{e^{-2m\pi iz'}}{-2\pi i}(cz+d)^{-k}. \end{aligned}$$

The residue theorem yields

$$(5)\quad \int_{-1/2+iA'}^{1/2+iA'} G(\tau,z)\,d\tau = f_{k,m}(z) - \int_{-1/2+iA'}^{-1/2+iA} G(\tau,z)\,d\tau + \int_{1/2+iA'}^{1/2+iA} G(\tau,z)\,d\tau$$
$$+ 2\pi i \sum_{\substack{\tau\in D\\ \tau\equiv z \bmod \Gamma_0^+(p)}} \mathrm{Res}_\tau\, G(\tau,z)$$
$$= f_{k,m}(z) + 2\pi i \sum_{\substack{\tau\in D\\ \tau\equiv z \bmod \Gamma_0^+(p)}} \mathrm{Res}_\tau\, G(\tau,z).$$

Proof of Lemma 3.6. (a) We put $z = \frac{1}{\sqrt{5}}e^{i\theta}$ $\left(\theta \in \left(\frac{\pi}{2}, \frac{\pi}{2}+\alpha_5\right)\right)$ and

$$A' = \begin{cases} \frac{1}{5} & \text{if } \theta \in \left(\frac{\pi}{2}, \frac{13}{5}\right], \\ \frac{3}{20} & \text{if } \theta \in \left(\frac{13}{5}, \frac{\pi}{2}+\alpha_5\right). \end{cases}$$

We can check that the poles of $G(\tau,z)$ on D occur only at

$$(6)\qquad \begin{cases} z \text{ and } \frac{-1}{5z} & \text{if } \theta \in \left(\frac{\pi}{2}, \frac{13}{5}\right], \\ z,\ \frac{-1}{5z},\ \frac{-2z-1}{5z+2},\ \text{and } \frac{5z+2}{10z+5} & \text{if } \theta \in \left(\frac{13}{5}, \frac{\pi}{2}+\alpha_5\right). \end{cases}$$

By (4), we have

$$(7)\quad \begin{aligned} \mathrm{Res}_z\, G(\tau,z) &= \frac{e^{\frac{2}{\sqrt5}m\pi\sin\theta}\,e^{-\frac{2}{\sqrt5}m\pi i\cos\theta}}{-2\pi i}, \\ \mathrm{Res}_{\frac{-1}{5z}}\, G(\tau,z) &= \frac{e^{-ik\theta}e^{\frac{2}{\sqrt5}m\pi\sin\theta}\,e^{\frac{2}{\sqrt5}m\pi i\cos\theta}}{-2\pi i}, \\ \mathrm{Res}_{\frac{-2z-1}{5z+2}}\, G(\tau,z) &= \frac{(\sqrt5\, e^{i\theta}+2)^{-k} e^{2m\pi\frac{\sin\theta}{9\sqrt5+20\cos\theta}}\, e^{2m\pi i\frac{4\sqrt5+9\cos\theta}{9\sqrt5+20\cos\theta}}}{-2\pi i}, \\ \mathrm{Res}_{\frac{5z+2}{10z+5}}\, G(\tau,z) &= \frac{(2e^{i\theta}+\sqrt5)^{-k} e^{2m\pi\frac{\sin\theta}{9\sqrt5+20\cos\theta}}\, e^{-2m\pi i\frac{4\sqrt5+9\cos\theta}{9\sqrt5+20\cos\theta}}}{-2\pi i}. \end{aligned}$$

Hence, (5) yields

$$(8)\quad F^1_{5,k,m}(\theta) - 2\cos\left(\tfrac{k\theta}{2} - \tfrac{2}{\sqrt5}m\pi\cos\theta\right)$$
$$= e^{-\frac{2}{\sqrt5}m\pi\sin\theta}e^{ik\theta/2}\int_{-1/2+iA'}^{1/2+iA'} G(\tau,z)\,d\tau + B_{k,m}(\theta)$$

where

$$B_{k,m}(\theta) = \begin{cases} 0 & \text{if } \theta \in \left(\frac{\pi}{2}, \frac{13}{5}\right], \\ -2\pi i e^{-\frac{2}{\sqrt{5}}m\pi\sin\theta} e^{\frac{ik\theta}{2}} \sum_{\tau=\frac{-5z-2}{2z+1}, \frac{-5z-2}{10z+5}} \mathrm{Res}_\tau\, G(\tau, z) & \text{if } \theta \in \left(\frac{13}{5}, \frac{\pi}{2}+\alpha_5\right). \end{cases}$$

The right hand side of (8) is bounded above by

$$\int_{-1/2+iA'}^{1/2+iA'} e^{-\frac{2}{\sqrt{5}}m\pi\sin\theta} |G(\tau,z)|\, d|\tau| + |B_{k,m}(\theta)|. \tag{9}$$

The integral in (9) tends to 0 as $m \to \infty$ uniformly in z. Actually, the integrand can be bounded above by $Ce^{-\frac{2}{\sqrt{5}}m\pi\sin(\frac{\pi}{2}+\alpha_5)}$ for some constant $C > 0$. Moreover,

$$\begin{aligned} |B_{k,m}(\theta)| &\le (|\sqrt{5}\,e^{i\theta}+2|^{|k|} + |2e^{i\theta}+\sqrt{5}|^{|k|}) e^{-2m\pi(\frac{1}{\sqrt{5}}\sin\theta - \frac{\sin\theta}{9\sqrt{5}+20\cos\theta})} \\ &= 2(9+4\sqrt{5}\cos\theta)^{\frac{|k|}{2}} e^{-2m\pi(\frac{1}{\sqrt{5}}\sin\theta - \frac{\sin\theta}{9\sqrt{5}+20\cos\theta})} \\ &= 2\{(9+4\sqrt{5}\cos\theta)^{\frac{|k|}{2m}} e^{-2\pi(\frac{1}{\sqrt{5}}\sin\theta - \frac{\sin\theta}{9\sqrt{5}+20\cos\theta})}\}^m. \end{aligned} \tag{10}$$

Put

$$\widetilde{B}_{k,m}(\theta) = (9+4\sqrt{5}\cos\theta)^{\frac{|k|}{2m}} e^{-2\pi(\frac{1}{\sqrt{5}}\sin\theta - \frac{\sin\theta}{9\sqrt{5}+20\cos\theta})}.$$

We can check that if m is sufficiently large, then $\widetilde{B}_{k,m}$ is increasing in $\left[\frac{13}{5}, \frac{\pi}{2}+\alpha_5\right]$ and the Taylor series of $\widetilde{B}_{k,m}(\theta)$ around $\theta = \frac{\pi}{2}+\alpha_5$ is given by

$$\widetilde{B}_{k,m}(\theta) = 1 + \left(\frac{8\pi}{5} - \frac{2|k|}{m}\right)\left(\theta - \frac{\pi}{2} - \alpha_5\right) + O\left(\left(\theta - \frac{\pi}{2} - \alpha_5\right)^2\right).$$

Hence, for all $\theta \in \left[\frac{13}{5}, \frac{\pi}{2}+\alpha_5 - \frac{1}{2m}\right]$, we have

$$\widetilde{B}_{k,m}(\theta) \le \widetilde{B_{k,m}}\left(\frac{\pi}{2}+\alpha_5-\frac{1}{2m}\right) \le 1 - \frac{4\pi}{5m} + \frac{C}{m^2}. \tag{11}$$

Here, the constant C does not depend on m. By (10) and (11), we have

$$|B_{k,m}(\theta)| \le 2\left(1 - \frac{4\pi}{5m} + \frac{C}{m^2}\right)^m.$$

The right hand side tends to $2e^{-4\pi/5}$ (< 0.1621) as $m \to \infty$. This completes the proof of (a).

(b) We put $z = -\frac{1}{2} + \frac{1}{2\sqrt{5}}e^{i\theta}$ ($\theta \in (\alpha_5, \frac{\pi}{2})$) and

$$A' = \begin{cases} \frac{3}{20} & \text{if } \theta \in \left(\alpha_5, \frac{3}{2}\right], \\ \frac{1}{5} & \text{if } \theta \in \left(\frac{3}{2}, \frac{\pi}{2}\right). \end{cases}$$

We can check that the poles of $G(\tau, z)$ on D occur only at

$$\begin{cases} z,\ \frac{-1}{5z},\ \frac{-2z-1}{5z+2},\ \text{and } \frac{5z+2}{10z+5} & \text{if } \theta \in \left(\alpha_5, \frac{3}{2}\right], \\ z \text{ and } \frac{-1}{5z} & \text{if } \theta \in \left(\frac{3}{2}, \frac{\pi}{2}\right). \end{cases}$$

By (4), we have

$$\mathrm{Res}_z\, G(\tau,z) = \frac{e^{\frac{1}{\sqrt{5}}m\pi\sin\theta}e^{-\frac{1}{\sqrt{5}}m\pi i\cos\theta+m\pi i}}{-2\pi i},$$
$$\mathrm{Res}_{\frac{-1}{5z}}\, G(\tau,z) = \frac{e^{-ik\theta}e^{\frac{1}{\sqrt{5}}m\pi\sin\theta}e^{\frac{1}{\sqrt{5}}m\pi i\cos\theta-m\pi i}}{-2\pi i},$$
$$\mathrm{Res}_{\frac{-2z-1}{5z+2}}\, G(\tau,z) = \frac{\left(\frac{\sqrt{5}}{2}e^{i\theta}-\frac{1}{2}\right)^{-k}e^{2m\pi\frac{\sin\theta}{3\sqrt{5}-5\cos\theta}}e^{-2m\pi i\frac{-\sqrt{5}+\cos\theta}{3\sqrt{5}-5\cos\theta}}}{-2\pi i},$$
$$\mathrm{Res}_{\frac{5z+2}{10z+5}}\, G(\tau,z) = \frac{\left(\frac{1}{2}e^{i\theta}-\frac{\sqrt{5}}{2}\right)^{-k}e^{2m\pi\frac{\sin\theta}{3\sqrt{5}-5\cos\theta}}e^{2m\pi i\frac{-\sqrt{5}+\cos\theta}{3\sqrt{5}-5\cos\theta}}}{-2\pi i}.$$

Thus, we can prove the inequality by a similar calculation to (a). ■

Proof of Lemma 3.7. (a) We define a function $H:\mathbb{R}\to\mathbb{R}$ by

$$H_{k,m}(t) = 2\cos\!\left(u_5-\tfrac{2\pi}{5}t\right)+2e^{-\frac{8\pi}{5}t}\cos\!\left(u_5+\tfrac{k\pi}{2}+\tfrac{2\pi}{5}t\right).$$

We can calculate that

$$\begin{cases} H_{k,m}(0)=4\cos u_5 & \text{if } k\equiv 0 \pmod 4,\\ H_{k,m}(0)=0 \text{ and } H'_{k,m}(0)=\frac{8\pi}{\sqrt{5}}\sin(u_5+\alpha_5) & \text{if } k\equiv 2 \pmod 4.\end{cases}$$

Hence if $r_k = 0$ (resp. $r_k = 6$), the signs of $H_{k,m}(t)$ and $\cos u_5$ (resp. $\sin(u_5+\alpha_5)$) coincide for sufficiently small $t>0$.

Because u_5 depends only on $m \pmod 5$ when k is fixed, we can find $\varepsilon>0$ and t_1 with $0<t_1<1/2$ such that for any $m\in\mathbb{Z}$,

$$\begin{cases} H_{k,m}(t_1)\cos u_5>\varepsilon & \text{if } r_k=0,\\ H_{k,m}(t_1)\sin(u_5+\alpha_5)>\varepsilon & \text{if } r_k=6.\end{cases}$$

Therefore, it suffices to show that

$$\lim_{m\to\infty}\left(F^1_{5,k,m}\!\left(\tfrac{\pi}{2}+\alpha_5-\tfrac{t_1}{m}\right)-H_{k,m}(t_1)\right)=0. \tag{12}$$

To do this, we use (5) and estimate the residues of $G(\tau,z)$. By (7),

$$\begin{aligned} -2\pi i e^{\frac{ik\theta}{2}}e^{-\frac{2}{\sqrt{5}}m\pi\sin\theta}\,\mathrm{Res}_z\, G(\tau,z) &= e^{\frac{ik\theta}{2}}e^{-\frac{2}{\sqrt{5}}m\pi i\cos\theta}\\ &= e^{iu_5}e^{\frac{ik}{2}(\theta-\frac{\pi}{2}-\alpha_5)}\left\{e^{-2\pi i(\frac{2}{5}+\frac{1}{\sqrt{5}}\cos\theta)}\right\}^m.\end{aligned}$$

Put $\theta=\frac{\pi}{2}+\alpha_5-\frac{t_1}{m}$. An estimation of the Taylor series around $\theta=\frac{\pi}{2}+\alpha_5$ shows that $e^{\frac{ik}{2}(\theta-\frac{\pi}{2}-\alpha_5)}\{e^{-2\pi i(\frac{2}{5}+\frac{1}{\sqrt{5}}\cos\theta)}\}^m$ converges to $e^{-\frac{2\pi i}{5}t_1}$ as $m\to\infty$. Hence we have

$$\lim_{m\to\infty}\left(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{5}}m\pi\sin\theta}\,\mathrm{Res}_z\, G(\tau,z)-e^{i(u_5-\frac{2\pi}{5}t_1)}\right)=0.$$

Similarly, we can prove that

$$\lim_{m\to\infty}\Big(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{5}}m\pi\sin\theta}\operatorname{Res}_{\frac{-1}{5z}}G(\tau,z)-e^{-i(u_5-\frac{2\pi}{5}t_1)}\Big)=0,$$

$$\lim_{m\to\infty}\Big(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{5}}m\pi\sin\theta}\operatorname{Res}_{\frac{-2z-1}{5z+2}}G(\tau,z)-e^{-\frac{8\pi}{5}t_1+i(u_5+\frac{k\pi}{2}+\frac{2\pi}{5}t_1)}\Big)=0,$$

$$\lim_{m\to\infty}\Big(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{5}}m\pi\sin\theta}\operatorname{Res}_{\frac{5z+2}{10z+5}}G(\tau,z)-e^{-\frac{8\pi}{5}t_1-i(u_5+\frac{k\pi}{2}+\frac{2\pi}{5}t_1)}\Big)=0.$$

Therefore, (12) follows from (5) and (6).

(b) can be proved in a similar way. ∎

The proofs of Lemmas 3.8 and 3.9 are similar to the case of $p=5$.

Proof of Lemma 3.8. (a) We put $z=\frac{1}{\sqrt{7}}e^{i\theta}$ $(\theta\in(\frac{\pi}{2},\frac{\pi}{2}+\alpha_7))$ and

$$A'=\begin{cases}\frac{13}{100} & \text{if } \theta\in\big(\frac{\pi}{2},\frac{11}{4}\big],\\ \frac{1}{10} & \text{if } \theta\in\big(\frac{11}{4},\frac{\pi}{2}+\alpha_7\big).\end{cases}$$

We can check that the poles of $G(\tau,z)$ on D occur only at

$$\begin{cases} z \text{ and } \frac{-1}{7z} & \text{if } \theta\in\big(\frac{\pi}{2},\frac{11}{4}\big],\\ z,\ \frac{-1}{7z},\ \frac{-3z-1}{7z+2},\ \frac{7z+3}{14z+7},\ \frac{-2z-1}{7z+3},\ \text{and } \frac{7z+2}{21z+7} & \text{if } \theta\in\big(\frac{11}{4},\frac{\pi}{2}+\alpha_7\big).\end{cases}\tag{13}$$

By (4), we have

$$\begin{aligned}
\operatorname{Res}_z G(\tau,z)&=\frac{e^{\frac{2}{\sqrt{7}}m\pi\sin\theta}e^{-\frac{2}{\sqrt{7}}m\pi i\cos\theta}}{-2\pi i},\\
\operatorname{Res}_{\frac{-1}{7z}} G(\tau,z)&=\frac{e^{-ik\theta}e^{\frac{2}{\sqrt{7}}m\pi\sin\theta}e^{\frac{2}{\sqrt{7}}m\pi i\cos\theta}}{-2\pi i},\\
\operatorname{Res}_{\frac{-3z-1}{7z+2}} G(\tau,z)&=\frac{(\sqrt{7}e^{i\theta}+2)^{-k}e^{2m\pi\frac{\sin\theta}{11\sqrt{7}+28\cos\theta}}e^{2m\pi i\frac{5\sqrt{7}+13\cos\theta}{11\sqrt{7}+28\cos\theta}}}{-2\pi i},\\
\operatorname{Res}_{\frac{7z+3}{14z+7}} G(\tau,z)&=\frac{(2e^{i\theta}+\sqrt{7})^{-k}e^{2m\pi\frac{\sin\theta}{11\sqrt{7}+28\cos\theta}}e^{-2m\pi i\frac{5\sqrt{7}+13\cos\theta}{11\sqrt{7}+28\cos\theta}}}{-2\pi i},\\
\operatorname{Res}_{\frac{-2z-1}{7z+3}} G(\tau,z)&=\frac{(\sqrt{7}e^{i\theta}+3)^{-k}e^{2m\pi\frac{\sin\theta}{16\sqrt{7}+42\cos\theta}}e^{2m\pi i\frac{5\sqrt{7}+13\cos\theta}{16\sqrt{7}+42\cos\theta}}}{-2\pi i},\\
\operatorname{Res}_{\frac{7z+2}{21z+7}} G(\tau,z)&=\frac{(3e^{i\theta}+\sqrt{7})^{-k}e^{2m\pi\frac{\sin\theta}{16\sqrt{7}+42\cos\theta}}e^{-2m\pi i\frac{5\sqrt{7}+13\cos\theta}{16\sqrt{7}+42\cos\theta}}}{-2\pi i}.
\end{aligned}\tag{14}$$

Hence, (5) yields

$$\begin{aligned}
&e^{-\frac{2}{\sqrt{7}}\pi m\sin\theta}e^{\frac{ik\theta}{2}}f_{k,m}\big(\tfrac{1}{\sqrt{7}}e^{i\theta}\big)-2\cos\big(\tfrac{k\theta}{2}-\tfrac{2}{\sqrt{7}}\pi m\cos\theta\big)\\
&\quad=e^{-\frac{2}{\sqrt{7}}\pi m\sin\theta}e^{ik\theta/2}\int_{-1/2+iA'}^{1/2+iA'}G(\tau,z)\,d\tau+B_{k,m}(\theta)+C_{k,m}(\theta)
\end{aligned}\tag{15}$$

where

$$B_{k,m}(\theta) = \begin{cases} 0 & \text{if } \theta \in \left(\frac{\pi}{2}, \frac{11}{4}\right], \\ -2\pi i e^{-\frac{2}{\sqrt{7}}\pi m \sin\theta} e^{ik\theta/2} \sum_{\tau=\frac{-3z-1}{7z+2}, \frac{7z+3}{14z+7}} \operatorname{Res}_\tau G(\tau, z) & \text{if } \theta \in \left(\frac{11}{4}, \frac{\pi}{2}+\alpha_7\right), \end{cases}$$

$$C_{k,m}(\theta) = \begin{cases} 0 & \text{if } \theta \in \left(\frac{\pi}{2}, \frac{11}{4}\right], \\ -2\pi i e^{-\frac{2}{\sqrt{7}}\pi m \sin\theta} e^{ik\theta/2} \sum_{\tau=\frac{-2z-1}{7z+3}, \frac{7z+2}{14z+7}} \operatorname{Res}_\tau G(\tau, z) & \text{if } \theta \in \left(\frac{11}{4}, \frac{\pi}{2}+\alpha_7\right). \end{cases}$$

The right hand side of (15) is bounded above by

$$\int_{-1/2+iA'}^{1/2+iA'} e^{-\frac{2}{\sqrt{7}}\pi m \sin\theta} |G(\tau, z)|\, d|\tau| + |B_{k,m}(\theta)| + |C_{k,m}(\theta)|. \tag{16}$$

It is easily checked that the integral in (16) tends to 0 as $m \to \infty$ uniformly in z. Moreover,

$$\begin{aligned} |B_{k,m}(\theta)| &\leq (|\sqrt{7}\, e^{i\theta} + 2|^{|k|} + |2e^{i\theta} + \sqrt{7}|^{|k|}) e^{-2\pi m(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{11\sqrt{7}+28\cos\theta})} \\ &= 2(11 + 4\sqrt{7}\cos\theta)^{\frac{|k|}{2}} e^{-2\pi m(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{11\sqrt{7}+28\cos\theta})} \\ &= 2\Big\{(11 + 4\sqrt{7}\cos\theta)^{\frac{|k|}{2m}} e^{-2\pi(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{11\sqrt{7}+28\cos\theta})}\Big\}^m, \end{aligned} \tag{17}$$

$$\begin{aligned} |C_{k,m}(\theta)| &\leq (|\sqrt{7}\, e^{i\theta} + 3|^{|k|} + |3e^{i\theta} + \sqrt{7}|^{|k|}) e^{-2\pi m(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{16\sqrt{7}+42\cos\theta})} \\ &= 2(16 + 6\sqrt{7}\cos\theta)^{\frac{|k|}{2}} e^{-2\pi m(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{16\sqrt{7}+42\cos\theta})} \\ &= 2\Big\{(16 + 6\sqrt{7}\cos\theta)^{\frac{|k|}{2m}} e^{-2\pi(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{16\sqrt{7}+42\cos\theta})}\Big\}^m. \end{aligned} \tag{18}$$

Put

$$\begin{aligned} \widetilde{B}_{k,m}(\theta) &= (11 + 4\sqrt{7}\cos\theta)^{\frac{|k|}{2m}} e^{-2\pi(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{11\sqrt{7}+28\cos\theta})}, \\ \widetilde{C}_{k,m}(\theta) &= (16 + 6\sqrt{7}\cos\theta)^{\frac{|k|}{2m}} e^{-2\pi(\frac{1}{\sqrt{7}} \sin\theta - \frac{\sin\theta}{16\sqrt{7}+42\cos\theta})}. \end{aligned}$$

We can check that if m is sufficiently large, $\widetilde{B}_{k,m}$ and $\widetilde{C}_{k,m}$ are then increasing in $\left[\frac{11}{4}, \frac{\pi}{2}+\alpha_7\right]$ and the Taylor series of $\widetilde{B}_{k,m}(\theta)$ and $\widetilde{C}_{k,m}(\theta)$ around $\theta = \frac{\pi}{2}+\alpha_7$ are given by

$$\begin{aligned} \widetilde{B}_{k,m}(\theta) &= 1 + \left(\tfrac{6\pi}{7} - \tfrac{\sqrt{3}|k|}{m}\right)\left(\theta - \tfrac{\pi}{2} - \alpha_7\right) + O\left(\left(\theta - \tfrac{\pi}{2} - \alpha_7\right)^2\right), \\ \widetilde{C}_{k,m}(\theta) &= 1 + \left(\tfrac{9\pi}{7} - \tfrac{3\sqrt{3}|k|}{2m}\right)\left(\theta - \tfrac{\pi}{2} - \alpha_7\right) + O\left(\left(\theta - \tfrac{\pi}{2} - \alpha_7\right)^2\right). \end{aligned}$$

Hence, for all $\theta \in \left[\frac{11}{4}, \frac{\pi}{2} + \alpha_5 - \frac{70}{53m}\right]$, we have

$$\widetilde{B}_{k,m}(\theta) \le \widetilde{B}_{k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{70}{53m}\right) \le 1 - \tfrac{60\pi}{53m} + \tfrac{C_1}{m^2}, \tag{19}$$

$$\widetilde{C}_{k,m}(\theta) \le \widetilde{C}_{k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{70}{53m}\right) \le 1 - \tfrac{90\pi}{53m} + \tfrac{C_2}{m^2}. \tag{20}$$

Here, the constants C_1 and C_2 do not depend on m. By (17)–(20), we have

$$|B_{k,m}(\theta)| + |C_{k,m}(\theta)| \le 2\left(1 - \tfrac{60\pi}{53m} + \tfrac{C_1}{m^2}\right)^m + 2\left(1 - \tfrac{90\pi}{53m} + \tfrac{C_2}{m^2}\right)^m.$$

The right hand side tends to $2e^{-\frac{60\pi}{53}} + 2e^{-\frac{90\pi}{53}}$ (< 0.06672) as $m \to \infty$. This completes the proof of (a).

(b) We put $z = -\frac{1}{2} + \frac{1}{2\sqrt{7}}e^{i\theta}$ $(\theta \in (\alpha_7 - \frac{\pi}{6}, \frac{\pi}{2}))$ and

$$A' = \begin{cases} \frac{1}{20} & \text{if } \theta \in \left(\alpha_7 - \frac{\pi}{6}, 1\right], \\ \frac{13}{100} & \text{if } \theta \in \left(1, \frac{\pi}{2}\right). \end{cases}$$

We can check that the poles of $G(\tau, z)$ on D occur only at

$$\begin{cases} z,\ \frac{-1}{7z},\ \frac{-3z-1}{7z+2},\ \frac{7z+2}{21z+7},\ \frac{-2z-1}{7z+3},\ \text{and } \frac{7z+3}{14z+7} & \text{if } \theta \in \left(\alpha_7 - \frac{\pi}{6}, 1\right], \\ z \text{ and } \frac{-1}{7z} & \text{if } \theta \in \left(1, \frac{\pi}{2}\right). \end{cases}$$

By (4), we have

$$\mathrm{Res}_z\, G(\tau, z) = \frac{e^{\frac{1}{\sqrt{7}}m\pi\sin\theta}e^{-\frac{1}{\sqrt{7}}m\pi i\cos\theta + m\pi i}}{-2\pi i},$$

$$\mathrm{Res}_{\frac{-1}{7z}}\, G(\tau, z) = \frac{e^{-ik\theta}e^{\frac{1}{\sqrt{7}}m\pi\sin\theta}e^{\frac{1}{\sqrt{7}}m\pi i\cos\theta - m\pi i}}{-2\pi i},$$

$$\mathrm{Res}_{\frac{-3z-1}{7z+2}}\, G(\tau, z) = \frac{\left(\frac{\sqrt{7}}{2}e^{i\theta} - \frac{3}{2}\right)^{-k}e^{2m\pi\frac{\sin\theta}{8\sqrt{7}-21\cos\theta}}e^{2m\pi i\frac{3\sqrt{7}-8\cos\theta}{8\sqrt{7}-21\cos\theta}}}{-2\pi i},$$

$$\mathrm{Res}_{\frac{7z+2}{21z+7}}\, G(\tau, z) = \frac{\left(\frac{3}{2}e^{i\theta} - \frac{\sqrt{7}}{2}\right)^{-k}e^{2m\pi\frac{\sin\theta}{8\sqrt{7}-21\cos\theta}}e^{-2m\pi i\frac{3\sqrt{7}-8\cos\theta}{8\sqrt{7}-21\cos\theta}}}{-2\pi i},$$

$$\mathrm{Res}_{\frac{-2z-1}{7z+3}}\, G(\tau, z) = \frac{\left(\frac{\sqrt{7}}{2}e^{i\theta} - \frac{1}{2}\right)^{-k}e^{2m\pi\frac{\sin\theta}{4\sqrt{7}-7\cos\theta}}e^{2m\pi i\frac{\sqrt{7}-\cos\theta}{4\sqrt{7}-7\cos\theta}}}{-2\pi i},$$

$$\mathrm{Res}_{\frac{7z+3}{14z+7}}\, G(\tau, z) = \frac{\left(\frac{1}{2}e^{i\theta} - \frac{\sqrt{7}}{2}\right)^{-k}e^{2m\pi\frac{\sin\theta}{4\sqrt{7}-7\cos\theta}}e^{-2m\pi i\frac{\sqrt{7}-\cos\theta}{4\sqrt{7}-7\cos\theta}}}{-2\pi i}.$$

Then we can prove the inequality by a similar calculation to that (a). ∎

Proof of Lemma 3.9. (a) We define $H_{k,m} : \mathbb{R} \to \mathbb{R}$ by

$$H_{k,m}(t) = 2\cos\left(u_7 - \tfrac{\sqrt{3}\pi}{7}t\right) + 2e^{-\frac{6\pi}{7}t}\cos\left(u_7 - \tfrac{2k\pi}{3} + \tfrac{3\sqrt{3}\pi}{7}t\right)$$
$$+ 2e^{-\frac{9\pi}{7}t}\cos\left(u_7 - \tfrac{k\pi}{3} - \tfrac{2\sqrt{3}\pi}{7}t\right).$$

We can calculate that

$$\begin{cases} H_{k,m}(0) = 6\cos u_7 & \text{if } k \equiv 0,6 \pmod{12}, \\ H_{k,m}(0) = 0 \text{ and } H'_{k,m}(0) = \frac{12\pi}{\sqrt{7}}\sin(u_7+\alpha_7) & \text{if } k \equiv 4,10 \pmod{12}, \\ H_{k,m}(0) = 0, H'_{k,m}(0) = 0, \text{ and } H''_{k,m}(0) = \frac{24\pi^2}{7}\sin(u_7+2\alpha_7-\pi/2) & \\ & \text{if } k \equiv 2,8 \pmod{12}. \end{cases}$$

Hence if $r_k = 0,6$ (resp. $r_k = 4,10$, $r_k = 8,14$), the signs of $H_{k,m}(t)$ and $\cos u_7$ (resp. $\sin(u_7+\alpha_7)$, $\sin(u_7+2\alpha_7-\pi/2)$) coincide for sufficiently small $t > 0$.

Because u_7 depends only on $m \pmod{14}$ when k is fixed, we can find $\varepsilon > 0$ and t_1 with $0 < t_1 < \frac{70}{53}$ such that for any $m \in \mathbb{Z}$,

$$\begin{cases} H_{k,m}(t_1)\cos u_7 > \varepsilon & \text{if } r_k = 0,6, \\ H_{k,m}(t_1)\sin(u_7+\alpha_7) > \varepsilon & \text{if } r_k = 4,10, \\ H_{k,m}(t_1)\sin(u_7+2\alpha_7-\pi/2) > \varepsilon & \text{if } r_k = 8,14. \end{cases}$$

Therefore, it suffices to show that

$$\lim_{m\to\infty}\bigl(F^1_{7,k,m}\bigl(\tfrac{\pi}{2}+\alpha_7-\tfrac{t_1}{m}\bigr) - H_{k,m}(t_1)\bigr) = 0. \tag{21}$$

To do this, we use (5) and estimate the residues of $G(\tau,z)$. By (14),

$$\begin{aligned} -2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi\sin\theta}\operatorname{Res}_z G(\tau,z) &= e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi i\cos\theta} \\ &= e^{iu_7}e^{\frac{ik}{2}(\theta-\pi/2-\alpha_7)}\{e^{-2\pi i(\frac{5}{14}+\frac{1}{\sqrt{7}}\cos\theta)}\}^m. \end{aligned}$$

Put $\theta = \frac{\pi}{2}+\alpha_7-\frac{t_1}{m}$. An estimation of the Taylor series around $\theta = \frac{\pi}{2}+\alpha_7$ shows that $e^{\frac{ik}{2}(\theta-\frac{\pi}{2}-\alpha_7)}\{e^{-2\pi i(\frac{5}{14}+\frac{1}{\sqrt{7}}\cos\theta)}\}^m$ converges to $e^{-\frac{\sqrt{3}\pi i}{7}t_1}$ as $m\to\infty$. Hence we have

$$\lim_{m\to\infty}\bigl(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi\sin\theta}\operatorname{Res}_z G(\tau,z) - e^{i(u_7-\frac{\sqrt{3}\pi}{7}t_1)}\bigr) = 0.$$

Similarly, we can prove that

$$\lim_{m\to\infty}\bigl(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi\sin\theta}\operatorname{Res}_{\frac{-1}{7z}} G(\tau,z) - e^{-i(u_7-\frac{\sqrt{3}\pi}{7}t_1)}\bigr) = 0,$$

$$\lim_{m\to\infty}\bigl(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi\sin\theta}\operatorname{Res}_{\frac{-3z-1}{7z+2}} G(\tau,z) - e^{-\frac{6\pi}{7}t+i(u_7-\frac{2k\pi}{3}+\frac{3\sqrt{3}\pi}{7}t_1)}\bigr) = 0,$$

$$\lim_{m\to\infty}\bigl(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi\sin\theta}\operatorname{Res}_{\frac{7z+3}{14z+7}} G(\tau,z) - e^{-\frac{6\pi}{7}t-i(u_7-\frac{2k\pi}{3}+\frac{3\sqrt{3}\pi}{7}t_1)}\bigr) = 0,$$

$$\lim_{m\to\infty}\bigl(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi\sin\theta}\operatorname{Res}_{\frac{-2z-1}{7z+3}} G(\tau,z) - e^{-\frac{9\pi}{7}t+i(u_7-\frac{k\pi}{3}-\frac{2\sqrt{3}\pi}{7}t_1)}\bigr) = 0,$$

$$\lim_{m\to\infty}\bigl(-2\pi i e^{ik\theta/2}e^{-\frac{2}{\sqrt{7}}m\pi\sin\theta}\operatorname{Res}_{\frac{7z+2}{21z+7}} G(\tau,z) - e^{-\frac{9\pi}{7}t-i(u_7-\frac{k\pi}{3}-\frac{2\sqrt{3}\pi}{7}t_1)}\bigr) = 0.$$

Therefore, (21) follows from (5) and (13).

(b) can be proved in a similar way. ∎

4. The proof of Theorem 2.1

Proof of Theorem 2.1 for $p = 5$. Throughout this proof, we assume that $m \in \mathbb{Z}$ is sufficiently large. We put

$$A = \#\left\{\theta \in \left(\tfrac{\pi}{2}, \tfrac{\pi}{2} + \alpha_5\right) \mid F^1_{5,k,m}(\theta) = 0\right\},$$
$$B = \#\left\{\theta \in \left(\alpha_5, \tfrac{\pi}{2}\right) \mid F^2_{5,k,m}(\theta) = 0\right\}.$$

Let $n = n_{k,m} \in \mathbb{Z}$, and let $u_5 \in [0, 2\pi)$ be the angle such that $\frac{k\pi}{4} + \frac{k\alpha_5}{2} + \frac{4m\pi}{5} = 2n\pi + u_5$.

Put $\beta_1(\theta) = \frac{k\theta}{2} - \frac{2}{\sqrt{5}} m\pi \cos\theta$ and $\beta_2(\theta) = \frac{k\theta}{2} - \frac{1}{\sqrt{5}} m\pi \cos\theta + m\pi$. Then Lemma 3.6 implies that

$$\theta \in \left[\tfrac{\pi}{2}, \tfrac{\pi}{2} + \alpha_5 - \tfrac{1}{2m}\right] \implies |F^1_{5,k,m}(\theta) - 2\cos\beta_1(\theta)| < 2,$$
$$\theta \in \left[\alpha_5 + \tfrac{2}{3m}, \tfrac{\pi}{2}\right] \implies |F^2_{5,k,m}(\theta) - 2\cos\beta_2(\theta)| < 2.$$

Therefore if $|\cos\beta_j(\theta)| = 1$ (i.e. $\beta_j(\theta)$ is a multiple of an integer and π), we have

$$\operatorname{sgn}(F^j_{5,k,m}(\theta)) = \operatorname{sgn}(\cos\beta_j(\theta)) \tag{22}$$

for $j = 1, 2$. Moreover, we can check that

$$\beta_1\left(\left[\tfrac{\pi}{2}, \tfrac{\pi}{2} + \alpha_5 - \tfrac{1}{2m}\right]\right) \supset \left[\tfrac{k\pi}{4}, 2n\pi + u_5 - \tfrac{\pi}{3}\right], \tag{23}$$
$$\beta_2\left(\left[\alpha_5 + \tfrac{2}{3m}, \tfrac{\pi}{2}\right]\right) \supset \left[2n\pi - \tfrac{k\pi}{4} + u_5 + \tfrac{\pi}{3}, m\pi + \tfrac{k\pi}{4}\right]. \tag{24}$$

We estimate the number of zeros of $f_{k,m}$ on the arcs for the cases of $r_k = 0, 6$ separately.

(i) *The case of $r_k = 0$* (i.e. $k = 4\ell_k$). By Lemmas 3.2 and 3.4, it is enough to prove that

$$A + B = m + \ell_k.$$

By Lemma 3.7, there exist $0 < t_1 < \frac{1}{2}$ and $0 < t_2 < \frac{2}{3}$ such that

(25)
$$\operatorname{sgn}\left(F^1_{5,k,m}\left(\tfrac{\pi}{2} + \alpha_5 - \tfrac{t_1}{m}\right)\right) = \begin{cases} 1 & \text{if } 0 < u_5 < \pi/2 \text{ or } 3\pi/2 < u_5 < 2\pi, \\ -1 & \text{if } \pi/2 < u_5 < 3\pi/2, \end{cases}$$

(26)
$$\operatorname{sgn}\left(F^2_{5,k,m}\left(\alpha_5 + \tfrac{t_2}{m}\right)\right) = \begin{cases} (-1)^{\ell_k} & \text{if } 0 < u_5 < \pi/2 \text{ or } 3\pi/2 < u_5 < 2\pi, \\ (-1)^{\ell_k+1} & \text{if } \pi/2 < u_5 < 3\pi/2. \end{cases}$$

Hence (22)–(26) and the intermediate value theorem imply that

$$A \geq \begin{cases} 2n-\ell_k & \text{if } 0<u_5<\pi/2 \\ 2n-\ell_k+1 & \text{if } \pi/2<u_5<3\pi/2, , \\ 2n-\ell_k+2 & \text{if } 3\pi/2<u_5<2\pi \end{cases}$$

$$B \geq \begin{cases} m+2\ell_k-2n & \text{if } 0<u_5<\pi/2, \\ m+2\ell_k-2n-1 & \text{if } \pi/2<u_5<3\pi/2, \\ m+2\ell_k-2n-2 & \text{if } 3\pi/2<u_5<2\pi. \end{cases}$$

In each case, we have $A+B\geq m+\ell_k$. On the other hand, $A+B\leq m+\ell_k$ from Lemma 3.2. Therefore, $A+B=m+\ell_k$.

(ii) *The case of* $r_k=6$ (i.e. $k=4\ell_k+6$). By Lemmas 3.2 and 3.4, it is enough to prove that

$$A+B=m+\ell_k.$$

By Lemma 3.7, there exist $0<t_1<1/2$ and $0<t_2<2/3$ such that

(27)

$$\operatorname{sgn}\bigl(F^1_{5,k,m}\bigl(\tfrac{\pi}{2}+\alpha_5-\tfrac{t_1}{m}\bigr)\bigr)=\begin{cases} 1 & \text{if } 0<u_5<\pi-\alpha_5 \text{ or } 2\pi-\alpha_5<u_5<2\pi, \\ -1 & \text{if } \pi-\alpha_5<u_5<2\pi-\alpha_5, \end{cases}$$

(28)

$$\operatorname{sgn}\bigl(F^2_{5,k,m}\bigl(\alpha_5+\tfrac{t_2}{m}\bigr)\bigr)=\begin{cases} (-1)^{\ell_k+1} & \text{if } 0<u_5<\pi-\alpha_5 \text{ or } 2\pi-\alpha_5<u_5<2\pi, \\ (-1)^{\ell_k} & \text{if } \pi-\alpha_5<u_5<2\pi-\alpha_5. \end{cases}$$

Hence (22)–(24), (27), (28), and the intermediate value theorem imply that

$$A \geq \begin{cases} 2n-\ell_k-2 & \text{if } 0<u_5<\pi-\alpha_5, \\ 2n-\ell_k-1 & \text{if } \pi-\alpha_5<u_5<2\pi-\alpha_5, \\ 2n-\ell_k & \text{if } 2\pi-\alpha_5<u_5<2\pi, \end{cases}$$

$$B \geq \begin{cases} m+2\ell_k-2n+2 & \text{if } 0<u_5<\pi-\alpha_5, \\ m+2\ell_k-2n+1 & \text{if } \pi-\alpha_5<u_5<2\pi-\alpha_5, \\ m+2\ell_k-2n & \text{if } 2\pi-\alpha_5<u_5<2\pi. \end{cases}$$

In each case, we have $A+B\geq m+\ell_k$. On the other hand, $A+B\leq m+\ell_k$ from Lemmas 3.2 and 3.4. ∎

Proof of Theorem 2.1 for $p=7$. Throughout this proof, we assume that $m\in\mathbb{Z}$ is sufficiently large. We put

$$A=\#\bigl\{\theta\in\bigl(\tfrac{\pi}{2},\tfrac{\pi}{2}+\alpha_7\bigr)\ \big|\ F^1_{7,k,m}(\theta)=0\bigr\},$$
$$B=\#\bigl\{\theta\in\bigl(\alpha_7-\tfrac{\pi}{6},\tfrac{\pi}{2}\bigr)\ \big|\ F^2_{7,k,m}(\theta)=0\bigr\}.$$

Let $n = n_{k,m} \in \mathbb{Z}$, and let $u_7 \in [0, 2\pi)$ be the angle such that $\frac{k\pi}{4} + \frac{k\alpha_7}{2} + \frac{5m\pi}{7} = 2n\pi + u_7$.

Put $\beta_1(\theta) = \frac{k\theta}{2} - \frac{2}{\sqrt{7}} m\pi\cos\theta$ and $\beta_2(\theta) = \frac{k\theta}{2} - \frac{1}{\sqrt{7}} m\pi\cos\theta + m\pi$. Then Lemma 3.8 implies that

$$\theta \in \left[\tfrac{\pi}{2}, \tfrac{\pi}{2} + \alpha_7 - \tfrac{70}{53m}\right] \implies |F^1_{7,k,m}(\theta) - 2\cos\beta_1(\theta)| < 2,$$
$$\theta \in \left[\alpha_7 - \tfrac{\pi}{6} + \tfrac{70}{53m}, \tfrac{\pi}{2}\right] \implies |F^2_{7,k,m}(\theta) - 2\cos\beta_2(\theta)| < 2.$$

Therefore if $|\cos\beta_j(\theta)| = 1$ (i.e. $\beta_j(\theta)$ is a multiple of an integer and π), we have

$$\operatorname{sgn}(F^j_{7,k,m}(\theta)) = \operatorname{sgn}(\cos\beta_j(\theta)) \tag{29}$$

for $j = 1, 2$. Moreover, we can check that

$$\beta_1\left(\left[\tfrac{\pi}{2}, \tfrac{\pi}{2} + \alpha_7 - \tfrac{70}{53m}\right]\right) \supset \left[\tfrac{k\pi}{4}, 2n\pi + u_7 - \tfrac{\pi}{3}\right], \tag{30}$$
$$\beta_2\left(\left[\alpha_7 + \tfrac{70}{53m}, \tfrac{\pi}{2}\right]\right) \supset \left[2n\pi - \tfrac{k\pi}{3} + u_7 + \tfrac{\pi}{3}, m\pi + \tfrac{k\pi}{4}\right]. \tag{31}$$

We estimate the number of zeros of $f_{k,m}$ on the arcs for the cases $r_k = 0, 4, 6, 8, 10, 14$ separately.

(i) *The case of* $r_k = 0$ (i.e. $k = 12\ell_k$). By Lemmas 3.3 and 3.5, it is enough to prove that

$$A + B = m + 4\ell_k.$$

By Lemma 3.9, there exist $0 < t_1 < \frac{70}{53}$ and $0 < t_2 < \frac{70}{53}$ such that

$$\operatorname{sgn}\left(F^1_{7,k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{t_1}{m}\right)\right) = \begin{cases} 1 & \text{if } 0 < u_7 < \pi/2 \text{ or } 3\pi/2 < u_7 < 2\pi, \\ -1 & \text{if } \pi/2 < u_7 < 3\pi/2, \end{cases} \tag{32}$$

$$\operatorname{sgn}\left(F^2_{7,k,m}\left(\alpha_7 - \tfrac{\pi}{6} + \tfrac{t_2}{m}\right)\right) = \begin{cases} 1 & \text{if } 0 < u_7 < \pi/2 \text{ or } 3\pi/2 < u_7 < 2\pi, \\ -1 & \text{if } \pi/2 < u_7 < 3\pi/2. \end{cases} \tag{33}$$

Hence (29)–(31), (32), (33), and the intermediate value theorem imply that

$$A \geq \begin{cases} 2n - 3\ell_k & \text{if } 0 < u_7 < \pi/2, \\ 2n - 3\ell_k + 1 & \text{if } \pi/2 < u_7 < 3\pi/2, \\ 2n - 3\ell_k + 2 & \text{if } 3\pi/2 < u_7 < 2\pi, \end{cases}$$

$$B \geq \begin{cases} m + 7\ell_k - 2n & \text{if } 0 < u_7 < \pi/2, \\ m + 7\ell_k - 2n - 1 & \text{if } \pi/2 < u_7 < 3\pi/2, \\ m + 7\ell_k - 2n - 2 & \text{if } 3\pi/2 < u_7 < 2\pi. \end{cases}$$

In each case, we have $A + B \geq m + 4\ell_k$. On the other hand, $A + B \leq m + 4\ell_k$ from Lemma 3.3.

(ii) *The case of* $r_k = 4$ (i.e. $k = 12\ell_k + 4$). By Lemmas 3.3 and 3.5, it is enough to prove that

$$A + B = m + 4\ell_k + 1.$$

By Lemma 3.9, there exist $0 < t_1 < \frac{70}{53}$ and $0 < t_2 < \frac{70}{53}$ such that

$$\begin{aligned}(34)\quad &\operatorname{sgn}\left(F^1_{7,k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{t_1}{m}\right)\right)\\ &= \begin{cases} 1 & \text{if } 0 < u_7 < \pi - \alpha_7 \text{ or } 2\pi - \alpha_7 < u_7 < 2\pi, \\ -1 & \text{if } \pi - \alpha_7 < u_7 < 2\pi - \alpha_7, \end{cases}\end{aligned}$$

$$\begin{aligned}(35)\quad &\operatorname{sgn}\left(F^2_{7,k,m}\left(\alpha_7 - \tfrac{\pi}{6} + \tfrac{t_2}{m}\right)\right)\\ &= \begin{cases} -1 & \text{if } 0 < u_7 < \pi - \alpha_7 \text{ or } 2\pi - \alpha_7 < u_7 < 2\pi, \\ 1 & \text{if } \pi - \alpha_7 < u_7 < 2\pi - \alpha_7. \end{cases}\end{aligned}$$

Hence (29)–(31), (34), (35), and the intermediate value theorem imply that

$$A \geq \begin{cases} 2n - 3\ell_k - 1 & \text{if } 0 < u_7 < \pi - \alpha_7, \\ 2n - 3\ell_k & \text{if } \pi - \alpha_7 < u_7 < 2\pi - \alpha_7, \\ 2n - 3\ell_k + 1 & \text{if } 2\pi - \alpha_7 < u_7 < 2\pi, \end{cases}$$

$$B \geq \begin{cases} m + 7\ell_k - 2n + 2 & \text{if } 0 < u_7 < \pi - \alpha_7, \\ m + 7\ell_k - 2n + 1 & \text{if } \pi - \alpha_7 < u_7 < 2\pi - \alpha_7, \\ m + 7\ell_k - 2n & \text{if } 2\pi - \alpha_7 < u_7 < 2\pi. \end{cases}$$

In each case, we have $A + B \geq m + 4\ell_k + 1$. On the other hand, $A + B \leq m + 4\ell_k + 1$ from Lemmas 3.3 and 3.5.

(iii) *The case of* $r_k = 6$ (i.e. $k = 12\ell_k + 6$). By Lemmas 3.3 and 3.5, it is enough to prove that

$$A + B = m + 4\ell_k + 1.$$

By Lemma 3.9, there exist $0 < t_1 < \frac{70}{53}$ and $0 < t_2 < \frac{70}{53}$ such that

$$(36)\quad \operatorname{sgn}\left(F^1_{7,k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{t_1}{m}\right)\right) = \begin{cases} 1 & \text{if } 0 < u_7 < \pi/2 \text{ or } 3\pi/2 < u_7 < 2\pi, \\ -1 & \text{if } \pi/2 < u_7 < 3\pi/2, \end{cases}$$

$$(37)\quad \operatorname{sgn}\left(F^2_{7,k,m}\left(\alpha_7 - \frac{\pi}{6} + \frac{t_2}{m}\right)\right) = \begin{cases} 1 & \text{if } 0 < u_7 < \pi/2 \text{ or } 3\pi/2 < u_7 < 2\pi, \\ -1 & \text{if } \pi/2 < u_7 < 3\pi/2. \end{cases}$$

Hence (29)–(31), (36), (37), and the intermediate value theorem imply that

$$A \geq \begin{cases} 2n - 3\ell_k - 2 & \text{if } 0 < u_7 < \pi/2, \\ 2n - 3\ell_k - 1 & \text{if } \pi/2 < u_7 < 3\pi/2, \\ 2n - 3\ell_k & \text{if } 3\pi/2 < u_7 < 2\pi, \end{cases}$$

$$B \geq \begin{cases} m + 7\ell_k - 2n + 3 & \text{if } 0 < u_7 < \pi/2, \\ m + 7\ell_k - 2n + 2 & \text{if } \pi/2 < u_7 < 3\pi/2, \\ m + 7\ell_k - 2n + 1 & \text{if } 3\pi/2 < u_7 < 2\pi. \end{cases}$$

In each case, we have $A+B \geq m+4\ell_k+1$. On the other hand, $A+B \leq m+4\ell_k+1$ from Lemmas 3.3 and 3.5.

(iv) *The case of* $r_k=8$ (i.e. $k=12\ell_k+8$). By Lemmas 3.3 and 3.5, it is enough to prove that

$$A+B=m+4\ell_k+2.$$

By Lemma 3.9, there exist $0<t_1<\frac{70}{53}$ and $0<t_2<\frac{70}{53}$ such that

$$\begin{aligned}(38)\quad &\operatorname{sgn}\bigl(F^1_{7,k,m}\bigl(\tfrac{\pi}{2}+\alpha_7-\tfrac{t_1}{m}\bigr)\bigr)\\ &\qquad=\begin{cases}1 & \text{if } 0<u_7<3\pi/2-2\alpha_7 \text{ or } 5\pi/2-2\alpha_7<u_7<2\pi,\\ -1 & \text{if } 3\pi/2-2\alpha_7<u_7<5\pi/2-2\alpha_7,\end{cases}\end{aligned}$$

$$\begin{aligned}(39)\quad &\operatorname{sgn}\bigl(F^2_{7,k,m}\bigl(\alpha_7-\tfrac{\pi}{6}+\tfrac{t_2}{m}\bigr)\bigr)\\ &\qquad=\begin{cases}1 & \text{if } 0<u_7<3\pi/2-2\alpha_7 \text{ or } 5\pi/2-2\alpha_7<u_7<2\pi,\\ -1 & \text{if } 3\pi/2-2\alpha_7<u_7<5\pi/2-2\alpha_7.\end{cases}\end{aligned}$$

Hence (29)–(31), (38), (39), and the intermediate value theorem imply that

$$A\geq\begin{cases}2n-3\ell_k-2 & \text{if } 0<u_7<3\pi/2-2\alpha_7,\\ 2n-3\ell_k-1 & \text{if } 3\pi/2-2\alpha_7<u_7<5\pi/2-2\alpha_7,\\ 2n-3\ell_k & \text{if } 5\pi/2-2\alpha_7<u_7<2\pi,\end{cases}$$

$$B\geq\begin{cases}m+7\ell_k-2n+4 & \text{if } 0<u_7<3\pi/2-2\alpha_7,,\\ m+7\ell_k-2n+3 & \text{if } 3\pi/2-2\alpha_7<u_7<5\pi/2-2\alpha_7,\\ m+7\ell_k-2n+2 & \text{if } 5\pi/2-2\alpha_7<u_7<2\pi.\end{cases}$$

In each case, we have $A+B \geq m+4\ell_k+2$. On the other hand, $A+B \leq m+4\ell_k+2$ from Lemmas 3.3 and 3.5.

(v) *The case of* $r_k=10$ (i.e. $k=12\ell_k+10$). By Lemmas 3.3 and 3.5, it is enough to prove that

$$A+B=m+4\ell_k+2.$$

By Lemma 3.9, there exist $0<t_1<\frac{70}{53}$ and $0<t_2<\frac{70}{53}$ such that

$$\begin{aligned}(40)\quad &\operatorname{sgn}\bigl(F^1_{7,k,m}\bigl(\tfrac{\pi}{2}+\alpha_7-\tfrac{t_1}{m}\bigr)\bigr)\\ &\qquad=\begin{cases}1 & \text{if } 0<u_7<\pi-\alpha_7 \text{ or } 2\pi-\alpha_7<u_7<2\pi,\\ -1 & \text{if } \pi-\alpha_7<u_7<2\pi-\alpha_7,\end{cases}\end{aligned}$$

$$\begin{aligned}(41)\quad &\operatorname{sgn}\bigl(F^2_{7,k,m}\bigl(\alpha_7-\tfrac{\pi}{6}+\tfrac{t_2}{m}\bigr)\bigr)\\ &\qquad=\begin{cases}-1 & \text{if } 0<u_7<\pi-\alpha_7 \text{ or } 2\pi-\alpha_7<u_7<2\pi,\\ 1 & \text{if } \pi-\alpha_7<u_7<2\pi-\alpha_7.\end{cases}\end{aligned}$$

Hence (29)–(31), (40), (41), and the intermediate value theorem imply that

$$A \geq \begin{cases} 2n - 3\ell_k - 3 & \text{if } 0 < u_7 < \pi - \alpha_7, \\ 2n - 3\ell_k - 2 & \text{if } \pi - \alpha_7 < u_7 < 2\pi - \alpha_7, \\ 2n - 3\ell_k - 1 & \text{if } 2\pi - \alpha_7 < u_7 < 2\pi, \end{cases}$$

$$B \geq \begin{cases} m + 7\ell_k - 2n + 5 & \text{if } 0 < u_7 < \pi - \alpha_7, \\ m + 7\ell_k - 2n + 4 & \text{if } \pi - \alpha_7 < u_7 < 2\pi - \alpha_7, \\ m + 7\ell_k - 2n + 3 & \text{if } 2\pi - \alpha_7 < u_7 < 2\pi. \end{cases}$$

In each case, we have $A + B \geq m + 4\ell_k + 2$. On the other hand, $A + B \leq m + 4\ell_k + 2$ from Lemmas 3.3 and 3.5.

(vi) *The case of* $r_k = 14$ (i.e. $k = 12\ell_k + 14$). By Lemmas 3.3 and 3.5, it is enough to prove that

$$A + B = m + 4\ell_k + 3.$$

By Lemma 3.9, there exist $0 < t_1 < \frac{70}{53}$ and $0 < t_2 < \frac{70}{53}$ such that

$$\text{(42)} \quad \operatorname{sgn}\left(F^1_{7,k,m}\left(\tfrac{\pi}{2} + \alpha_7 - \tfrac{t_1}{m}\right)\right) = \begin{cases} 1 & \text{if } 0 < u_7 < 3\pi/2 - 2\alpha_7 \text{ or } 5\pi/2 - 2\alpha_7 < u_7 < 2\pi, \\ -1 & \text{if } 3\pi/2 - 2\alpha_7 < u_7 < 5\pi/2 - 2\alpha_7, \end{cases}$$

$$\text{(43)} \quad \operatorname{sgn}\left(F^2_{7,k,m}\left(\alpha_7 - \tfrac{\pi}{6} + \tfrac{t_2}{m}\right)\right) = \begin{cases} 1 & \text{if } 0 < u_7 < 3\pi/2 - 2\alpha_7 \text{ or } 5\pi/2 - 2\alpha_7 < u_7 < 2\pi, \\ -1 & \text{if } 3\pi/2 - 2\alpha_7 < u_7 < 5\pi/2 - 2\alpha_7. \end{cases}$$

Hence (29)–(31), (42), (43), and the intermediate value theorem imply that

$$A \geq \begin{cases} 2n - 3\ell_k - 4 & \text{if } 0 < u_7 < 3\pi/2 - 2\alpha_7, \\ 2n - 3\ell_k - 3 & \text{if } 3\pi/2 - 2\alpha_7 < u_7 < 5\pi/2 - 2\alpha_7, \\ 2n - 3\ell_k - 2 & \text{if } 5\pi/2 - 2\alpha_7 < u_7 < 2\pi, \end{cases}$$

$$B \geq \begin{cases} m + 7\ell_k - 2n + 7 & \text{if } 0 < u_7 < 3\pi/2 - 2\alpha_7, \\ m + 7\ell_k - 2n + 6 & \text{if } 3\pi/2 - 2\alpha_7 < u_7 < 5\pi/2 - 2\alpha_7, \\ m + 5\ell_k - 2n + 5 & \text{if } 5\pi/2 - 2\alpha_7 < u_7 < 2\pi. \end{cases}$$

In each case, we have $A + B \geq m + 4\ell_k + 3$. On the other hand, $A + B \leq m + 4\ell_k + 3$ from Lemmas 3.3 and 3.5. ∎

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