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Relations between Reeb graphs, systems of hypersurfaces and epimorphisms onto free groups

by

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Abstract. We construct a correspondence between epimorphisms $\varphi\colon \pi_1(M) \to F_r$ from the fundamental group of a compact manifold M onto the free group of rank r, and systems of r framed non-separating hypersurfaces in M, which induces a bijection onto framed cobordism classes of such systems. In consequence, for closed manifolds any such φ can be represented by the Reeb epimorphism of a Morse function $f\colon M \to \mathbb{R}$, i.e. by the epimorphism induced by the quotient map $M \to \mathcal{R}(f)$ onto the Reeb graph of f. Applying this construction we discuss the problem of classification up to (strong) equivalence of epimorphisms onto free groups, providing a new purely geometrical-topological proof of the solution of this problem for surface groups.

1. Introduction. The Reeb graph $\mathcal{R}(f)$ of a Morse function $f\colon M \to \mathbb{R}$ on a closed manifold M, as an invariant of the pair (M, f), is a tool of global analysis attracting more attention recently due to its applications to computer graphics as well as its importance in purely mathematical problems (for more details see [3, 6, 8, 9, 16, 19, 22, 32]). The graph $\mathcal{R}(f)$ is constructed by contracting the connected components of level sets of the function f. Since it is a finite graph, its fundamental group is a free group F_r of a finite rank $r \geq 0$. This work is motivated by a natural question: is any epimorphism $\pi_1(M) \to F_r$ represented as the canonical epimorphism $q_\#\colon \pi_1(M) \to \pi_1(\mathcal{R}(f))$, induced by the quotient map $q\colon M \to \mathcal{R}(f)$ for a Morse function f? The epimorphism $q_\#$ is called the Reeb epimorphism of f. We give an affirmative answer to this question in Theorem 4.17. Below we summarize the main results obtained in this work and the methods used.

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One of the main ingredients in the proof is the correspondence, given by an extended Pontryagin–Thom construction, between homomorphisms $\pi_1(W) \to F_r$ and systems of hypersurfaces $\mathcal{N} = (N_1, \ldots, N_r)$ consisting of framed and properly embedded submanifolds N_i of codimension 1 in a compact manifold W, possibly with boundary. A system $\mathcal{N}$ is *independent* if it is non-separating, and it is *regular* if each N_i is connected. It is an easy observation that an independent system of hypersurfaces induces a surjective homomorphism $\pi_1(W) \to F_r$. The converse is the first substantial result of this work (Theorem 2.10). It provides, for any epimorphism $\varphi\colon \pi_1(W) \to F_r$, the construction of a regular and independent system of hypersurfaces which induces φ.

Having these geometric tools, we study the problem of classification of epimorphisms $G \to F_r$ up to equivalence and strong equivalence defined in [10, 11, 12]. Briefly, on the set $\mathrm{Hom}(G, F_r)$ of homomorphisms there are the natural actions of the automorphism groups $\mathrm{Aut}(G)$ and $\mathrm{Aut}(F_r)$ by composition. Two homomorphisms are *strongly equivalent* (resp. *equivalent*) if they are in the same orbit of the action of $\mathrm{Aut}(G)$ (resp. $\mathrm{Aut}(G) \times \mathrm{Aut}(F_r)$). First, note that two systems induce the same homomorphism if and only if they are framed cobordant as systems of hypersurfaces (see Definition 2.3). This leads to a correspondence between strong equivalence classes of epimorphisms $\pi_1(M) \to F_r$ and elements of $\mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M)$, the set of framed cobordism classes of independent and regular systems of size r in M up to diffeomorphisms which preserve the basepoint. It is a one-to-one correspondence if the natural homomorphism $\mathrm{Diff}_\bullet(M) \to \mathrm{Aut}(\pi_1(M))$ is surjective. For example, this holds when M is a closed surface (by the Dehn–Nielsen Theorem) or when M is a hyperbolic manifold of dimension at least 3 (by the Mostow Rigidity Theorem). As an application of the methods we develop, in the proof of Theorem 3.15 we determine the elements of $\mathcal{H}_r^{\mathrm{fr}}(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)$ for a closed surface Σ. This provides a classification up to strong equivalence of epimorphisms $\pi_1(\Sigma) \to F_r$, which was originally shown by R. Grigorchuk, P. Kurchanov and H. Zieschang [10, 11, 12] by using more algebraic, but also topological methods (see Theorem 3.1).

Transition from strong equivalence classes to equivalence classes is obtained by considering the action of $\mathrm{Aut}(F_r)$, which is generated by elementary Nielsen transformations. We define analogous operations on $\mathcal{H}_r^{\mathrm{fr}}(M)$ which cause the same change of an inducing epimorphism as its composition with the corresponding Nielsen transformation. These operations allow us to compute equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$ (see Theorem 3.20) as in the Grigorchuk–Kurchanov–Zieschang Theorem.

Next, we exhibit relations to Reeb graph theory. Extending the methods of the second author from [23] we assign in Theorem 4.6 a Morse function f on W and its Reeb graph to any system of hypersurfaces without boundary in such a way that the induced homomorphism factorizes through the Reeb epimorphism of f. Moreover, if the system is independent, this gives the construction of the initial graph (see Figure 4) as the Reeb graph such that submanifolds from the system are components of the same level set of f. Subsequently, one of the main results of the paper, Theorem 4.15, provides, for a regular and independent system $\mathcal{N}$ of hypersurfaces and a graph Γ with natural necessary conditions, the construction of a Morse function realizing Γ as its Reeb graph such that submanifolds from $\mathcal{N}$ correspond to prescribed edges of Γ outside a spanning tree.

Prescribed components of level sets are an additional ingredient to the realization theorems for Reeb graphs. The classical result of V. Sharko [33] provides a realization of any graph with the so-called good orientation as the Reeb graph of a function on a surface. Recently, the second author [22, 23] resolved the realization problem with an arbitrary fixed closed manifold. In the case of surfaces the realization is up to isomorphism of graphs with a detailed description of Reeb graphs of Morse functions. For higher-dimensional manifolds it is up to homeomorphism, and the construction relies on combinatorial modifications of Reeb graphs. It is known that any graph Γ with good orientation is obtained from the initial graph by using a finite sequence of combinatorial modifications. In this work, we extend these results to the situation when the manifold W has a boundary and one can prescribe connected components of level sets of the function corresponding to edges of the graph outside a spanning tree.

The principal significance of Theorem 4.15 is that it allows one to represent any epimorphism

$$\varphi\colon \pi_1(M) \to \pi_1(\Gamma)$$

as the Reeb epimorphism of a Morse function whose Reeb graph is homeomorphic to Γ (Corollary 4.18). Theorems 4.14 and 4.17 provide an answer to the initial question for a manifold W with boundary. An epimorphism $\pi_1(W) \to \pi_1(\Gamma)$ is represented as the Reeb epimorphism if and only if it is induced by a system of hypersurfaces without boundary. Equivalently, it factorizes through $\pi_1(W)/\langle\pi_1(\partial W)\rangle^{\pi_1(W)}$, where $\langle\pi_1(\partial W)\rangle^{\pi_1(W)}$ is the smallest normal subgroup of $\pi_1(W)$ containing the classes of all loops from ∂W.

Note that the problem of representability of an epimorphism as the Reeb epimorphism was also considered independently by O. Saeki [30]; for a finite graph Γ without loops and a closed manifold he constructs a smooth function with finitely many critical values such that its Reeb graph is isomorphic to Γ and under this identification its Reeb epimorphism is φ. Thus Saeki realizes not only the topological structure of Γ, but also the combinatorial one, at the cost of losing the non-degeneracy of critical points. Note that the number of vertices of degree 2 in the Reeb graph of a Morse function cannot be arbitrary (see, for instance [22, Theorem 5.6]), and thus we focus on the homeomorphism type. Our results are also different in that they deal with manifolds with boundary and allow us to control the system of hypersurfaces in connected components of level sets of the constructed function.

Another subject of this paper is the maximum values of some related quantities. The corank of a finitely generated group G is the maximum rank r for which there exists an epimorphism $G \to F_r$. As defined in [22] for closed manifolds, the Reeb number $\mathcal{R}(W)$ of W is the maximum cycle rank of Reeb graphs of Morse functions $f\colon W \to \mathbb{R}$ which are constant on each connected component of ∂W. In other words, $\mathcal{R}(W)$ is the maximum rank of the Reeb epimorphism of such a Morse function on W. For closed manifolds we have $\mathcal{R}(M) = \operatorname{corank}(\pi_1(M))$ (see [23, 9]). In Theorem 4.14 we establish the corresponding formula for manifolds with boundary:

$$\mathcal{R}(W) = \operatorname{corank}(\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}).$$

$\mathcal{R}(W)$ is also equal to the maximum size of an independent system of hypersurfaces without boundary in W. The last quantity we consider is the maximum size of an independent system of hypersurfaces in W, which was denoted by $C(W)$ by O. Cornea [5]. It is always equal to the corank of $\pi_1(W)$.

Relations between these numbers have already been studied by other authors. The equality $C(W) = \operatorname{corank}(\pi_1(W))$ was established by O. Cornea [5] for closed smooth manifolds and by W. Jaco [15] for combinatorial manifolds with boundary. The equality $R(M) = \operatorname{corank}(\pi_1(M))$ was proved by the second author [23] and independently by I. Gelbukh [8] for orientable manifolds by using foliation theory and later in [9] without the orientability assumption by other methods. It is worth emphasizing that, while these papers contain geometric descriptions of the corank of $\pi_1(M)$, no correspondence between epimorphisms, systems of hypersurfaces and Reeb graphs was given. This work fills this gap.

The paper is organized as follows. In Section 2 we describe the correspondence between systems of hypersurfaces and homomorphisms onto free groups. Next, in Section 3 we deal with the problem of classification of epimorphisms onto free groups up to equivalence and strong equivalence. Section 4 establishes the representation of epimorphisms onto free groups as the Reeb epimorphisms of Morse functions. The rest of the section is devoted to some applications concerning description of the corank and Reeb number, the problem of extendability of independent systems and connections with topological conjugacy of functions.

2. Systems of hypersurfaces and induced homomorphisms. We assume that all manifolds considered are smooth of dimension $n \geq 2$. Hereafter, M and W are connected and compact smooth manifolds with fixed basepoints and M is closed, unless otherwise stated.

We use the following model of F_r, the free group on r generators. Consider the circle S^1 as the quotient $[-1,1]/\{-1,1\}$ and take $F_r := \pi_1(\bigvee_{i=1}^r \mathrm{S}^1_i)$, the fundamental group of the wedge product of $r \geq 1$ copies of the circle. By convention, $\bigvee_{i=1}^0 \mathrm{S}^1_i = \mathrm{pt}$, thus $F_0 = 1$ is the trivial group.

We will omit basepoints from the notation.

2.1. Systems of hypersurfaces. Let W be a compact manifold. A submanifold N of W is called *proper* if $N \cap \partial W = \partial N$. A *framing* of a submanifold N in W is a smooth function ν which assigns to each $x \in N$ a basis of the normal bundle of N at x. The pair (N, ν) is called a *framed submanifold.* If N is of codimension 1, then its framing is just a nonzero section of the normal bundle of N. Thus N has a closed product neighbourhood $P(N) \cong N \times [-1,1]$ and it is called 2*-sided.* We assume that $P(N)$ is compatible with the framing. Denote by $P_t(N)$ the submanifold corresponding to $N \times \{t\}$. The positive side of N containing $P_t(N)$ for $t \in (0,1]$ agrees with the side determined by the framing.

A *system of hypersurfaces* in W is a tuple $\mathcal{N} = (N_1, \dots, N_r)$ of disjoint, proper, 2-sided submanifolds N_i together with their framings ν_i. The number r is called the *size* of the system $\mathcal{N}$. Denote by

$$W|\mathcal{N} := W \setminus \bigcup_{i=1}^r \operatorname{Int} P(N_i)$$

the complement of the system $\mathcal{N}$ for sufficiently small product neighbourhoods of N_i's. It will cause no confusion if we use $\mathcal{N}$ to designate also $\bigcup_{i=1}^r N_i$, the union of all submanifolds from the system. Of course, the framings ν_i of the submanifolds N_i form a framing ν of $\mathcal{N}$ such that $\nu|_{N_i} = \nu_i$. Unless it is necessary, we will not write a framing of a system explicitly.

A system $\mathcal{N}$ is called *independent* if $W|\mathcal{N}$ is connected, and it is called *regular* if each N_i is connected. The system $\mathcal{N}$ is *without boundary* if $\partial\mathcal{N} = \varnothing$. Note that we do not assume that the submanifolds N_i are connected, unless $\mathcal{N}$ is regular.

Now we define the extended Pontryagin–Thom construction for a system of hypersurfaces.

Definition 2.1. The *homomorphism* $\varphi_{\mathcal{N}} \colon \pi_1(W) \to F_r$ *induced by a system* $\mathcal{N} = (N_1, \dots, N_r)$ omitting the basepoint is defined as follows. Fix product neighbourhoods $P(N_i) \cong N_i \times [-1,1]$ which are disjoint. We define the map

$$f_{\mathcal{N}} \colon W \to \bigvee_{i=1}^r \mathrm{S}^1_i$$

which maps $W|\mathcal{N}$ to the basepoint and each $P(N_i)$ onto the ith circle $\mathrm{S}^1_i = [-1,1]/\{-1,1\}$ by mapping $P_t(N_i)$ to t. It is clear that $f_{\mathcal{N}}$ is continuous, so let $\varphi_{\mathcal{N}} := (f_{\mathcal{N}})_{\#}$ be the homomorphism induced by $f_{\mathcal{N}}$ on the fundamental groups.

By the definition of a system of hypersurfaces, $\varphi_{\mathcal{N}}$ is well-defined and it is clear that it does not depend on the choice of $P(N_i)$'s and a given framing, but on the orientation of the normal bundle of $\mathcal{N}$.

Proposition 2.2. *Any homomorphism $\varphi\colon \pi_1(W) \to F_r$ is induced by a system of hypersurfaces. If a system $\mathcal{N}$ is independent, then $\varphi_{\mathcal{N}}$ is an epimorphism.*

Proof. Since $\bigvee_{i=1}^r \mathrm{S}_i^1$ is an Eilenberg–MacLane space $\mathrm{K}(F_r,1)$, there is a map $f\colon W \to \bigvee_{i=1}^r \mathrm{S}_i^1$ such that $f_{\#} = \varphi$. Smooth it outside the inverse image of the basepoint and take regular values $a_i \in \mathrm{S}_i^1$ of both f and $f|_{\partial W}$. Since W is compact, there is a neighbourhood $[a_i - \varepsilon, a_i + \varepsilon]$ consisting of regular values, and thus $N_i := f^{-1}(a_i)$ is a 2-sided, proper submanifold with a product neighbourhood $f^{-1}([a_i - \varepsilon, a_i + \varepsilon]) \cong N_i \times [a_i - \varepsilon, a_i + \varepsilon]$. Take the map $h\colon \bigvee_{i=1}^r \mathrm{S}_i^1 \to \bigvee_{i=1}^r \mathrm{S}_i^1$ which contracts

$$\bigvee_{i=1}^{r} \mathrm{S}_i^1 \setminus \bigcup_{i=1}^{r} [a_i - \varepsilon, a_i + \varepsilon]$$

to the basepoint and maps linearly $[a_i - \varepsilon, a_i + \varepsilon]$ onto S_i^1, preserving orientation. It is clear that $(h \circ f)_{\#} = \varphi$ is induced by $\mathcal{N} = (N_1, \ldots, N_r)$ with framings compatible with the orientations of $[a_i - \varepsilon, a_i + \varepsilon]$.

If $\mathcal{N}$ is independent, then for any i there is a loop α_i in $(W|\mathcal{N}) \cup P(N_i)$ such that $f_{\mathcal{N}} \circ \alpha_i$ represents the generator of $\pi_1(\bigvee \mathrm{S}_i^1)$ corresponding to S_i^1. Thus $\varphi_{\mathcal{N}}$ is surjective. ∎

There is a quite easy characterization, using a special notion of framed cobordism, of systems in a closed manifold M which induce the same homomorphism to a free group.

Recall (cf. [25]) that submanifolds N and N' in M are *cobordant* if there exists a proper compact submanifold $W \subset M \times [0,1]$, called a *cobordism* between N and N', and $\epsilon \in (0,1)$ such that $W \cap (M \times [0,\varepsilon]) = N \times [0,\varepsilon]$ and $W \cap (M \times [1-\varepsilon,1]) = N' \times [1-\varepsilon,1]$. Framed submanifolds (N,ν) and (N',ν') are *framed cobordant* if there is a cobordism $W \subset M \times [0,1]$ between N and N' with a framing ϑ such that $\vartheta(x,t) = (\nu(x),0)$ for $(x,t) \in N \times [0,\varepsilon]$ and $\vartheta(x,t) = (\nu'(x),0)$ for $(x,t) \in N' \times [1-\varepsilon,1]$.

Definition 2.3. Let $\mathcal{N} = (N_1, \ldots, N_r)$ and $\mathcal{N}' = (N_1', \ldots, N_r')$ be two systems in M of the same size r. We say that $\mathcal{N}$ and $\mathcal{N}'$ are *framed cobordant* (as systems of hypersurfaces) if there are r disjoint framed cobordisms $W_i \subset M \times [0,1]$ between N_i and N_i'.

In other words, the systems $\mathcal{N}$ and $\mathcal{N}'$ are framed cobordant if the framed submanifolds $\mathcal{N}$ and $\mathcal{N}'$ are framed cobordant by the cobordism W which has a partition $W = W_1 \sqcup \dots \sqcup W_r$ such that $\partial W_i = N_i \times \{0\} \sqcup N_i' \times \{1\}$. Clearly, this is an equivalence relation in the family of systems of hypersurfaces in M of size r. Note that the cobordisms W_i form the system $\mathcal{W} = (W_1, \dots, W_r)$ of hypersurfaces in $M \times [0,1]$.

Note that the notion of framed cobordism between systems of hypersurfaces of size 1 is the same as ordinary framed cobordism.

Proposition 2.4. *Systems $\mathcal{N}$ and $\mathcal{N}'$ of hypersurfaces in M are framed cobordant if and only if $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$.*

Proof. If the systems $\mathcal{N}$ and $\mathcal{N}'$ are framed cobordant by framed cobordisms $W_1, \dots, W_r$ which form the system $\mathcal{W}$, then as in Definition 2.1 this leads to a map $f_{\mathcal{W}} \colon M \times [0,1] \to \bigvee_{i=1}^r \mathrm{S}_i^1$ for a fixed product neighbourhood $P(\mathcal{W})$. It is clear that $f_{\mathcal{W}}|_{M \times \{0\}} = f_{\mathcal{N}}$ and $f_{\mathcal{W}}|_{M \times \{1\}} = f_{\mathcal{N}'}$ for product neighbourhoods $P(\mathcal{N}) = P(\mathcal{W}) \cap M \times \{0\}$ and $P(\mathcal{N}') = P(\mathcal{W}) \cap M \times \{1\}$, respectively. Thus $f_{\mathcal{W}}$ is a homotopy between $f_{\mathcal{N}}$ and $f_{\mathcal{N}'}$, so $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$.

Conversely, if $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$, then $f_{\mathcal{N}}$ and $f_{\mathcal{N}'}$ are homotopic by a map $f \colon M \times [0,1] \to \bigvee_{i=1}^r \mathrm{S}_i^1$ which is smooth outside the preimage of the basepoint since $\bigvee_{i=1}^r \mathrm{S}_i^1$ is an Eilenberg–MacLane space $\mathrm{K}(F_r,1)$. As in the proof of Proposition 2.2, take regular values $a_i \in \mathrm{S}_i^1$ and framed submanifolds $W_i = f^{-1}(a_i)$ which form a system of hypersurfaces in $M \times [0,1]$. They are framed cobordisms between $f_{\mathcal{N}}^{-1}(a_i) \cong N_i$ and $f_{\mathcal{N}'}^{-1}(a_i) \cong N_i'$. By the construction of $f_{\mathcal{N}}$ and $f_{\mathcal{N}'}$ it is clear that the system $(f_{\mathcal{N}}^{-1}(a_1), \dots, f_{\mathcal{N}}^{-1}(a_r))$ is framed cobordant to $\mathcal{N}$ and $(f_{\mathcal{N}'}^{-1}(a_1), \dots, f_{\mathcal{N}'}^{-1}(a_r))$ is framed cobordant to $\mathcal{N}'$. The statement follows by transitivity of framed cobordism. ∎

Remark 2.5. It is easy to check that if two systems of hypersurfaces differ only in their framings, but the positive sides are the same, then they are framed cobordant. Thus the induced homomorphism depends only on the choice of sides of submanifolds from the system, not on particular framings.

2.2. Epimorphisms and independence of inducing systems. The aim of this section is to prove that any epimorphism onto a free group is induced by an independent and regular system.

Let $\mathcal{N} = (N_1, \dots, N_r)$ be a system of hypersurfaces in a compact and connected manifold W. Note that any class of loops in W can be represented by a loop in $\operatorname{Int} W$.

Lemma 2.6. *For any class of loops $\omega \in \pi_1(W)$, either w can be represented by a loop in $W|\mathcal{N}$ or there is a loop $\alpha \in \omega$ which can be written as the concatenation of paths $\alpha_1 \cdot \ldots \cdot \alpha_k$ whose ends lie in $W|\mathcal{N}$ and $\alpha_i \cap P_t(\mathcal{N})$ is a single point for any $t \in [-1,1]$. Thus writing $a_i := [\mathrm{S}_i^1]$ for the generators of $F_r = \pi_1(\bigvee_{i=1}^r \mathrm{S}_i^1)$ we have $\varphi_{\mathcal{N}}(\omega) = a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$, where $\epsilon_j \in \{-1,+1\}$ and i_j is the unique index for which $\alpha_j \cap N_{i_j}$ is nonempty.*

Proof. Take any loop α in ω which is in general position relative to $\mathcal{N}$. Then the intersection of α and $\mathcal{N}$ is a finite set. Now, cut α into paths α_i as required. ∎

LEMMA 2.7. *Suppose there is a path $\gamma\colon [0,1] \to W$ such that $\gamma \cap \mathcal{N} = \gamma \cap N_j = \{x, y\}$, where $x = \gamma(0) \in X$ and $y = \gamma(1) \in Y$ are in the different connected components X and Y of N_j, and which joins x and y from the same side, i.e. $\gamma \cap P_t(N_j) = \varnothing$ for any $t \in [-1, 0)$ or for any $t \in (0, 1]$. Then there is a system $\mathcal{N}' = (N_1', \dots, N_r')$ such that $N_i = N_i'$ for $i \neq j$, N_j' has one connected component less than N_j and $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$.*

Proof. First, change $\gamma\colon [0,1] \to W$ to an embedded arc in $\operatorname{Int} W$ with the same properties as in the statement. Take a small, closed tubular neighbourhood $P(\gamma)$ of γ parametrized by $\gamma \times \mathrm{D}_3^{n-1}$ such that $P(\gamma) \cap \mathcal{N} = P(\gamma) \cap N_j$, where $\mathrm{D}_t^{n-1} = \{x \in \mathbb{R}^{n-1} \ : \ ||x|| \leq t\}$ is a closed disc of radius t. We may assume that $P(\gamma) \cap X = \{x\} \times \mathrm{D}_3^{n-1}$ and $P(\gamma) \cap Y = \{y\} \times \mathrm{D}_3^{n-1}$. Now, take the connected sum of X and Y along γ in W, i.e. define

$$A = X \#_\gamma Y := \big(X \setminus (\{x\} \times \mathrm{D}_2^{n-1})\big) \cup (\gamma \times \partial\, \mathrm{D}_2^{n-1}) \cup \big(Y \setminus (\{y\} \times \mathrm{D}_2^{n-1})\big).$$

Obviously, A is a topological manifold, smoothly embedded outside $\{x, y\} \times \partial\, \mathrm{D}_2^{n-1}$. Take an open ε-neighbourhood U of $\{x, y\} \times \partial\, \mathrm{D}_2^{n-1}$ and smooth the corners inside U. Hence we may assume that A is a 2-sided smooth submanifold of W with a product neighbourhood $P(A)$ such that

$$\begin{aligned} P(A \setminus U) = {} & P\big(X \cup Y \setminus (\{x, y\} \times \mathrm{D}_2^{n-1}) \setminus U\big) \\ & \cup \big(\gamma([\varepsilon, 1-\varepsilon]) \times (\mathrm{D}_3^{n-1} \setminus \operatorname{Int} \mathrm{D}_1^{n-1})\big). \end{aligned}$$

Since γ joins X and Y from the same side, the orientations of their normal bundles induce the orientation of $P(A)$, and thus a framing of A.

Let $\mathcal{N}' = (N_1', \dots, N_r')$ be a system such that $N_i = N_i'$ for $i \neq j$ and $N_j' = (N_j \setminus (X \cup Y)) \cup A$. We will show that $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$. Let $[\alpha] \in \pi_1(W)$ be any class of loops in W with basepoint outside $P(\mathcal{N})$ and $P(\gamma)$. We may assume that α does not intersect $(\{x, y\} \times \mathrm{D}_2^{n-1}) \cup U$ and it is in general position relative to $\mathcal{N}'$. Write $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_k$ as in Lemma 2.6 with respect to the system $\mathcal{N}'$, so $\varphi_{\mathcal{N}'}([\alpha]) = a_{i_1}^{\epsilon_1} \dots a_{i_k}^{\epsilon_k}$. Note that $\varphi_{\mathcal{N}}([\alpha])$ is obtained from $\varphi_{\mathcal{N}'}([\alpha]) = a_{i_1}^{\epsilon_1} \dots a_{i_k}^{\epsilon_k}$ by removing those $a_{i_j}^{\epsilon_j}$ which correspond to α_j such that $\alpha_j \cap (\gamma \times \partial\, \mathrm{D}_2^{n-1}) \neq \varnothing$. However, if α_j intersects $\gamma \times \partial\, \mathrm{D}_2^{n-1}$ and goes inside $\gamma \times \mathrm{D}_2^{n-1}$ (i.e. it has an end point in $\gamma \times \mathrm{D}_2^{n-1}$), then α_{j+1} also intersects $\gamma \times \partial\, \mathrm{D}_2^{n-1}$, since it needs to leave $\gamma \times \mathrm{D}_2^{n-1}$ and does not intersect $\{x, y\} \times \mathrm{D}_2^{n-1}$. Thus $a_{i_j} = a_{i_{j+1}}$ and $\epsilon_{j+1} = -\epsilon_j$. Therefore $\varphi_{\mathcal{N}}([\alpha]) = \varphi_{\mathcal{N}'}([\alpha])$, so $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$. ∎

We call the constructed submanifold $X \#_\gamma Y$ the *connected sum* of X and Y *along* γ.

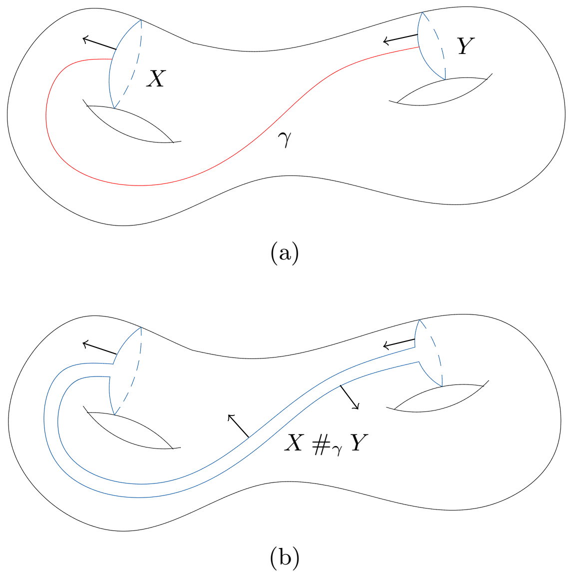

Fig. 1. An example of a connected sum of submanifolds X and Y along a curve γ which joins them from the same side. The arrows indicate normal vectors from the framing.

PROPOSITION 2.8. *Let $\mathcal{N} = (N_1, \ldots, N_r)$ be a system of hypersurfaces in W such that $\varphi_{\mathcal{N}}$ is an epimorphism and there are no paths as in the statement of Lemma 2.7. Then there is a unique independent and regular system $\mathcal{A} = (A_1, \ldots, A_r)$ in W such that for each j the submanifold A_j is a component of N_j and there is a loop τ_j such that $\tau_j \cap \mathcal{N} = \tau_j \cap A_j$ is a single point. In particular, if $\mathcal{N}$ is regular, then it is independent.*

Proof. Since $\varphi_{\mathcal{N}}$ is an epimorphism, for any j there is a loop τ_j in W such that $f_{\mathcal{N}} \circ \tau_j$ represents the generator of $F_r = \pi_1(\bigvee_{i=1}^r \mathrm{S}^1_i)$ which corresponds to S^1_j. As in Lemma 2.6, we may consider τ_j as the concatenation of paths $\alpha^j_1, \ldots, \alpha^j_k$ such that $a_j = \varphi_{\mathcal{N}}([\tau_j]) = a^{\epsilon_1}_{i_1} \ldots a^{\epsilon_k}_{i_k}$, where $a_i = [\mathrm{S}^1_i]$. If $k > 1$, then there is some cancellation in the word $a^{\epsilon_1}_{i_1} \ldots a^{\epsilon_k}_{i_k}$, so for some l both α^j_l and α^j_{l+1} intersect the same submanifold N_{i_l}. If they intersect two different components of N_{i_l}, one obtains a path as in the statement of Lemma 2.7, a contradiction. However, if they intersect N_{i_l} in the same connected component X, then we may assume that the starting point of α^j_l and the endpoint of α^j_{l+1} are in $P_t(X)$ for some $t \in [-2,2] \setminus [-1,1]$ by reparameterizing $P(N_{i_l})$. Since X is connected, we may replace the paths α^j_l and α^j_{l+1} in τ_j by an arc in $P_t(X)$ joining these two points, which provides a loop for which the number of paths in its representation from Lemma 2.6 is reduced. Proceeding inductively we may assume that $\tau_j \cap \mathcal{N} = \tau_j \cap N_j$ is a single point.

Note that if there were two components X and Y of N_j with loops τ_X and τ_Y with the same basepoint intersecting $\mathcal{N}$ only in single points of X and Y, respectively, then they would determine a path joining X and Y as in Lemma 2.7. Thus for any j there is a unique connected component A_j of N_j with this property.

The system $\mathcal{A} = (A_1, \ldots, A_r)$ is regular by definition and independent by the above property of A_j's. The uniqueness of $\mathcal{A}$ follows by the uniqueness of its components.

If $\mathcal{N}$ is regular, then $\mathcal{N} = \mathcal{A}$, so it is independent. ∎

Remark 2.9. Using the techniques of Cornea [5] one can show that for a closed manifold M, if $\mathcal{N}$ is not regular and $\varphi_{\mathcal{N}}$ is surjective, then there is an independent and regular system $\mathcal{N}' = (N_1', \ldots, N_r')$ in M such that $\mathcal{N}' \subset \mathcal{N}$, without the assumption on the existence of paths as in Lemma 2.7.

Theorem 2.10. *Any epimorphism $\varphi\colon \pi_1(W) \to F_r$ is induced by a regular and independent system of hypersurfaces.*

Proof. Let $\mathcal{N} = (N_1, \ldots, N_r)$ be a system inducing $\varphi = \varphi_{\mathcal{N}}$ given by Proposition 2.2. By Lemma 2.7 we may assume that there is no path as in the statement of the lemma. Thus Proposition 2.8 yields a regular and independent system $\mathcal{A} = (A_1, \ldots, A_r)$ such that A_j is a component of N_j and for each j there is a loop τ_j such that $\tau_j \cap \mathcal{N} = \tau_j \cap A_j$ is a single point. Therefore $\varphi_{\mathcal{N}}([\tau_j]) = \varphi_{\mathcal{A}}([\tau_j])$ for each j and $\varphi_{\mathcal{A}}\colon \pi_1(W) \to F_r$ is surjective. We will show that $\ker \varphi_{\mathcal{N}} \subset \ker \varphi_{\mathcal{A}}$.

Let $[\alpha] \in \ker \varphi_{\mathcal{N}}$ and write $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_k$ as in Lemma 2.6 with respect to the system $\mathcal{N}$. We proceed by induction on k, which is even since $\varphi_{\mathcal{N}}([\alpha]) = 1$. If $k = 0$, then $\alpha \cap \mathcal{N} = \varnothing$, so $\alpha \in W|\mathcal{N} \subset W|\mathcal{A}$ and therefore $[\alpha] \in \ker \varphi_{\mathcal{A}}$. Suppose that any element in $\ker \varphi_{\mathcal{N}}$ represented by a loop which can be written as the concatenation of less than k paths as in Lemma 2.6 is also contained in $\ker \varphi_{\mathcal{A}}$. Let $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_k$ for $[\alpha] \in \ker \varphi_{\mathcal{N}}$. Since $1 = \varphi_{\mathcal{N}}([\alpha]) = a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$, there is an index m such that $a_{i_m} = a_{i_{m+1}}$ and $\epsilon_{m+1} = -\epsilon_m$, so $i_m = i_{m+1} =: j$. Thus both the paths α_m and α_{m+1} intersect the same component X of N_j since there are no paths as in Lemma 2.7. Obviously, we may extend the tubular neighbourhood of X slightly and assume that the beginning of the path α_m and the end of α_{m+1} are in $P_t(X)$ for some $t \notin [-1, 1]$. Since X is connected, so also is $P_t(X)$, and there is an arc γ in $P_t(X)$ joining these two points. Thus we may define the loop

$$\beta = \alpha_1 \cdot \ldots \cdot \alpha_{m-1} \cdot (\gamma \cdot \alpha_{m+2}) \cdot \alpha_{m+3} \cdot \ldots \cdot \alpha_k$$

which has $k - 2$ paths as in Lemma 2.6. Write $\varphi_{\mathcal{N}}([\alpha]) = \omega \cdot a_j^{\epsilon_m} a_j^{-\epsilon_m} \cdot \omega'$. Evidently, $\varphi_{\mathcal{N}}([\beta]) = \omega\omega' = 1$ and by induction hypothesis $\varphi_{\mathcal{A}}([\beta]) = 1$. It is clear that in both the cases $X = A_j$ and $X \neq A_j$we get $\varphi_{\mathcal{A}}([\alpha]) = \varphi_{\mathcal{A}}([\beta]) = 1$, so $[\alpha] \in \ker \varphi_{\mathcal{A}}$. By induction $\ker \varphi_{\mathcal{N}} \subset \ker \varphi_{\mathcal{A}}$.

Therefore $\varphi_{\mathcal{A}} = \eta \circ \varphi_{\mathcal{N}}$ for some epimorphism $\eta\colon F_r \to F_r$. Since free groups are Hopfian (see [2]), η is an isomorphism, so $\ker \varphi_{\mathcal{N}} = \ker \varphi_{\mathcal{A}}$. Because $[\tau_j]$'s generate a subgroup of $\pi_1(W)$ mapped isomorphically onto F_r by $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{A}}$ on which they are equal, we obtain $\varphi_{\mathcal{A}} = \varphi_{\mathcal{N}}$ everywhere and the theorem is proved. ∎

3. Systems of hypersurfaces in the classification of epimorphisms. Hereafter, Σ_g and S_g denote respectively an orientable and a non-orientable closed surface of genus g.

Let G be a finitely generated group and $\varphi\colon G \to F_r$ be an epimorphism. The number r is called the *rank* of φ. The *corank* of G is defined as the largest rank of an epimorphism from G onto a free group and it is denoted by $\operatorname{corank}(G)$. Since G is finitely generated, its corank is well-defined and

$$\operatorname{corank}(G) \leq \operatorname{rank}_{\mathbb{Z}} \operatorname{Ab}(G),$$

where $\operatorname{Ab}(G)$ is the abelianization of G. For more information about the corank and its properties we refer to [5, 7, 9, 15, 23]. When $G = \pi_1(X)$ the corank of G is also called the first non-commutative Betti number of X (cf. [7]). We only recall that $\operatorname{corank}(\pi_1(\Sigma_g)) = g$ and $\operatorname{corank}(\pi_1(S_g)) = \lfloor g/2 \rfloor$.

Grigorchuk, Kurchanov and Zieschang [10, 12] studied epimorphisms onto free groups from fundamental groups of compact surfaces. As in their papers, we call two homomorphisms $\varphi, \psi\colon G \to H$ *equivalent*, and write $\varphi \sim \psi$, if there exist isomorphisms $\nu\colon G \to G$ and $\eta\colon H \to H$ such that $\varphi \circ \nu = \eta \circ \psi$; and φ, ψ are *strongly equivalent* if one can choose $\eta = \mathrm{id}_H$, in which case we write $\varphi \simeq \psi$. Obviously, $\varphi \simeq \psi$ implies $\varphi \sim \psi$. We are interested in the case $H = F_r$.

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\nu"', "\cong"] & F_r \arrow[d, "\eta", "\cong"'] \\ G \arrow[r, "\varphi"] & F_r \end{tikzcd}$$

(a)

$$\begin{tikzcd} G \arrow[dr, "\psi"] \arrow[dd, "\nu"', "\cong"] & \\ & F_r \\ G \arrow[ur, "\varphi"'] & \end{tikzcd}$$

(b)

Fig. 2. Equivalence (a) and strong equivalence (b) of epimorphisms onto free groups.

In this section we apply the results of the previous section to the problem of classification of epimorphisms onto free groups up to equivalence and strong equivalence. In particular, we give an alternative proof of the following theorem.

THEOREM 3.1 (Grigorchuk–Kurchanov–Zieschang [10, 11, 12]). *If Σ is a closed surface of Euler characteristic $\chi(\Sigma) = 2 - k$ and $1 \leq r \leq \lfloor \frac{k}{2} \rfloor =$ $\operatorname{corank}(\pi_1(\Sigma))$, then there are finitely many, say p and q, classes of epimorphisms $\pi_1(\Sigma) \to F_r$ with respect to equivalence and strong equivalence, respectively. More precisely,*

(1) *if Σ is orientable, then $p = q = 1$,*
(2) *if $\Sigma = S_k$ is nonorientable, then*
   (a) *$p = q = 1$ if $k = 2m + 1$ is odd,*
   (b) *$p = 2$ and $q = 2^r$ if $k = 2m$ is even and $r < m$,*
   (c) *$p = 1$ and $q = 2^r - 1$ if $k = 2m$ is even and $r = m$.*

PROPOSITION 3.2 ([11]). *For $m \geq r$ there exists only one class of epimorphisms $F_m \to F_r$ up to strong equivalence.*

Note that the Poincaré conjecture is equivalent to the classification of some pairs of epimorphisms onto free groups, which shows the importance of such studies.

THEOREM 3.3 (Stallings–Jaco–Waldhausen–Hempel, [13, 14]). *The Poincaré conjecture holds if and only if for each $g \geq 2$ any two epimorphisms $\pi_1(\Sigma_g) \to F_g \times F_g$ are equivalent.*

3.1. Systems of hypersurfaces up to framed cobordism and diffeomorphism. Let us denote by $\mathcal{H}_r(M)$ the set of all independent and regular systems of hypersurfaces in M of size r which omit the basepoint, and by $\mathcal{H}_r^{\mathrm{fr}}(M)$ the set of framed cobordism classes of elements of $\mathcal{H}_r(M)$. On each of these sets there is a natural action of $\mathrm{Diff}_\bullet(M)$, the set of self-diffeomorphisms of M which preserve the basepoint, so we may form the orbit space $\mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M)$. Note that if $h \in \mathrm{Diff}_\bullet(M)$, then a system $\mathcal{N} = (N_1, \ldots, N_r)$ and its image $h(\mathcal{N}) = (h(N_1), \ldots, h(N_r))$ induce strongly equivalent homomorphisms.

Moreover, for groups G and H denote by $\mathrm{Epi}(G, H)$ the set of all epimorphisms $G \to H$.

We have the natural map $\Theta\colon \mathcal{H}_r(M) \to \mathrm{Epi}(\pi_1(M), F_r)$ which sends a system $\mathcal{N}$ into the induced epimorphism $\varphi_{\mathcal{N}}$. By Proposition 2.4 it factorizes through an injective map $\overline{\Theta}\colon \mathcal{H}_r^{\mathrm{fr}}(M) \to \mathrm{Epi}(\pi_1(M), F_r)$. Theorem 2.10 states that both these mappings are also surjective.

COROLLARY 3.4. *The map $\overline{\Theta}\colon \mathcal{H}_r^{\mathrm{fr}}(M) \to \mathrm{Epi}(\pi_1(M), F_r)$ is a bijection between the set of all framed cobordism classes of regular and independent systems of hypersurfaces of size r in M and the set of all epimorphisms from $\pi_1(M)$ onto the free group of rank r.* ∎

Now, let us consider the strong equivalence relation $\simeq$ on $\mathrm{Epi}(G, F_r)$. The composition

$$\mathcal{H}_r^{\mathrm{fr}}(M) \to \mathrm{Epi}(\pi_1(M), F_r) \to \mathrm{Epi}(\pi_1(M), F_r)/{\simeq}$$

is still surjective and it factorizes through a map $\overline{\overline{\Theta}}\colon \mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M) \to \mathrm{Epi}(\pi_1(M), F_r)/{\simeq}$.

COROLLARY 3.5. *The number of strong equivalence classes of epimorphisms $\pi_1(M) \to F_r$ is not greater than the cardinality of $\mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M)$.*

The question is when the latter set is finite. It is for example the case for the surface groups.

PROPOSITION 3.6. *For a closed surface Σ the map $\overline{\overline{\Theta}}\colon \mathcal{H}^{\mathrm{fr}}(\Sigma)/\mathrm{Diff}_\bullet(\Sigma) \to \mathrm{Epi}(\pi_1(\Sigma), F_r)/{\simeq}$ is a bijection.*

Proof. We know that the map is surjective. For injectivity it suffices to note that by the Dehn–Nielsen Theorem (see [4]) any automorphism of $\pi_1(\Sigma)$ can be represented by a self-diffeomorphism of Σ. If $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}'}$ are strongly equivalent via $\eta = h_\#$ induced by $h \in \mathrm{Diff}_\bullet(\Sigma)$, then $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'} \circ \eta = (f_{\mathcal{N}'} \circ h)_\# = (f_{h^{-1}(\mathcal{N}')})_\# = \varphi_{h^{-1}(\mathcal{N}')}$, so $\mathcal{N}$ and $h^{-1}(\mathcal{N}')$ are framed cobordant. ∎

REMARK 3.7. The same is true for any manifold M for which any automorphism of $\pi_1(M)$ is induced by some element of $\mathrm{Diff}_\bullet(M)$. By the Mostow Rigidity Theorem it is the case for hyperbolic manifolds of dimension at least 3.

Now, our aim is to calculate $\mathcal{H}_r^{\mathrm{fr}}(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)$. We need the following series of three lemmas.

LEMMA 3.8. *Let Σ be a non-orientable compact surface with $\partial\Sigma \neq \varnothing$ and $S \subset \partial\Sigma$ be a connected component. Then there exists $h \in \mathrm{Diff}_\bullet(\Sigma)$ such that $h(S) = S$, $h|_S$ is orientation-reversing and $h|_{\partial\Sigma\setminus S} = \mathrm{id}_{\partial\Sigma\setminus S}$.*

Proof. First, assume that Σ is the projective plane $\mathbb{R}\mathrm{P}^2$ with one disc B removed, i.e. $\Sigma = \mathbb{R}\mathrm{P}^2 \setminus \mathrm{Int}\, B$ and $S = \partial B$. Let $D \subset \mathrm{Int}\,\Sigma$ be another disc and $\Sigma' = (\Sigma\setminus D)\cup B$ be a Möbius band. Fix a parametrization $\Sigma' \cong [-1,1]\times[0,1]/(t,0) \sim (-t,1)$ for $t \in [-1,1]$ such that $S \subset \mathrm{Int}\,\Sigma'$ is symmetric with respect to the core $\{0\} \times [0,1]$, i.e. if $(t,x) \in S$, then $(-t,x)$ is also in S. Then $h'\colon \Sigma' \to \Sigma'$ defined by $h'(t,x) = (-t,x)$ is a self-diffeomorphism such that $h'|_S\colon S \to S$ has degree -1, so it is orientation-reversing, but on $\partial\Sigma'$ it is orientation-preserving, so isotopic to the identity. Thus we can extend h' to $h\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ such that $h(B) = B$ and $h|_S$ has degree -1, and take $h|_\Sigma$.

In the general case, glue a disc B and Σ along S and take a diffeomorphism $\Sigma \cup_S B \to \Sigma'' \# \mathbb{R}\mathrm{P}^2$ such that $B \subset \Sigma' \subset \mathbb{R}\mathrm{P}^2$ as before. The lemma follows from the first case. ∎

LEMMA 3.9. *Let $\mathcal{N} = (N_1 \cup N_2)$ be a system of size* 1 *in a manifold M such that N_1 and N_2 are connected. If $M \setminus N_1$ and $M \setminus N_2$ are connected, but $M|\mathcal{N}$ is disconnected, then $\varphi_{\mathcal{N}} \colon \pi_1(M) \to \mathbb{Z}$ is not surjective.*

Proof. Assume that $\varphi_{\mathcal{N}}$ is an epimorphism. Then there is a loop α in general position relative to $\mathcal{N}$ such that $\varphi_{\mathcal{N}}([\alpha]) = \pm 1$. As in Lemma 2.6 write α as a concatenation $\alpha_1 \ldots \alpha_k$ of paths α_i, each of which intersects $\mathcal{N}$ in a single point. Therefore

$$1 \equiv \pm 1 = \varphi_{\mathcal{N}}([\alpha]) \equiv k \bmod 2,$$

so k is odd. By assumption $M|\mathcal{N}$ has exactly two components. Since N_1, N_2, $M \setminus N_1$ and $M \setminus N_2$ are connected, each α_i joins both the components of $M|\mathcal{N}$. Thus k is even, because α is a loop, so it starts and ends at the same point. This gives a contradiction, so $\varphi_{\mathcal{N}}$ is not surjective. ∎

REMARK 3.10. While we know that independent systems induce surjective homomorphisms, non-independent systems can induce both surjective and non-surjective homomorphisms. The above lemma shows when $\varphi_{\mathcal{N}}$ is not an epimorphism, and it can be generalized to other similar situations.

LEMMA 3.11. *For an independent system $\mathcal{N} = (N_1 \cup \cdots \cup N_r)$ of size* 1 *in a manifold M there exists a regular and independent system $\mathcal{N}' = (N')$ which is framed cobordant to $\mathcal{N}$, so $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$. Moreover:*

(1) *The complement $M|\mathcal{N}'$ can be non-orientable if $M|\mathcal{N}$ is non-orientable.*
(2) *The complement $M|\mathcal{N}'$ is orientable if $M|\mathcal{N}$ is orientable and $M|\mathcal{N} \cup P(N_i)$ is non-orientable for each i.*

Proof. The construction of $\mathcal{N}'$ is performed as in the proof of Lemma 2.7 by using arcs γ connecting components of $\mathcal{N}$. They can be found since $\mathcal{N}$ is independent.

Consider a two-sheeted orientation cover $\pi \colon \widetilde{M} \to M$, where

$$\widetilde{M} := \{\mu_x \mid x \in M \text{ and } \mu_x \in H_n(M, M \setminus \{x\}) \text{ is a local orientation at } x\}.$$

For (1), if $M|\mathcal{N}$ is non-orientable, then there is a loop α in $M|\mathcal{N}$ which reverses orientation, which means that it lifts to a path in $\widetilde{M}$ which joins two different local orientations at the basepoint. Since $M|\mathcal{N} \setminus \operatorname{Im} \alpha$ is connected, we may perform the construction of $\mathcal{N}'$ in this space. Then α is also contained in $M|\mathcal{N}'$, so it is non-orientable.

Now, assume that $M|\mathcal{N}$ is orientable, but $M|\mathcal{N} \cup P(N_i)$ is non-orientable for each i. To obtain a contradiction, suppose that $M|\mathcal{N}'$ is non-orientable, so there is a loop α in $M|\mathcal{N}'$ which reverses orientation and we may assume that it is in general position relative to $\mathcal{N}$. Using Lemma 2.6 write α as a concatenation $\alpha_1 \ldots \alpha_k$ of paths α_i, each intersecting $\mathcal{N}$ in a single point. Note that since α is in $M|\mathcal{N}'$, it intersects $\mathcal{N}$ only when it goes into or leaves a tubular neighbourhood $P(\gamma)$ of some arc γ, as mentioned at the beginning of the proof. Therefore, as in the proof of Lemma 2.7, if α intersects $\mathcal{N}$ going inside $P(\gamma)$, then it needs to leave $P(\gamma)$ again intersecting $\mathcal{N}$. Thus k is even.

For any i consider α_i which intersects N_i in a point x and take a small closed disc D in M around x such that the cover π is trivial over D and $\partial D \cap \operatorname{Im} \alpha_i = \{x_1, x_2\}$, where x_1 and x_2 lie on different sides of N_i such that α_i goes from x_1 to x_2. By the assumption, there is a reversing-orientation loop β_i in $M|\mathcal{N} \cup P(N_i)$ intersecting $\mathcal{N}$ only once at x and we may assume that its image agrees with the image of α_i on D. We take a loop α' which differs from α only on the segment of α_i between x_1 and x_2, where it goes like β_i outside D. Note that the local orientations at x_2 assigned by lifts of α and α' are opposite. Repeating this for each α_i we obtain a loop α'' which omits $\mathcal{N}$ and which is still orientation-reversing since we changed the local orientations by β_i an even number of times. This contradicts the fact that $M|\mathcal{N}$ is orientable and proves (2). ∎

Remark 3.12. In fact, in (1) the complement $M|\mathcal{N}'$ is always non-orientable if $M|\mathcal{N}$ is non-orientable. Indeed, if $\mathcal{N}''$ is any other regular and independent system framed cobordant to $\mathcal{N}$ such that $M|\mathcal{N}''$ is orientable, then it is also framed cobordant to $\mathcal{N}'$, but this is a contradiction by the next proposition.

Proposition 3.13. *Let M be a non-orientable manifold and let $\mathcal{N}$ and $\mathcal{N}'$ be two regular and independent systems of hypersurfaces in M of the same size r such that $M|\mathcal{N}$ is orientable, but $M|\mathcal{N}'$ is non-orientable. Then $\mathcal{N}$ and $\mathcal{N}'$ are not framed cobordant.*

Proof. Let $\mathcal{N} = (N_1, \dots, N_r)$ and $\mathcal{N}' = (N'_1, \dots, N'_r)$. It is clear that we may assume that $\mathcal{N}$ satisfies the conditions in Lemma 3.11(2) since framed cobordism between $\mathcal{N}$ and $\mathcal{N}'$ implies framed cobordism between $\mathcal{N}_* = (N_{i_1}, \dots, N_{i_k})$ and $\mathcal{N}'_* = (N'_{i_1}, \dots, N'_{i_k})$, where $i_1 < \dots < i_k$ are all indices such that $M|\mathcal{N} \cup P(N_{i_j})$ is non-orientable. We will show that $\mathcal{N}_*$ and $\mathcal{N}'_*$ are not framed cobordant even as submanifolds, let alone as systems of hypersurfaces. For this we use Lemma 3.11 for $(N_{i_1} \cup \dots \cup N_{i_k})$ and $(N'_{i_1} \cup \dots \cup N'_{i_k})$ to assume that $r = 1$.

So now, each of $\mathcal{N}$ and $\mathcal{N}'$ is just a non-separating connected 2-sided submanifold in M, $M|\mathcal{N}$ is orientable and $M|\mathcal{N}'$ is non-orientable. Suppose that $W \subset M \times [0,1]$ is a framed cobordism between $\mathcal{N}$ and $\mathcal{N}'$. Consider the orientation cover $\pi\colon \widetilde{M} \to M$ and the lifts $\widetilde{\mathcal{N}} := \pi^{-1}(\mathcal{N})$ and $\widetilde{\mathcal{N}'} := \pi^{-1}(\mathcal{N}')$. Moreover, by the property of π the complement $\widetilde{M}|\widetilde{\mathcal{N}}$ has two connected components since $M|\mathcal{N}$ is orientable, and $\widetilde{M}|\widetilde{\mathcal{N}'}$ is connected since $M|\mathcal{N}'$ is non-orientable. The cobordism W lifts to the framed cobordism $\widetilde{W} := (\pi \times \mathrm{id}_{[0,1]})^{-1}(W) \subset \widetilde{M} \times [0,1]$ between $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$. Therefore $\varphi_{\widetilde{\mathcal{N}}} =$

$\varphi_{\widetilde{\mathcal{N}'}}\colon \pi_1(\widetilde{M}) \to \mathbb{Z}$ and $\varphi_{\widetilde{\mathcal{N}'}}$ is surjective, because $\widetilde{\mathcal{N}'}$ is independent. However, we will show $\varphi_{\widetilde{\mathcal{N}}}$ is not surjective, which gives a contradiction.

To see this, note that $\widetilde{\mathcal{N}}$ can have one or two components. If $\widetilde{\mathcal{N}}$ is connected, then $\varphi_{\widetilde{\mathcal{N}}}$ is evidently not surjective, since $\widetilde{M|\mathcal{N}}$ is not connected. If $\widetilde{\mathcal{N}}$ has two components, we use Lemma 3.9 together with the fact that $\widetilde{M}|\widetilde{\mathcal{N}} \to M|\mathcal{N}$, the restriction of π, is also the orientation cover of $M|\mathcal{N}$.

Thus $\mathcal{N}$ and $\mathcal{N}'$ are not framed cobordant. ∎

Remark 3.14. The above proposition is easily seen not to be true for non regular systems of hypersurfaces. To construct an example, it suffices to take a system $\mathcal{N} = (N_1 \cup N_2)$ of size 1 consisting of two non-separating framed circles in $M = \Sigma_1 \# S_2$, the connected sum of the torus and the Klein bottle, such that $N_1 \subset \Sigma_1$ and $N_2 \subset S_2$ are disjoint from the discs used in the connected sum operation. Since N_2 has trivial normal bundle and $S_2|N_2$ is orientable, there is an orientation-reversing loop β in S_2 intersecting N_2 in a single point. Obviously, there is also an orientation-preserving loop α in Σ_1 intersecting N_1 in a single point. Moreover, one can take an arc γ in M joining $\alpha \cap N_1$ and $\beta \cap N_2$ and disjoint from α and β outside these points. Performing the connected sum of N_1 and N_2 along γ we obtain $\mathcal{N}' = (N')$ such that $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$ by Lemma 2.7, so $\mathcal{N}$ and $\mathcal{N}'$ are framed cobordant. However, $M|\mathcal{N}$ is orientable, but $M|\mathcal{N}'$ is not because the concatenation $\alpha \cdot \gamma \cdot \beta$ is an orientation-reversing loop which can be homotoped to lie in $M|\mathcal{N}'$.

Now, we can provide an alternative proof of the Grigorchuk–Kurchanov–Zieschang Theorem (Theorem 3.1) for strong equivalence. First, let us make a short preparation.

Let $\mathcal{N} = (N_1, \ldots, N_r)$ and $\mathcal{N}' = (N_1', \ldots, N_r')$ be two arbitrary regular and independent systems of hypersurfaces in a closed surface Σ. Thus all N_i, N_i' are circles. Assume that $\Sigma|\mathcal{N}$ and $\Sigma|\mathcal{N}'$ are diffeomorphic. By homogeneity, take a diffeomorphism $h'\colon \Sigma|\mathcal{N} \to \Sigma|\mathcal{N}'$ which sends $P_{\pm 1}(N_i)$ onto $P_{\pm 1}(N_i')$. Glue all tubes $P(N_i) \cong [-1,1] \times N_i$ to $\Sigma|\mathcal{N}$ along $\{-1\} \times N_i$ to obtain a surface $\overline{\Sigma}$ with $2r$ boundary components $\{1\} \times N_i$ and $P_1(N_i)$, $i = 1, \ldots, r$. Let

$$\xi_i\colon P_1(N_i) \to \{1\} \times N_i$$

be a gluing map which leads to Σ. Analogously, we define $\overline{\Sigma'}$ and take ξ_i' for $\mathcal{N}'$. Extend h' to a diffeomorphism $\bar{h}\colon \overline{\Sigma} \to \overline{\Sigma'}$ using $P(N_i) \cong [-1,1] \times \mathrm{S}^1 \cong P(N_i')$, so $\bar{h}(\mathcal{N}) = \mathcal{N}'$. It follows easily that $\bar{h}$ induces $h \in \mathrm{Diff}_\bullet(\Sigma)$ after performing gluing operations via ξ_i and ξ_i' if and only if $\bar{h}^{-1} \circ \xi_i' \circ \bar{h}|_{P_1(N_i)}$ is isotopic to ξ_i for each $i = 1, \ldots, r$. If that is the case, then $h(\mathcal{N}) = \mathcal{N}'$, so $\mathcal{N}$ and $\mathcal{N}'$ are the same elements in $\mathcal{H}_r^{\mathrm{fr}}(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)$.

THEOREM 3.15. *Let Σ be a closed surface, let r be an integer such that $1 \le r \le \operatorname{corank}(\pi_1(\Sigma))$ and set $q = |\mathcal{H}^{\mathrm{fr}}_r(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)|$.*

(1) *If Σ is orientable or non-orientable of odd genus, then $q = 1$.*
(2) *If $\Sigma = S_{2m}$ is non-orientable of genus $2m$, then*
- *if $r < m$, then $q = 2^r$,*
- *if $r = m$, then $q = 2^r - 1$.*

As a consequence, q is the number of strong equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$ as in Theorem 3.1.

Proof. We use the above notation. If Σ is orientable, then $\Sigma|\mathcal{N}$ and $\Sigma|\mathcal{N}'$ are diffeomorphic surfaces and we may assume that the diffeomorphism h' is orientation-preserving. Since Σ is orientable, all maps ξ_i and ξ_i' are also orientation-preserving, so we obtain $h \in \mathrm{Diff}_\bullet(\Sigma)$ such that $h(N_i) = N_i'$. Therefore $q = 1$.

Now assume that Σ is non-orientable of odd genus. Then $\Sigma|\mathcal{N}$ and $\Sigma|\mathcal{N}'$ are compact surfaces with $2r$ boundary components and of the same odd Euler characteristic, so they are also non-orientable. Using Lemma 3.8 we may change h', by composing it with another diffeomorphism, so that $\bar{h}^{-1} \circ \xi_i' \circ \bar{h}|_{P_1(N_i)}$ and ξ_i are isotopic. As before, this implies that $q = 1$.

Finally, let $\Sigma = S_{2m}$ be non-orientable of even genus $2m$. For any non-empty subset $I \subset \{1, \dots, r\}$ it is easy to construct a system $\mathcal{N}_I$ such that $\Sigma|\mathcal{N}_I$ is orientable and the gluing maps ξ_i^I (defined as before) are orientation-reversing only for $i \in I$. We omit the case when $I = \varnothing$ since then Σ would be orientable. Moreover, for $r < m$ we denote by $\mathcal{N}_0$ a system for which $\Sigma|\mathcal{N}_0$ is non-orientable. Note that if $r = m$, then $\Sigma|\mathcal{N}$ is always a sphere with $2r$ open discs removed, so it is orientable.

By the previous considerations it is clear that the systems $\mathcal{N}_I$ for $\varnothing \ne I \subset \{1, \dots, r\}$ and $\mathcal{N}_0$ for $r < m$ represent all elements of $\mathcal{H}^{\mathrm{fr}}_r(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)$ (when $\Sigma|\mathcal{N}$ is non-orientable we use Lemma 3.8 as before). Thus $q \le 2^r$ for $r < m$ and $q \le 2^r - 1$ for $r = m$. We will show that they are different elements of $\mathcal{H}^{\mathrm{fr}}_r(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)$. This will follow if we show that the systems are not framed cobordant to each other.

By Proposition 3.13 we know that $\mathcal{N}_0$ is not framed cobordant to any $\mathcal{N}_I$. If we have two systems $\mathcal{N}_I = (N_1^I, \dots, N_r^I)$ and $\mathcal{N}_J = (N_1^J, \dots, N_r^J)$ for $I \ne J$, then we may assume that there is an index $1 \le j \le r$ such that $j \notin I$, but $j \in J$, so ξ_j^I is orientation-preserving, but ξ_j^J is orientation-reversing. If $I = \{i_1, \dots, i_k\}$, form the systems $\mathcal{N}_I^* = (N_{i_1}^I, \dots, N_{i_k}^I)$ and $\mathcal{N}_J^* = (N_{i_1}^J, \dots, N_{i_k}^J)$. By construction, $\Sigma|\mathcal{N}_I^*$ is orientable, but $\Sigma|\mathcal{N}_J^*$ is not. Again Proposition 3.13 shows that $\mathcal{N}_I^*$ and $\mathcal{N}_J^*$ are not framed cobordant, so $\mathcal{N}_I$ and $\mathcal{N}_J$ cannot be framed cobordant either and the proof is complete.

The last statement follows by Proposition 3.6. ∎

Corollary 3.16. *With the above notation,*

$$\mathcal{H}_r^{\mathrm{fr}}(S_{2m})/\mathrm{Diff}_\bullet(S_{2m}) = \begin{cases} \{[\mathcal{N}_0], [\mathcal{N}_I] : \varnothing \neq I \subset \{1,\dots,r\}\} & \text{for } r < m, \\ \{[\mathcal{N}_I] : \varnothing \neq I \subset \{1,\dots,r\}\} & \text{for } r = m. \end{cases} \blacksquare$$

3.2. Analogues of Nielsen transformations for systems of hypersurfaces. We have found that strong equivalence classes of epimorphisms $\pi_1(M) \to F_r$ can be described by elements of $\mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M)$. In this section we show how to get equivalence classes from them.

It is known that the automorphism group $\mathrm{Aut}(F_r)$ of a finitely generated free group F_r is generated by *elementary Nielsen transformations* (see e.g. [2]). On a given ordered basis $(a_1,\dots,a_r)$ on F_r we define them as follows:

(T1) $n_\sigma \colon (a_1,\dots,a_r) \mapsto (a_{\sigma(1)},\dots,a_{\sigma(r)})$ for some permutation $\sigma \in S_r$;
(T2) $n_i \colon (a_1,\dots,a_r) \mapsto (a_1,\dots,a_{i-1},a_i^{-1},a_{i+1},\dots,a_r)$ for $i \in \{1,\dots,r\}$;
(T3) $n_{ij} \colon (a_1,\dots,a_r) \mapsto (a_1,\dots,a_{i-1},a_ia_j,a_{i+1},\dots,a_r)$ which replaces a_i by a_ia_j for some $i \neq j$.

Note that (T1) can be obtained from the other two, but it is convenient to use. Thus we have three types of automorphisms: $n_\sigma, n_i, n_{ij} \in \mathrm{Aut}(F_r)$.

Definition 3.17. Let $\mathcal{N} = (N_1,\dots,N_r)$ be an independent and regular system of hypersurfaces in a closed manifold M. We define analogous operations on $\mathcal{H}_r(M)$:

(H1) $\mathcal{N} \mapsto \mathcal{N}^\sigma := (N_{\sigma(1)},\dots,N_{\sigma(r)})$ for some permutation $\sigma \in S_r$;
(H2) $\mathcal{N} \mapsto \mathcal{N}^i$ is obtained by changing the framing of the submanifold N_i to the one with opposite orientation;
(H3) $\mathcal{N} \mapsto \mathcal{N}^{ij}$ is obtained for $i \neq j$ by replacing N_j by $N_j \#_\gamma P_1(N_i)$, where γ is an arc as in Lemma 2.7 which intersects $\mathcal{N}$ only in two points and joins N_j and $P_1(N_i)$ from the same side.

An arc γ in (H3) always exists since $\mathcal{N}$ is independent. For the resulting system $\mathcal{N}^{ij}$ we take smaller tubular neighbourhoods to be disjoint, e.g. $P_{[-1,1/2]}(N_i) \cong [-1,1/2] \times N_i$. By Lemma 2.7 the homomorphism $\varphi_{\mathcal{N}^{ij}}$ is the same as the one induced by the system $(N_1,\dots,N_i,\dots,N_j \cup P_1(N_i),\dots,N_r)$, so it is clear by the definition that $\varphi_{\mathcal{N}^{ij}} = n_{ij} \circ \varphi_{\mathcal{N}}$. Therefore $\varphi_{\mathcal{N}^{ij}}$ is surjective and since obviously $\mathcal{N}^{ij}$ is regular, by Proposition 2.8 it is also independent, so operation (H3) on $\mathcal{H}_r(M)$ is well defined: it does not depend on the choice of γ up to framed cobordism.

In the same way, (H1) and (H2) are analogues of (T1) and (T2):

$$\varphi_{\mathcal{N}^\sigma} = n_\sigma \circ \varphi_{\mathcal{N}} \quad \text{and} \quad \varphi_{\mathcal{N}^i} = n_i \circ \varphi_{\mathcal{N}}.$$

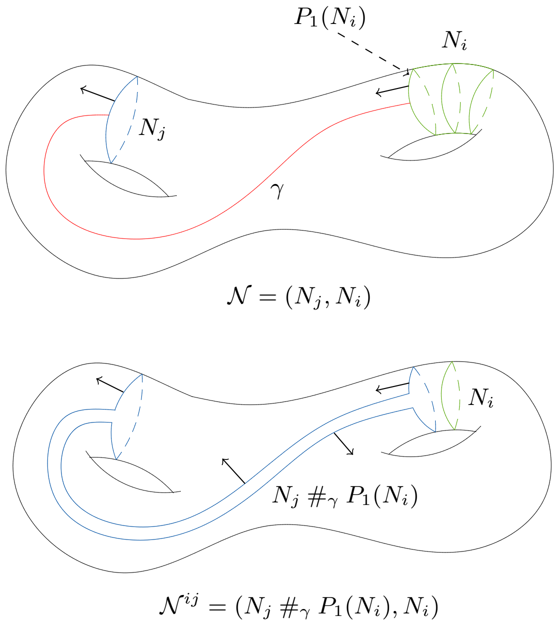

Fig. 3. Operation (H3) which transforms $\mathcal{N} = (N_j, N_i)$ into $\mathcal{N}^{ij} = (N_j \#_\gamma P_1(N_i), N_i)$.

Since elementary Nielsen transformations generate $\mathrm{Aut}(F_r)$, we have the following straightforward conclusion.

PROPOSITION 3.18. *Two epimorphisms $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}'}$ induced by $\mathcal{N}, \mathcal{N}' \in \mathcal{H}_r(M)$ are equivalent if and only if $\mathcal{N}'$ can be transformed by applying a finite number of operations* (H1)–(H3) *to a system $\mathcal{N}''$ such that $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}''}$ are strongly equivalent.* ■

In particular, if M is a manifold for which the map

$$\overline{\overline{\Theta}}\colon \mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M) \to \mathrm{Epi}(\pi_1(M), F_r)/\simeq$$

is a bijection, then $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}'}$ are equivalent if and only if $\mathcal{N}''$ can be obtained so as to represent the same element of $\mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M)$ as $\mathcal{N}$.

LEMMA 3.19. *Operations* (H1)–(H3) *on $\mathcal{N}$ do not change the orientability of $M|\mathcal{N}$.*

Proof. This is clear for (H1) and (H2). For (H3), if α is an orientation-reversing loop in $M|\mathcal{N}$, then $M|\mathcal{N} \setminus \mathrm{Im}\,\alpha$ is also connected and a path γ between N_j and $P_1(N_i)$ can be taken to be disjoint from α, so $M|\mathcal{N}^{ij}$ is also non-orientable by Proposition 3.13. If $M|\mathcal{N}$ is orientable, but α is an orientation-reversing loop in $M|\mathcal{N}^{ij}$, then it intersects N_j and $P_1(N_i)$ in $P(\gamma)$, the tubular neighbourhood of γ. When α intersects N_j and goes into

$P(\gamma)$, it may pass through $P(\gamma)$ and $P_{[0,1]}(N_i) \cong [0,1] \times N_i$ or again intersect N_j. Note that $P_{[0,1]}(N_i)$ is orientable, because $P_1(N_i)$ is orientable as a submanifold of the orientable manifold $M|\mathcal{N}$. Thus α may be changed to another orientation-reversing loop lying outside $\mathcal{N}$, a contradiction. Therefore $M|\mathcal{N}^{ij}$ is also orientable. ∎

THEOREM 3.20. *Let Σ be a closed surface and let $1 \le r \le \operatorname{corank}(\pi_1(\Sigma))$ be an integer. Denote by p the number of equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$.*

(1) *If Σ is orientable or non-orientable of odd genus, then $p = 1$.*
(2) *If $\Sigma = S_{2m}$ is non-orientable of genus $2m$, then*
   - *if $r < m$, then $p = 2$,*
   - *if $r = m$, then $p = 1$.*

Proof. For the first part note that $1 \le p \le q$, where q is the number of strong equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$, and $q = 1$ if Σ is orientable or non-orientable of odd genus. If Σ is non-orientable of genus $2m$, then by Theorem 3.15 and Proposition 3.18 we need to investigate operations (H1)–(H3) on the systems $\mathcal{N}_0$ and $\mathcal{N}_I$ for $\varnothing \neq I \subset \{1, \dots, r\}$. Since by the above lemma the operations do not change the orientability of complements of systems, $\varphi_{\mathcal{N}_0}$ and $\varphi_{\mathcal{N}_I}$ cannot be equivalent for any I, so $p \ge 2$ if $r < m$. We will show that all $\mathcal{N}_I$ induce equivalent epimorphisms.

Apply operation (H3) to $\mathcal{N}_J = \mathcal{N} = (N_1, \dots, N_r)$, for $i \notin J$ and $j \in J$, to obtain the system $\mathcal{N}^{ij}$ which represents the same element in $\mathcal{H}_r^{\mathrm{fr}}(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)$ as $\mathcal{N}_I$ for some I. We will show that $I = J \cup \{i\}$.

First, note that $l \in J$ if and only if $\Sigma|\mathcal{N} \cup P(N_l)$ is non-orientable. Let us divide the proof into four steps:

- $j \in I$: Use the fact that $\Sigma|\mathcal{N}^{ij} \cup P(N_j \#_\gamma P_1(N_i)) = \Sigma|\mathcal{N} \cup P(N_j)$ is non-orientable, since $j \in J$.
- $J \setminus \{j\} \subset I$: Let $l \in J \setminus \{j\}$. Thus there is an orientation-reversing loop α in $\Sigma|\mathcal{N} \cup P(N_l)$ which intersects N_l in a single point. Since a tubular neighbourhood of α is a Möbious band, $\Sigma|\mathcal{N} \setminus \operatorname{Im}\alpha$ is also connected and γ in (H3) can be taken to be disjoint from α. Thus $\operatorname{Im}\alpha \subset \Sigma|\mathcal{N}^{ij} \cup P(N_l)$, so the latter subspace is non-orientable. Therefore $l \in I$ since (H3) does not depend on the choice of γ up to framed cobordism.
- $i \in I$: Take an orientation-reversing loop α in $\Sigma|\mathcal{N} \cup P(N_j)$ intersecting N_j in a single point x, which is the starting point of an arc γ joining N_j and $P_1(N_i)$, and intersecting $\partial P(\gamma)$ in a single point y. Thus we may write $\alpha = \alpha_1 \cdot \alpha_2$, where α_1 is a path outside $P(\gamma)$ joining y and x. Let the endpoint of γ in $P_1(N_i)$ correspond to $(1, z) \in \{1\} \times N_i$ and take a path $\tau\colon [-1,1] \to P(N_i) \cong [-1,1] \times N_i$ defined by $\tau(t) = (-t, z)$. Moreover, take a path β from $\tau(1) \in P_{-1}(N_i)$ to y, which is contained in $\Sigma|\mathcal{N} \setminus \operatorname{Im}\gamma$ (such a path

exists since γ does not disconnect $\Sigma|\mathcal{N})$. Now, form the loop $\alpha' = \alpha_1 \cdot \gamma \cdot \tau \cdot \beta$, which is contained in $\Sigma|\mathcal{N}^{ij} \cup P_{[-1,1/2]}(N_i)$ by taking a smaller tubular neighbourhood of γ used in the connected sum $N_j \#_\gamma P_1(N_i)$. Remember that the neighbourhood of N_i should be considered smaller than $P(N_i)$, say $P_{[-1,1/2]}(N_i)$, after performing (H3). The loop α' is orientation-reversing since it is homotopic to $\alpha \cdot \overline{\alpha_2} \cdot \gamma \cdot \tau \cdot \beta$, where $\overline{\alpha_2}$ is the inverse path for α_2, and $\overline{\alpha_2} \cdot \gamma \cdot \tau \cdot \beta$ is orientation-preserving as it can be homotoped to lie in $\Sigma|\mathcal{N} \cup P(N_i)$, which is orientable. Therefore $i \in I$.

- $l \notin I$ if $l \notin J \cup \{i\}$: If $\Sigma|\mathcal{N}^{ij} \cup P(N_l)$ contains an orientation-reversing loop, then by Lemma 3.19 so does $\Sigma|N \cup P(N_l)$, a contradiction.

Thus using (H3) we may transform any $\mathcal{N}_J$ to be the same element of $\mathcal{H}_r^{\mathrm{fr}}(\Sigma)/\mathrm{Diff}_\bullet(\Sigma)$ as $\mathcal{N}_{\{1,\ldots,r\}}$, so they all induce equivalent epimorphisms. ∎

4. Reeb graphs and Reeb epimorphisms. In this section we establish relations between Reeb graph theory and the earlier results. First, let us recall basic notions.

A *smooth triad* is a triple (W, W_-, W_+), where W is a manifold with boundary $\partial W = W_- \sqcup W_+$ (possibly $W_\pm = \varnothing$). A smooth function $f \colon W \to [a,b]$ is a *function on the smooth triad* (W, W_-, W_+) if $f^{-1}(a) = W_-$, $f^{-1}(b) = W_+$ and all its critical points are in $\operatorname{Int} W$.

Let $f \colon W \to \mathbb{R}$ be a function with finitely many critical points on a smooth triad (W, W_-, W_+). We say that $x, y \in W$ are in *Reeb relation* $\sim_\mathcal{R}$ if they belong to the same connected component of a level set of f. The quotient space $W/{\sim_\mathcal{R}}$ is denoted by $\mathcal{R}(f)$ and called the *Reeb graph* of f.

The Reeb graph of the function f as above is homeomorphic to a finite graph, i.e. to a one-dimensional finite CW-complex (see [28, 33]). The vertices of $\mathcal{R}(f)$ correspond to the components of $W_\pm$ and to the components of level sets of f containing critical points. The homomorphism $q_\# \colon \pi_1(W) \to \pi_1(\mathcal{R}(f)) \cong F_r$ induced by the quotient map $q \colon W \to \mathcal{R}(f)$ is surjective (see [16]) and is called the *Reeb epimorphism* of f. The number r as above is called the *cycle rank* of $\mathcal{R}(f)$ and it is equal to the first Betti number $\beta_1(\mathcal{R}(f))$.

For an oriented graph (i.e. each edge has a chosen direction), the *indegree* and *outdegree* of a vertex v are the numbers of incident edges which income to and outgo from v, respectively. The *degree* $\deg(v)$ is the number of all incident edges to v.

With $\mathcal{R}(f)$ equipped with the quotient topology, f induces a continuous function $\overline{f} \colon \mathcal{R}(f) \to \mathbb{R}$ such that $f = \overline{f} \circ q$. It is strictly monotonic on each edge of $\mathcal{R}(f)$ and has extrema only at vertices of degree 1. A function on a graph satisfying these properties induces an orientation of the graph called a *good orientation* (see Sharko [33]). Thus any Reeb graph is considered with a good orientation. The graphs in the figures in this paper are oriented from the bottom to the top.

Recall that $f\colon W \to \mathbb{R}$ is a *Morse function* if it is smooth and all its critical points are non-degenerate. A Morse function f is *simple* if each of its critical levels contains only one critical point, and it is *ordered* if for any two critical points p and p' of f, if $\operatorname{ind}(p) < \operatorname{ind}(p')$ then $f(p) < f(p')$, where $\operatorname{ind}(p)$ is the index of the non-degenerate critical point p. Note that the Reeb graph of an ordered Morse function on a manifold of dimension at least 3 is a tree. The same is also true for a surface W and a self-indexing Morse function $f\colon W \to \mathbb{R}$, i.e. $f(p) = \operatorname{ind}(p)$ for each critical point p (see [23]).

It is convenient to use a slightly different condition on a Morse function than its simplicity. Namely, we say that a Morse function f is *$\mathcal{R}$-simple* if any vertex of $\mathcal{R}(f)$ corresponds to exactly one critical point of f. Equivalently, each connected component of any level set of f contains at most one critical point. Obviously, a simple Morse function is $\mathcal{R}$-simple. Moreover, every $\mathcal{R}$-simple Morse function can easily be slightly perturbed to get a simple Morse function without changing the Reeb graph.

Let $f\colon W \to \mathbb{R}$ be an $\mathcal{R}$-simple Morse function on the triad (W, W_-, W_+) of dimension $n = \dim W$. The vertices of degree 1 in $\mathcal{R}(f)$ correspond to components of ∂W and to the extremum points of f, the critical points of indices 0 and n. If W is an orientable surface, then all other vertices have degree 3 (see [22]). However, if W is not an orientable surface, then the other vertices of $\mathcal{R}(f)$ have degrees 2 and 3. In addition, for $n \geq 3$, vertices of degree 3 correspond to critical points of index 1 (with indegree 2) or index $n-1$ (with outdegree 2), and vertices of degree 2 correspond to indices $1, \ldots, n-1$ (see [22, 23, 28]).

In [23] the second author defined *combinatorial modifications* of Reeb graphs of simple (or $\mathcal{R}$-simple) Morse functions, labelled (1)–(9), (11), (12), and a modification (10) transforming the Reeb graph of a simple Morse function to the Reeb graph of a Morse function, which accumulates some critical points to one connected component of a level set. We will use them extensively in the proofs. They were introduced for manifolds of dimension at least 3, but they can also be well-defined for non-orientable surfaces because of the same degree-index correspondence as mentioned above (with some caution for modification (7)). For orientable surfaces we have to omit modifications with vertices of degree 2. In fact, for orientable surfaces Fabio and Landi [6] introduced such operations called *elementary deformations*. In order not to separate these cases, we will use the term *combinatorial modification* for Reeb graphs of functions on any manifold, having in mind that for orientable surfaces there are no modifications with vertices of degree 2. Moreover, modification (6) for orientable surfaces works in both ways since it deals with critical points of the same index 1 (it corresponds to elementary deformations (K_2) and (K_3) of [6]).

The considerations in this section are motivated by and based on the following theorem of the second author [23] which summarized previous results.

Theorem 4.1 ([23, Theorem 5.2]). *Let M be a closed, smooth and connected manifold of dimension at least 2. The following are equivalent:*

(a) *There exists a Morse function $f\colon M \to \mathbb{R}$ (simple if M is not an orientable surface) such that $\beta_1(\mathcal{R}(f)) = r$.*
(b) *There exists an epimorphism $\pi_1(M) \to F_r$.*
(c) *There exists a regular and independent system of hypersurfaces of size r in M.*

Remark 4.2. The equivalence of conditions (b) and (c) has been established by Cornea [5, Theorem 1] and for combinatorial manifolds by Jaco [15, Theorem 2.1]. It is evident that condition (a) implies (b) and (c) (see [16]). Moreover, Gelbukh [8] showed the equivalence of (a) and (b) for orientable manifolds by using foliation theory. In [23, Theorem 5.2] there is a crucial proof that (c) implies (a). Theorems 4.6 and 2.10 provide a direct proof that (b) and (c) each imply (a).

Remark 4.3. A characteristic feature of orientable surfaces is that a simple (and also an $\mathcal{R}$-simple) Morse function on a triad of an orientable surface of genus g has Reeb graph with cycle rank g (see [3, 22]).

Combinatorial modifications of Reeb graphs together with the construction of the initial graph (see Figure 4) are the main ingredients in the solution of the realization problem for Reeb graphs in [23]. It was a natural problem to determine which graph can be the Reeb graph of a smooth function with isolated critical points on a given manifold (cf. [30]; see Remark 4.20 below).

Proposition 4.4 ([23]). *For any finite graph Γ with good orientation there is a finite sequence of combinatorial modifications* (1)–(12) *which transform the initial graph to Γ up to vertices of degree* 2.

Theorem 4.5 ([23, Theorem 6.4]). *Let M be a closed, connected manifold of dimension $n \geq 2$ and Γ be a finite connected oriented graph. There exists a Morse function $f\colon M \to \mathbb{R}$ such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ if and only if Γ has a good orientation and $\beta_1(\Gamma) \leq$* corank$(\pi_1(M))$. *Moreover, if M is not an orientable surface and the degree of each vertex in Γ is no greater than 3, then f can be taken to be simple.*

In this section we resolve the realization problem for a manifold with boundary together with representation of an epimorphism as the Reeb epimorphism of a Morse function. We do it by constructing a Morse function with certain components of level sets prescribed by a system of hypersurfaces.

4.1. The initial graph and factorization through a Reeb epimorphism. Recall that $\pi_0(X)$ is the set of path components of a space X and their number is denoted by $|\pi_0(X)|$. Since all the spaces X discussed by us are locally path connected, $\pi_0(X)$ is equal to the set of connected components of X.

We call a graph Γ *admissible* for a triad (W, W_-, W_+) if Γ has at least $|\pi_0(W_-)|$ vertices of degree 1 and indegree 0 and at least $|\pi_0(W_+)|$ vertices of degree 1 and outdegree 0.

The graph in Figure 4(a) is called the *initial graph* (with cycle rank r). Recall that by our convention it is oriented from the bottom to the top. We distinguish a spanning tree in the initial graph, coloured red in the figure. Moreover, we order the edges $e_1, \dots, e_r$ outside the tree as in the figure. We also define a version of the initial graph which is admissible for a triad (W, W_-, W_+) (see Figure 4(b)).

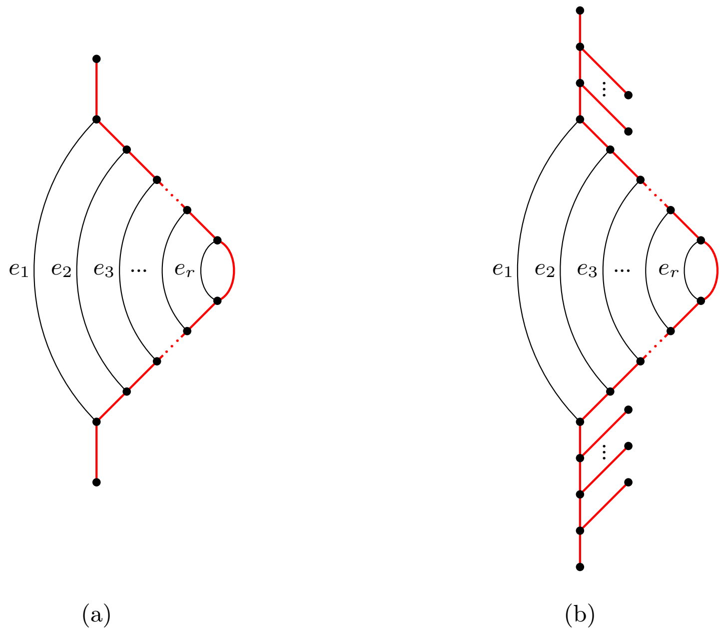

Fig. 4. (a) The initial graph with distinguished tree and ordered edges outside it. (b) The initial graph admissible for (W, W_-, W_+).

The initial graph with cycle rank g occurs easily as the Reeb graph of a height function on an orientable surface of genus g. In fact, by [22, Theorem 5.6] it can be the Reeb graph of a Morse function on any closed surface with Euler characteristic at most $2 - 2g$.

THEOREM 4.6. *Let $\partial W = W_- \sqcup W_+$ and $\mathcal{N} = (N_1, \dots, N_r)$ be a system of hypersurfaces without boundary in W. Then the induced homomorphism $\varphi_{\mathcal{N}}$ factorizes through the Reeb epimorphism of a simple Morse function $f\colon W \to \mathbb{R}$ on the smooth triad (W, W_-, W_+) such that all connected components of $\mathcal{N}$ are components of some regular level set $f^{-1}(c) = \mathcal{N} \sqcup V$, where V is a non-empty submanifold. Moreover, if we allow f to be not necessarily simple for $\dim W = 2$, then we can construct f in such a way that*

$$\beta_1(\mathcal{R}(f)) = |\pi_0(\mathcal{N})| - |\pi_0(W|\mathcal{N})| + 1.$$

Furthermore, if $\mathcal{N}$ is regular and independent, then $\mathcal{R}(f)$ can be taken to be homeomorphic to the initial graph admissible for (W, W_-, W_+), with cycle rank r and such that N_i corresponds to the edge e_i for each i as in Figure 4.

Proof. The construction of the desired function is analogous to that in Theorem 4.1 (c)⇒(a) (of [23]), but here $\mathcal{N}$ may not be independent and regular. For details on the existence of Morse functions and gluing operations we refer to [24].

Take an ordered Morse function $h\colon W|\mathcal{N} \to \mathbb{R}$ on the triad

$$(W|\mathcal{N}, P_{-1}(\mathcal{N}) \sqcup W_+, P_1(\mathcal{N}) \sqcup W_-)$$

and a regular value d from [23, Lemma 3.3] such that $V := h^{-1}(d)$ has the same number of connected components as $W|\mathcal{N}$. Let

$$P(V) := h^{-1}([d-\varepsilon, d+\varepsilon]) \cong V \times [-1,1],$$
$$Q_- := h^{-1}((-\infty, d-\varepsilon]) \quad \text{and} \quad Q_+ := h^{-1}([d+\varepsilon, \infty)).$$

Then

$$\partial Q_- = P_{-1}(V) \sqcup P_{-1}(\mathcal{N}) \sqcup W_+ \quad \text{and} \quad \partial Q_+ = P_1(V) \sqcup P_1(\mathcal{N}) \sqcup W_-.$$

Thus V, Q_+ and Q_- all have the same number of connected components. Now, take simple and ordered Morse functions

$$g_-\colon Q_- \to [-2,-1] \quad \text{on the triad } (Q_-, \emptyset, \partial Q_-),$$
$$g_+\colon Q_+ \to [1,2] \quad \text{on the triad } (Q_+, \partial Q_+, \emptyset).$$

Let us glue them together with suitable projections $P(\mathcal{N} \sqcup V) \to [-1,1]$ obtaining a simple Morse function $f\colon W \to \mathbb{R}$ with regular value 0 such that $f^{-1}(0) = \mathcal{N} \sqcup V$.

Let $q\colon W \to \mathcal{R}(f)$ and $g\colon \mathcal{R}(f) \to \mathcal{R}(f)/q(W|\mathcal{N}) = \bigvee_{i=1}^r \mathrm{S}^1_i$ be the quotient maps. The map g sends $q(W|\mathcal{N})$ to the basepoint, and $q(P(N_i)) \cong [-1,1]$ linearly and preserving orientation onto S^1_i. It is clear that $\varphi_{\mathcal{N}} = (f_{\mathcal{N}})_\# = (g \circ q)_\# = g_\# \circ q_\#$, so $\varphi_{\mathcal{N}}$ factorizes through the Reeb epimorphism $q_\#$ of f.

Now, let us compute $\beta_1(\mathcal{R}(f))$. The subset $q(W|\mathcal{N})$ of $\mathcal{R}(f)$ is homeomorphic to the Reeb graph $\mathcal{R}(f|_{W|\mathcal{N}})$, so it has $|\pi_0(W|\mathcal{N})|$ connected components. If $\dim W \geq 3$, then the components of $\mathcal{R}(f|_{W|\mathcal{N}})$ are trees, because the components of $\mathcal{R}(f|_{Q_\pm})$ are trees by [23, Proposition 3.2] (since the Morse functions on $Q_\pm$ are ordered) and they are gluing, through $\mathcal{R}(f|_{P(V)}) \cong [-1,1]$, one component of $\mathcal{R}(f|_{Q_\pm})$ with the unique component of $\mathcal{R}(f|_{P(V)})$. In the case of surfaces by the same fact we may define Morse functions (self-indexing, not simple) on components of $Q_\pm$ whose Reeb graphs are trees. In both cases, $q(W|\mathcal{N})$ has $|\pi_0(W|\mathcal{N})|$ components which are trees, and so they are contractible.

Thus the quotient $\mathcal{R}(f)/q(W|\mathcal{N})$ can be obtained from $\mathcal{R}(f)$ by first contracting each component of $q(W|\mathcal{N})$, and then gluing the resulting points to one point. The first operation does not change the first Betti number, but the second increases it by 1 for each gluing of two points. Hence $\mathcal{R}(f)/q(W|\mathcal{N})$ has cycle rank $\beta_1(\mathcal{R}(f)) + |\pi_0(W|\mathcal{N})| - 1$. On the other hand, it is clear that $\mathcal{R}(f)/q(W|\mathcal{N})$ is homeomorphic to the wedge product of $|\pi_0(\mathcal{N})|$ circles. Therefore

$$|\pi_0(\mathcal{N})| = \beta_1(\mathcal{R}(f)) + |\pi_0(W|\mathcal{N})| - 1.$$

Now, let $\mathcal{N}$ be a regular and independent system of hypersurfaces. Let $\dim W \geq 3$. Since $W|\mathcal{N}$ and V are connected, the manifolds $Q_\pm$ are also connected, so we may assume that $g_\pm$ has only one critical point, which is an extremum point. Then by [23, Proposition 3.2] the Reeb graph $\mathcal{R}(g_-)$ (resp. $\mathcal{R}(g_+)$) is a tree with one minimum (resp. maximum) and so all vertices of degree 3 have indegree 1 (resp. outdegree 1). By the above formula on the cycle rank we have $\beta_1(\mathcal{R}(f)) = r$. We proceed as in the proof of [23, Proposition 6.2]. By means of combinatorial modifications we move up (resp. move down) all vertices of degree 2 in $\mathcal{R}(g_-)$ (in $\mathcal{R}(g_+)$). Then by applying modification (4) to $f|_{Q_-} = g_-$ and (5) to $f|_{Q_+} = g_+$ we can obtain a simple Morse function on W whose Reeb graph is homeomorphic to the initial graph admissible for (W, W_-, W_+) with the desired correspondence between N_i and its edges e_i.

The last statement in the case of surfaces can be obtained similarly with some additional effort, or it follows from Theorem 4.15 (Theorem 4.6 is not used in the proof of Theorem 4.15 for surfaces). ∎

Remark 4.7. For $\dim W = 2$ note that the components of $Q_\pm$ can be either orientable or non-orientable, even if W is non-orientable. Since on an orientable component a simple Morse function has a Reeb graph with maximum cycle rank, simplicity of a Morse function in the theorem is also excluded if W is a non-orientable surface – the components of $\mathcal{R}(f|_{G_\pm})$ then could not be trees.

Using the above theorem we may easily prove the last part of Proposition 2.8. If $\mathcal{N}$ is regular and $\varphi_{\mathcal{N}}$ is surjective, then by Theorem 4.6 the induced epimorphism $\varphi_{\mathcal{N}}$ factorizes through a Reeb epimorphism of rank

$r' = r - |\pi_0(W|\mathcal{N})| + 1$. Since $r \leq r'$, this implies that $|\pi_0(W|\mathcal{N})| \leq 1$, so $\mathcal{N}$ is independent.

This theorem can also be used to easily prove the following known fact for orientable surfaces.

COROLLARY 4.8. *Any epimorphism* $\varphi\colon \pi_1(\Sigma_g) \to F_r$ *factorizes through an epimorphism* $\pi_1(\Sigma_g) \to F_g$.

Proof. By Theorem 4.6, φ factorizes through the Reeb epimorphism of a simple Morse function $f\colon \Sigma_g \to \mathbb{R}$, whose rank is equal to $\beta_1(\mathcal{R}(f))$. By [3] (cf. [22]) the Reeb graph of a simple Morse function on Σ_g always has cycle rank g, so $\beta_1(\mathcal{R}(f)) = g$. ∎

In fact, since any two epimorphisms $\pi_1(\Sigma_g) \to F_g$ are strongly equivalent by Theorem 3.1, for a fixed $\psi\colon \pi_1(\Sigma_g) \to F_g$ any $\varphi\colon \pi_1(\Sigma_g) \to F_r$ factorizes through $\psi \circ \eta$ for some $\eta \in \mathrm{Aut}(\pi_1(\Sigma_g))$.

4.2. Reeb number of manifolds with boundary. The *Reeb number* $\mathcal{R}(M)$ of a closed manifold M was an object of study in [22, 23] and without using this name in [3, 8, 9]. It is defined as the maximum cycle rank among the Reeb graphs of functions with finitely many critical points on M. By Theorem 4.1 ([23, Theorem 5.2]) and [22, Lemma 3.5] we have $\mathcal{R}(M) = \mathrm{corank}(\pi_1(M))$ (see also [9]).

For a compact manifold W, possibly with boundary, following Cornea [5] we define $C(W)$ to be the maximum number of connected components in a proper, 2-sided submanifold N of W such that $W \setminus N$ is connected. In other words, it is the maximum size of an independent and regular system in W. It is clear by Theorem 4.1 that $\mathcal{R}(M) = C(M)$ for a closed manifold M.

The following fact was proven by Jaco [15] for combinatorial manifolds. Cornea announced only inequality $C(W) \geq \mathrm{corank}(\pi_1(W)) - |\pi_0(\partial W)| + 1$ if $\partial W \neq \varnothing$, but the theorem also holds in the smooth category.

THEOREM 4.9. $C(W) = \mathrm{corank}(\pi_1(W))$.

Proof. If there is an independent and regular system $\mathcal{N}$ of size $k = C(W)$, then the induced homomorphism $\varphi_{\mathcal{N}}$ is onto F_k, so

$$C(W) \leq \mathrm{corank}(\pi_1(W)).$$

On the other hand, any epimorphism onto the free group of rank $\mathrm{corank}(\pi_1(W))$ is by Theorem 2.10 induced by a regular and independent system, so $C(W) = \mathrm{corank}(\pi_1(W))$. ∎

Now, we extend the definition of the Reeb number to manifolds with boundary. First, define $\mathcal{R}(W, W_-, W_+)$, where $\partial W = W_- \sqcup W_+$, to be the maximum cycle rank among the Reeb graphs of smooth functions with finitely many critical points on the smooth triad (W, W_-, W_+). By [22, Lemma 3.5] applied to smooth triads this maximum is attained by simple Morse functions.

Proposition 4.10. *$\mathcal{R}(W, W_-, W_+)$ is equal to the maximum size of an independent and regular system without boundary in W. Thus it does not depend on the partition $\partial W = W_- \sqcup W_+$.*

Proof. By Theorem 4.6, if $\mathcal{N}$ is a regular and independent system without boundary in W of size r, then $\varphi_{\mathcal{N}}$ factorizes through the Reeb epimorphism of rank $|\pi_0(\mathcal{N})| - |\pi_0(W|\mathcal{N})| + 1 = r$, which gives an inequality in one direction. On the other hand, if f is a simple Morse function on (W, W_-, W_+) such that $\mathcal{R}(f)$ has cycle rank $\mathcal{R}(W, W_-, W_+)$, then the components of the level sets of f corresponding to the edges of $\mathcal{R}(f)$ outside some spanning tree form a regular and independent system of hypersurfaces in W of size $\mathcal{R}(W, W_-, W_+)$. ∎

Definition 4.11. For a compact manifold W with boundary we define its *Reeb number* as $\mathcal{R}(W) := \mathcal{R}(W, W_-, W_+)$ for any $\partial W = W_- \sqcup W_+$.

It is obvious that $\mathcal{R}(W) \leq C(W)$.

Remark 4.12. Note that $\mathcal{R}(W)$ can be defined as the maximum cycle rank among the Reeb graphs of Morse functions on W which are constant on connected components of ∂W. We use triads for simplicity.

Let $\mathrm{Cone}(X) := X \times [0,1]/X \times \{1\}$ denote the cone over the space X. The point corresponding to $X \times \{1\}$ is called the *vertex* of the cone.

For a compact manifold W with the boundary divided into two parts, $\partial W = A \sqcup B$, if $A_1, \dots, A_k$ are all the connected components of A, we define

$$\mathrm{Cone}_A(W) := W \cup_A \bigcup_{i=1}^{k} \mathrm{Cone}(A_i),$$

that is, we glue the cones $\mathrm{Cone}(A_i)$ and W along A. Let v_i be the vertex of $\mathrm{Cone}(A_i)$. Clearly, we may identify

$$\mathrm{Cone}_A(W) \setminus \{v_1, \dots, v_k\} \cong W \setminus A.$$

Hereafter, we denote by $\langle \pi_1(A) \rangle^{\pi_1(W)}$ the normal subgroup of $\pi_1(W)$ generated by the images of $\pi_1(A_i)$ in $\pi_1(W)$ by the homomorphisms induced by the inclusions $A_i \subset W$. By the Seifert–van Kampen theorem,

$$\pi_1(\mathrm{Cone}_A(W)) \cong \pi_1(W)/\langle \pi_1(A) \rangle^{\pi_1(W)}.$$

It is clear that up to isomorphism this group is well-defined for any choice of basepoints.

The following proposition describes properties of an epimorphism onto a free group in terms of its factorization and a system of hypersurfaces, generalizing [34, Proposition 4.2] of Stallings.

PROPOSITION 4.13. *Let W be a compact manifold and $\partial W = A \sqcup B$. Then an epimorphism $\varphi\colon \pi_1(W) \to F_r$ factorizes through $\pi_1(W)/\langle \pi_1(A)\rangle^{\pi_1(W)}$ if and only if it is induced by an independent and regular system $\mathcal{N}$ such that $\mathcal{N} \cap A = \varnothing$.*

Proof. Set $H := \langle \pi_1(A)\rangle^{\pi_1(W)}$. If $\mathcal{N}$ is an independent and regular system such that $\mathcal{N} \cap A = \varnothing$, then clearly the images in $\pi_1(W)$ of loops in A are contained in the kernel of $\varphi_{\mathcal{N}}$, so $\varphi_{\mathcal{N}}$ factorizes through $\pi_1(W)/H$.

Conversely, assume that $\varphi = \psi \circ \eta$, where $\eta\colon \pi_1(W) \to \pi_1(W)/H$ and $\psi\colon \pi_1(W)/H \to F_r$. We proceed as in the proof of Proposition 2.2. Let ψ be induced by $f\colon \mathrm{Cone}_A(W) \to \bigvee_{i=1}^r \mathrm{S}^1_i$ which is a smooth map outside $\{v_1, \ldots, v_k\}$ and the inverse image of the basepoint. Take regular values $a_i \in \mathrm{S}^1_i$ and define

$$N_i = f^{-1}(a_i) \subset \mathrm{Cone}_A(W) \setminus \{v_1, \ldots, v_k\} \cong W \setminus A.$$

Thus $\mathcal{N} = (N_1, \ldots, N_r)$ is a system in $\mathrm{Cone}_A(W)$ which induces ψ such that $\mathcal{N} \cap A = \varnothing$. Clearly, as a system in W, it induces φ. It is easy to check that the procedures in the proofs of Lemma 2.7 and Theorem 2.10 give an independent and regular system $\mathcal{N}'$ inducing φ which also satisfies $\mathcal{N}' \cap A = \varnothing$. ∎

The following theorem is a generalization of [23, Theorem 4.1].

THEOREM 4.14. *For an epimorphism $\varphi\colon \pi_1(W) \to F_r$ the following are equivalent:*

(1) *$\varphi = \varphi_{\mathcal{N}}$ for an independent and regular system $\mathcal{N}$ without boundary;*
(2) *φ factorizes through $\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$;*
(3) *there is a Morse function f (simple if $\dim W \geq 3$) on any smooth triad (W, W_-, W_+) and a spanning tree T in $\mathcal{R}(f)$ such that $\varphi = (p_T \circ q)_\#$, where $q\colon W \to \mathcal{R}(f)$ and $p_T\colon \mathcal{R}(f) \to \mathcal{R}(f)/T = \bigvee_{i=1}^r \mathrm{S}^1$ are quotient maps.*

Thus

$$\mathcal{R}(W) = \mathrm{corank}(\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}).$$

Proof. The equivalence of (1) and (2) follows from the above proposition for $A = \partial W$. If $\varphi = \varphi_{\mathcal{N}}$ for an independent and regular system $\mathcal{N} = (N_1, \ldots, N_r)$ without boundary, then by Theorem 4.6 there is a Morse function f (simple if $\dim W \geq 3$) on (W, W_-, W_+) whose Reeb graph has cycle rank r and all components of $\mathcal{N}$ are components of the same level set $f^{-1}(c)$. Thus the edges corresponding to the components of $\mathcal{N}$ are outside some spanning tree T of $\mathcal{R}(f)$, and so $(p_T \circ q)_\# = \varphi_{\mathcal{N}}$. This proves that (1) implies (3), and the converse is clear.

By Proposition 4.10 we get $\mathcal{R}(W) = \mathrm{corank}\big(\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}\big)$. ∎

4.3. Realization of a system of hypersurfaces as components of level sets of a function. For a further study of Reeb epimorphisms of Morse functions we would like to have prescribed components of level sets of the function corresponding to edges outside a spanning tree of the graph.

THEOREM 4.15. *Let* (W, W_-, W_+) *be a smooth triad, let*

$$W_\pm = W_1^\pm \sqcup \cdots \sqcup W^\pm_{|\pi_0(W_\pm)|}$$

be a decomposition into connected components and consider a regular and independent system $\mathcal{N} = (N_1, \ldots, N_r)$ *of hypersurfaces without boundary in* W*. Let* Γ *be a finite connected graph with good orientation, whose cycle rank is equal to* r *and which is admissible for* (W, W_-, W_+)*. Distinguish vertices* $a_1^\pm, \ldots, a^\pm_{|\pi_0(W_\pm)|}$ *of degree 1 in* Γ*, where all* a_i^- *have indegree* 0 *and all* a_i^+ *have outdegree* 0*. Moreover, take a spanning tree* T *of* Γ *and order the edges outside* T *as* $e_1, \ldots, e_r$*. Then there is a Morse function* $f\colon W \to \mathbb{R}$ *on the triad* (W, W_-, W_+)*, such that* $\mathcal{R}(f)$ *is orientation-preserving homeomorphic to* Γ*, each* N_i *is a component of a level set of* f *which corresponds to the edge* e_i *and each* $W_i^\pm$ *corresponds to* $a_i^\pm$*. Moreover, if* $\dim W \geq 3$ *and the degree of each vertex in* Γ *is no greater than* 3*, then* f *can be taken to be simple.*

Proof. Let Γ' be a tree obtained from Γ by cutting along all edges e_i as in Figure 5. Denote by c_i^- and c_i^+ the vertices of Γ' of outdegree 0 and indegree 0, respectively, obtained by cutting Γ along the edge e_i.

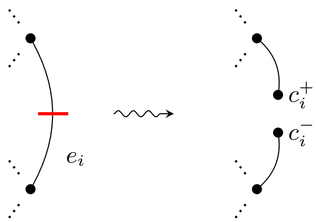

Fig. 5. Cutting along edge.

For $\dim W = 2$ consider the closed surface Q which is formed from $W|\mathcal{N}$ by attaching discs to all its boundary components. By [22, Theorem 5.4 and Remark 5.5] there is a Morse function $g\colon Q \to \mathbb{R}$ whose Reeb graph is orientation-preserving homeomorphic to Γ'.

For each vertex from $a_1^\pm, \ldots, a^\pm_{|\pi_0(W_\pm)|}$ and all $c_i^\pm$ consider a disc D_l in Q centred at the corresponding extremum point of g and whose boundary is a component of a level set of g. Clearly, there is a diffeomorphism

$$h\colon W|\mathcal{N} \to Q \setminus \bigcup_l \operatorname{Int} D_l$$

and by homogeneity we may assume that it maps $W_i^{\pm}$ (resp. the circle $P_{\pm 1}(N_i)$) to the boundary of D_l corresponding to $a_j^{\pm}$ (resp. corresponding to $c_i^{\pm}$). Thus we define a Morse function $f\colon W|\mathcal{N} \to \mathbb{R}$ by $f = g \circ h$. Obviously, we can rescale f to assume that its value on $P_{-1}(N_i)$ is smaller than its value on $P_{+1}(N_i)$ for each i. Thus we may extend f from $P(\mathcal{N})$ to a Morse function on W whose Reeb graph, by construction, is orientation-preserving homeomorphic to Γ, and N_i corresponds to the edge e_i for each i. Again, it can be rescaled to be a function on the triad (W, W_-, W_+).

Now, let $\dim W \geq 3$. We proceed as in the proof of Theorem 4.5 (i.e. [23, Theorem 6.4]) with the difference that the manifold has a boundary. However, we deal with the simplest case when the graph is a tree. Steps 1 and 2 of the proof in [23] reduce the problem to the case when Γ' has only vertices of degrees 1 and 3 and is *primitive*, i.e. there is no oriented (directed) path from a vertex with indegree 2 to a vertex with outdegree 2. By Theorem 4.6 for the empty system of hypersurfaces in $W|\mathcal{N}$ there is a simple Morse function $g\colon W|\mathcal{N} \to \mathbb{R}$ whose Reeb graph is the initial graph admissible for the triad $(W|\mathcal{N}, W_- \sqcup P_1(\mathcal{N}), W_+ \sqcup P_{-1}(\mathcal{N}))$. We may increase (or decrease if necessary) the number of vertices of degree 1 by using modifications (8) and (9), so that $\mathcal{R}(g)$ is the initial graph with the same numbers of vertices of indegree 0 and vertices of outdegree 0 as Γ'. Moreover, since Γ' and $\mathcal{R}(g)$ are primitive trees, this forces the same numbers of vertices of indegree 2 and of outdegree 2. Thus it suffices to appropriately rearrange vertices of degree 3 to produce Γ' from $\mathcal{R}(g)$.

For this purpose, we introduce a combinatorial modification (13) of Figure 6, which, loosely speaking allows us to transfer a vertex v of indegree 2 to the other edge outgoing from a vertex w of outdegree 2 adjacent to v. The analogous modification for graphs with opposite orientations is labelled (14). Since the graphs Γ' and $\mathcal{R}(g)$ are primitive, small neighbourhoods of two adjacent vertices of degree 3 look like in modifications (4), (5), (13) or (14). Note that these modifications are two-sided, i.e. they work in both directions. We will show that Γ' can be transformed to the initial graph by using them, and so $\mathcal{R}(g)$ can be transformed to Γ' obtaining a simple Morse function $f'\colon W|\mathcal{N} \to \mathbb{R}$ whose Reeb graph is orientation-preserving homeomorphic to Γ'. Moreover, previously rearranging vertices in the initial graph $\mathcal{R}(g)$ using modifications (4) and (5) we may ensure that the distinguished vertices of degree 1 in the homeomorphic graphs Γ' and $\mathcal{R}(f')$ correspond to appropriate components of $\partial(W|\mathcal{N})$. Again, as in the case of surfaces, we can rescale f' to assume that its value on $P_{-1}(N_i)$ is smaller than its value on $P_{+1}(N_i)$ for each i and extend f' on $P(\mathcal{N})$ to a Morse function f on W whose Reeb graph, by construction, is orientation-preserving homeomorphic to Γ, and which satisfies all the desired conditions.

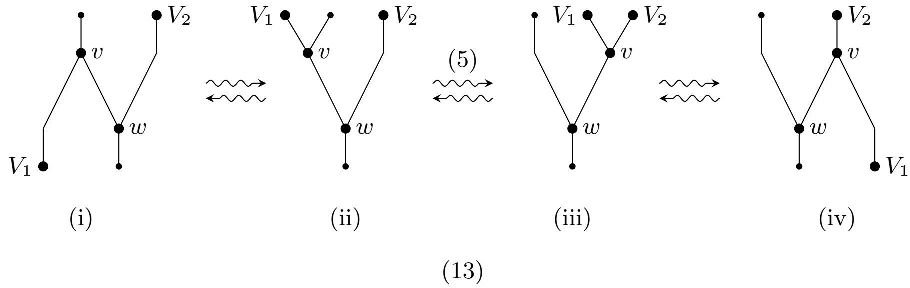

Fig. 6. The combinatorial modification (13) of the Reeb graph of a simple Morse function. It transfers a vertex v of degree 3 and indegree 2 to the other outgoing edge from a vertex w of degree 3 and outdegree 2. The situation with opposite directions of graphs leads to modification (14).

Thus, take an oriented path τ in Γ' between vertices of degree 1 and with maximum number of vertices of degree 3. Assume that there is a vertex v of degree 3 outside τ, which is adjacent to a vertex w of degree 3 on τ. Without loss of generality assume that w has outdegree 2. If v also has outdegree 2, then use modification (5) to move v on the path τ. For the second case when v has indegree 2, first use (5) to move w up along τ as far as possible. Then apply (13) to v and w to move v on τ. The resulting graph is still primitive and has more vertices of degree 3 on τ. Repeating this procedure as long as there is a vertex of degree 3 outside τ we obtain the initial graph.

It only remains to describe modification (13). Suppose $g\colon Q \to \mathbb{R}$ is a simple Morse function on a triad (Q, Q_-, Q_+) such that $\mathcal{R}(g)$ is isomorphic to the graph in Figure 6(i). Let v and w be adjacent vertices of $\mathcal{R}(g)$ as in the figure and let v_1 be a vertex of degree 1 in $\mathcal{R}(g)$ adjacent to v and corresponding to the submanifold V_1 of Q, so V_1 is a single point (an extremum point of g) or a component of Q_-. If it is an extremum point, then use modification (8) and then (9) to obtain the graph from Figure 6(ii). Now assume V_1 is a component of Q_-. First, rescale the function along the edge between v and v_1 so that the value at v_1 is greater than the value at w. Take a neighbourhood U of this edge containing no other vertices than v and v_1 such that the corresponding submanifold S of Q forms a triad $(S, V_1 \sqcup S_1, S_2)$. Change g on S by defining a new simple and ordered Morse function on $(S, S_1, V_1 \sqcup S_2)$ with no critical points being extremum points. By [23, Proposition 3.2] this produces a function with Reeb graph as in (ii). Modification (5) leads to case (iii), and an analogous argument allows us to pass to (iv). Finally, modification (14) for a simple Morse function g can be obtained from (13) for the function $-g$. ∎

Remark 4.16. Note that in the case of surfaces the function f constructed in the proof of the above theorem can be simple if $W|\mathcal{N}$ is non-orientable or has genus 0 since then the function g in the proof can be taken to be simple (see [22]). This is the case for example if W is non-orientable of odd genus.

4.4. Realization of epimorphisms onto free groups as Reeb epimorphisms. The previous theorem allows us to realize an epimorphism $\pi_1(W) \to \pi_1(\Gamma)$ as the Reeb epimorphism of a Morse function on (W, W_-, W_+). Note that by Theorem 4.14 we need to assume that the original epimorphism factorizes through $\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$.

THEOREM 4.17. *Let Γ be a finite connected graph with good orientation, let (W, W_-, W_+) be a smooth triad and assume that $\varphi\colon \pi_1(W) \to \pi_1(\Gamma)$ is an epimorphism factorizing through $\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$. Then there is a Morse function $f\colon W \to \mathbb{R}$ on (W, W_-, W_+) such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ and under this identification the Reeb epimorphism of f is equal to φ. Moreover, if W is not a surface and the degree of each vertex in Γ is no greater than 3, then f can be taken simple.*

Proof. Take a spanning tree T of Γ and label the edges outside T by $e_1, \dots, e_r$. Take the quotient map $p_T\colon \Gamma \to \Gamma/T = \bigvee_{i=1}^r \mathrm{S}^1$ which maps e_i onto the ith circle. By Theorem 4.14 the epimorphism $(p_T)_\# \circ \varphi$ is induced by an independent and regular system $\mathcal{N} = (N_1, \dots, N_r)$ of hypersurfaces without boundary in W of size r. By Theorem 4.15 there is a Morse function $f\colon W \to \mathbb{R}$ whose Reeb graph is Γ up to vertices of degree 2, and N_i corresponds to e_i. If $q\colon W \to \mathcal{R}(f)$ is the quotient map, then by construction $(p_T)_\# \circ q_\# = \varphi_{\mathcal{N}} = (p_T)_\# \circ \varphi$. Since $(p_T)_\#$ is an isomorphism, $\varphi = q_\#$ is the Reeb epimorphism of f. ∎

COROLLARY 4.18. *Let $\varphi\colon \pi_1(M) \to \pi_1(\Gamma)$ be an epimorphism, where M is a closed manifold and Γ is a finite connected graph with good orientation. Then there is a Morse function $f\colon M \to \mathbb{R}$ such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ and under this identification the Reeb epimorphism of f is equal to φ.* ∎

REMARK 4.19. The Reeb epimorphism of f does not represent a unique epimorphism $\pi_1(W) \to \pi_1(\Gamma)$ in general, because it depends on the homeomorphism between Γ and $\mathcal{R}(f)$. It is unique for oriented graphs such that the identity map is the only orientation-preserving automorphism. Otherwise, Theorem 4.15 provides a more rigorous representation of a Reeb epimorphism if some additional data is given. For instance, assume that there are distinguished edges $e_1, \dots, e_r$ outside a spanning tree T of Γ and a regular and independent system $\mathcal{N}$ of hypersurfaces inducing $(p_T)_\# \circ \varphi$. Then the condition that each N_i corresponds to e_i implies the uniqueness of an epimorphism $\pi_1(W) \to \pi_1(\Gamma)$ represented by the Reeb epimorphism of f.

REMARK 4.20. Independently, O. Saeki [30] has proven a similar result for a closed manifold M, which provides a representation of an epimorphism $\varphi\colon \pi_1(M) \to \pi_1(\Gamma)$, for any finite graph Γ without loops, as the Reeb epimorphism of a smooth function with finitely many critical values. In addition, since the function has degenerate critical points, it can realize Γ as the Reeb graph up to isomorphism of graphs. However, the number of vertices of degree 2 in the Reeb graph of a Morse function cannot be arbitrary (see for instance [22, Theorem 5.6]), thus we need to ignore them in our construction of a Morse function. Moreover, the above theorem together with Theorem 4.15 provides a more rigorous representation of φ as the Reeb epimorphism, since we control the components of level sets of the function. This may be crucial in applications of Reeb epimorphisms, e.g. when dealing with topological conjugacy of Morse functions (see Section 4.6). Finally, our result also covers manifolds with boundary.

We showed that for manifolds of dimension at least 3 any epimorphism $\pi_1(M) \to \pi_1(\Gamma)$ can be represented as the Reeb epimorphism of a simple Morse function provided that Γ satisfies some necessary conditions: it has a good orientation and the degrees of its vertices are ≤ 3. Now, let us investigate when in Theorem 4.17 one can take a simple Morse function in the case of surfaces.

LEMMA 4.21. *Let Σ be a closed surface of Euler characteristic $\chi(\Sigma) = 2 - 2g$, so it is orientable of genus g or non-orientable of genus $2g$, and let $f\colon \Sigma \to \mathbb{R}$ be a Morse function such that the degree of each vertex in $\mathcal{R}(f)$ is no greater than 3. Then $\beta_1(\mathcal{R}(f)) = g$ if and only if f is $\mathcal{R}$-simple and $\mathcal{R}(f)$ has no vertices of degree 2.*

Proof. Let V and E be the sets of vertices and edges of $\mathcal{R}(f)$, respectively. Then $\beta_1(\mathcal{R}(f)) = |E| - |V| + 1$. Moreover, if k_i is the number of critical points of f of index i, then $\chi(\Sigma) = k_0 - k_1 + k_2$ and the number of vertices of degree 1 in $\mathcal{R}(f)$ is $k_0 + k_2$. Let Δ_2 and Δ_3 be the numbers of vertices of degree 2 and 3, respectively. Note that $2|E| = \sum_{v\in V} \deg(v) = k_0 + k_2 + 2\Delta_2 + 3\Delta_3$ by Euler's handshaking lemma. Combining these equalities we obtain

$$2(\beta_1(\mathcal{R}(f)) - g) = 2|E| - 2|V| + \chi(\Sigma) = \Delta_3 - k_1.$$

Thus $\beta_1(\mathcal{R}(f)) = g$ if and only if $k_1 = \Delta_3$. However, $k_1 \geq \Delta_2 + \Delta_3$, so $k_1 = \Delta_3$ if and only if $\Delta_2 = 0$ and each vertex of $\mathcal{R}(f)$ corresponds to a single critical point of f, so f is $\mathcal{R}$-simple. ∎

Note that a simple Morse function on a non-orientable surface of odd genus always has a vertex of degree 2 in its Reeb graph.

We know that the Reeb graph of a simple Morse function on a closed orientable surface of genus g has cycle rank g. For a non-orientable surface Σ one can construct simple Morse functions with Reeb graphs having arbitrary cycle rank between 0 and $\mathcal{R}(\Sigma)$ (cf. Theorem 4.1). However, simple Morse functions on non-orientable surfaces of even genus also have a feature which can be described in terms of Reeb epimorphisms.

PROPOSITION 4.22. *Let Γ be a finite connected graph with good orientation such that $\beta_1(\Gamma) < g$. Then there is a unique strong equivalence class Ξ of epimorphisms $\pi_1(S_{2g}) \to \pi_1(\Gamma)$ such that for any simple Morse function $f\colon S_{2g} \to \mathbb{R}$ with $\mathcal{R}(f) = \Gamma$ its Reeb epimorphism belongs to Ξ.*

Proof. Let $r := \beta_1(\mathcal{R}(f))$ and $\mathcal{N} = (N_1, \dots, N_r)$ be an independent and regular system of hypersurfaces in S_{2g} which are connected components of level sets of f and which correspond to edges outside some spanning tree of $\mathcal{R}(f)$. We claim that $S_{2g}|\mathcal{N}$ is non-orientable. Indeed, assume it is orientable. Note that $f|_{S_{2g}|\mathcal{N}}$ is a simple Morse function and its Reeb graph $\mathcal{R}(f|_{S_{2g}|\mathcal{N}})$ is a tree. Therefore $S_{2g}|\mathcal{N}$ has genus 0 as a surface with boundary, i.e. it is a sphere with discs removed. This implies that $r = g$, a contradiction.

Therefore, as in the proof of Theorem 3.15, any two Reeb epimorphisms of simple Morse functions $S_{2g} \to \mathbb{R}$ with Reeb graph Γ are strongly equivalent and Ξ is represented by a system of hypersurfaces whose complement is non-orientable. ∎

Note that in fact the above proposition is also true for any non-orientable surface S_{2g+1} of odd genus since by Theorem 3.1 there is only one strong equivalence class of epimorphisms $\pi_1(S_{2g+1}) \to \pi_1(\Gamma)$.

The following corollary follows from Remark 4.16, Theorem 4.17, Lemma 4.21 and Proposition 4.22.

COROLLARY 4.23. *Let Γ be a finite connected graph with good orientation such that the degree of each vertex is no greater than 3, let Σ be a closed surface and $\mathcal{R}_{\mathrm{epi}}(\Sigma)$ be the set of all Reeb epimorphisms of simple Morse functions on Σ. Take an epimorphism $\psi\colon \pi_1(\Sigma) \to \pi_1(\Gamma)$.*

- *If Σ is orientable of genus g, then $\psi \in \mathcal{R}_{\mathrm{epi}}(\Sigma)$ if and only if $\beta_1(\Gamma) = g$.*
- *If Σ is non-orientable of odd genus, then $\psi \in \mathcal{R}_{\mathrm{epi}}(\Sigma)$.*
- *If Σ is non-orientable of even genus $2g$, then $\psi \in \mathcal{R}_{\mathrm{epi}}(\Sigma)$ if and only if either $\beta_1(\Gamma) = g$, or $\beta_1(\Gamma) < g$ and ψ belongs to a unique strong equivalence class Ξ of epimorphisms $\pi_1(\Sigma) \to \pi_1(\Gamma)$ represented by systems of hypersurfaces whose complements are non-orientable.* ∎

4.5. Extendability of independent systems of hypersurfaces. Let $\mathcal{N} = (N_1, \dots, N_r)$ be an independent and regular system of hypersurfaces in W. We say that $\mathcal{N}$ is *extended* by a system $\mathcal{N}'$ if $\mathcal{N}'$ is also a regular and independent system such that $\mathcal{N} \subset \mathcal{N}'$ and their framings determine the same orientation of the normal bundle of $\mathcal{N}$ in W.

Proposition 4.24. *Let $\mathcal{N}$ be an independent and regular system without boundary in W of size r. Then*

$$\operatorname{corank}\bigl(\pi_1(W)/\langle\pi_1(\mathcal{N})\rangle^{\pi_1(W)}\bigr)=\operatorname{corank}\bigl(\pi_1(W|\mathcal{N})/\langle\pi_1(\partial P(\mathcal{N}))\rangle^{\pi_1(W|\mathcal{N})}\bigr)+r$$

and it is the maximum size of an independent and regular system of hypersurfaces without boundary in W which extends $\mathcal{N}$. In particular, for a closed manifold M we get

$$\mathcal{R}(M|\mathcal{N}) = \operatorname{corank}\bigl(\pi_1(M)/\langle\pi_1(\mathcal{N})\rangle^{\pi_1(M)}\bigr) - r.$$

Proof. Suppose we have a 2-sided connected submanifold $N = N_1$ without boundary with product neighbourhood $P(N)$ in a compact manifold W such that $W|N$ is connected. Thus W is obtained from $W|N$ be gluing the components of $\partial(W|N) = P_{-1}(N) \sqcup P_1(N) \sqcup \partial W$ using a diffeomorphism $h\colon P_{-1}(N) \to P_1(N)$. It is known (see [20, Chapter IV]) that $\pi_1(W)$ is the HNN extension of $\pi_1(W|N)$ relative to $h_\#\colon H_{-1} \to H_1$, where $H_t = \pi_1(P_t(N)) < \pi_1(W|N)$. In other words, $\pi_1(W)$ is the free product $\pi_1(W|N) * \mathbb{Z}$ divided by the normal closure of $\{u\omega u^{-1}h_\#(\omega)^{-1} : \omega \in H_{-1}\}$, where u is the stable letter which generates $\mathbb{Z}$. The group $\pi_1(W|N)$ is a subgroup of $\pi_1(W)$ and the groups H_{-1} and H_1 are conjugate in $\pi_1(W)$. In fact, the normal subgroup $\pi_1(N)^{\pi_1(W)}$ in $\pi_1(W)$ is equal to $\langle H_{-1}\rangle^{\pi_1(W)} = \langle H_1\rangle^{\pi_1(W)} = \langle H_{-1}, H_1\rangle^{\pi_1(W|N)}$. Therefore $\pi_1(W)/\pi_1(N)^{\pi_1(W)}$ is isomorphic to

$$\pi_1(W)/\langle H_{-1}, H_1\rangle^{\pi_1(W|N)} \cong \pi_1(W|N)/\langle H_{-1}, H_1\rangle^{\pi_1(W|N)} * \mathbb{Z}.$$

This gives the first part of the proposition for $r = 1$ since $\operatorname{corank}(G * H) = \operatorname{corank}(G) + \operatorname{corank}(H)$ (see [5]). The general case follows by considering all submanifolds N_i simultaneously and HNN extension with r stable letters.

The description of the number on both sides of the first equality of the statement follows by Proposition 4.13. ∎

Example 4.25. The Reeb number of a compact manifold W with nonempty boundary can be smaller than $C(W)$. For example, let $M = \Sigma \times \mathrm{S}^1$, where Σ is a closed surface with $\chi(\Sigma) = 2 - k \le 0$. Then

$$\mathcal{R}(M) = \operatorname{corank}(\pi_1(\Sigma) \times \mathbb{Z}) = \max(\lfloor k/2\rfloor, 1) = \lfloor k/2\rfloor \ge 1$$

(see [7, 22]). Let $\mathcal{N} = (\Sigma \times \{1\})$ and $W := M|\mathcal{N} \cong \Sigma \times [0, 1]$. Then $C(W) = \operatorname{corank}(\pi_1(\Sigma)) = \lfloor k/2\rfloor$. However, $\mathcal{R}(W) = \operatorname{corank}((\pi_1(\Sigma)\times\mathbb{Z})/\pi_1(\Sigma)) - 1 = 0$ by Proposition 4.24. Therefore $\mathcal{N}$ cannot be extended to a regular and independent system of hypersurfaces in M of a larger size.

Example 4.26. Let $\Sigma_{g,h}$ and $S_{g,h}$ denote, respectively, an orientable and non-orientable surface of genus g with $h \ge 1$ open discs removed. Then

- $\mathcal{R}(\Sigma_{g,h}) = g$ and $C(\Sigma_{g,h}) = 2g + h - 1$,
- $\mathcal{R}(S_{g,h}) = \lfloor g/2\rfloor$ and $C(S_{g,h}) = g + h - 1$.

Indeed, by Proposition 4.13 we have $\mathcal{R}(W) = \operatorname{corank}(\pi_1(\operatorname{Cone}_{\partial W}(W)))$. This gives the desired Reeb numbers since $\operatorname{Cone}_{\partial \Sigma_{g,h}}(\Sigma_{g,h}) \cong \Sigma_g$ and $\operatorname{Cone}_{\partial S_{g,h}}(S_{g,h}) \cong S_g$. The calculation of $C(W)$ follows from Theorem 4.9 and the fact that $\pi_1(\Sigma_{g,h}) = F_{2g+h-1}$ and $\pi_1(S_{g,h}) = F_{g+h-1}$.

Corollary 4.27. *Any independent, regular system $\mathcal{N}$ of hypersurfaces without boundary in a compact surface Σ can be extended to such a system of size $\mathcal{R}(\Sigma)$.*

Proof. Let r be the size of $\mathcal{N}$. Since Σ is two-dimensional, $\mathcal{N}$ consists of circles in Σ. By the classification of compact surfaces, if $\Sigma = \Sigma_{g,h}$ then $\Sigma|\mathcal{N} \cong \Sigma_{g-r,h+2r}$, and if $\Sigma = S_{g,h}$ then $\Sigma|N \cong S_{g-2r,h+2r}$ or $\Sigma|N \cong \Sigma_{g/2-r,h+2r}$ (the latter case can only occur if g is even). By the above example, in all these cases we have $\mathcal{R}(\Sigma|\mathcal{N}) = \mathcal{R}(\Sigma) - r$, so $\mathcal{N}$ can be extended to the size $\mathcal{R}(\Sigma)$. ∎

4.6. Topological conjugacy of Morse functions. Now, we focus on relations between Reeb epimorphisms and Morse functions. The main issue is that in general different Reeb epimorphisms have different codomains. Although the fundamental groups of Reeb graphs with the same cycle ranks are isomorphic, they are not isomorphic in the canonical way. However, this ambiguity can be omitted for oriented graphs for which the identity map is the only orientation-preserving automorphism (see Remark 4.19).

Let us restrict our attention to the case of a closed manifold M. Functions f_1 and f_2 on M are called *topologically conjugate* if there are a self-homeomorphism $h\colon M \to M$ and an orientation-preserving homeomorphism $\eta\colon \mathbb{R} \to \mathbb{R}$ such that $f_1 = \eta \circ f_2 \circ h$. In this case h induces a unique homeomorphism $\overline{h}\colon \mathcal{R}(f_1) \to \mathcal{R}(f_2)$ such that $\overline{h} \circ q_1 = q_2 \circ h$ and $\overline{f}_1 = \eta \circ \overline{f}_2 \circ \overline{h}$.

Lemma 4.28. *If f_1 and f_2 are simple Morse functions topologically conjugate by h, then their Reeb graphs are isomorphic through $\overline{h}$.*

Proof. If there were a vertex with degree 2 in $\mathcal{R}(f_1)$ mapped by $\overline{h}$ to a point on an edge in $\mathcal{R}(f_2)$, then some smooth product triad would be mapped by h^{-1} homeomorphically onto a smooth triad with exactly one non-degenerate critical point. This is a contradiction by comparing the Euler characteristics. ∎

$$\begin{tikzcd}
M \arrow[r, "q_1"] \arrow[d, "h"'] & \mathcal{R}(f_1) \arrow[r, "\overline{f}_1"] \arrow[d, dashed, "\overline{h}"'] & \mathbb{R} \arrow[d, "\eta"] \\
M \arrow[r, "q_2"] & \mathcal{R}(f_2) \arrow[r, "\overline{f}_2"] & \mathbb{R}
\end{tikzcd}$$

Fig. 7. Topologically conjugate simple Morse functions have isomorphic Reeb graphs.

We cannot speak directly about equivalence and strong equivalence of Reeb epimorphisms since they have distinct codomains, even if the relevant Reeb graphs are isomorphic. By the diagram in Figure 7 we see that if f_1 and f_2 are topologically conjugate by h, then $\overline{h}_{\#} \circ (q_1)_{\#}$ and $(q_2)_{\#}$ are strongly equivalent. Thus we say that two Reeb epimorphisms

$$\varphi_i \colon \pi_1(M) \to \pi_1(\mathcal{R}(f_i))$$

are *strongly equivalent* if they are strongly equivalent with respect to an isomorphism $k \colon \mathcal{R}(f_1) \to \mathcal{R}(f_2)$, i.e. $k_{\#} \circ \varphi_1$ and φ_2 are strongly equivalent.

The following theorem is a classical result in the theory of Morse functions.

THEOREM 4.29 (Kulinich [18], Sharko [32]; cf. [6]). *Two simple Morse functions on a closed orientable surface Σ are topologically conjugate by $h \colon \Sigma \to \Sigma$ if and only if their Reeb graphs are isomorphic as oriented graphs through $\overline{h}$.*

This theorem allows us to give another proof of a part of Theorem 3.1 for orientable surfaces which uses Reeb graphs. First, for any two epimorphisms $\pi_1(\Sigma_g) \to F_r$ we need to take systems of hypersurfaces which induce them. Then we extend them to systems of maximum size $\mathcal{R}(\Sigma_g) = g$ by Corollary 4.27 and now we can represent the induced epimorphisms $\pi_1(\Sigma_g) \to F_g$ by Reeb epimorphisms of simple Morse functions whose Reeb graphs are the initial graphs (they have no vertices of degree 2). Thus by Theorem 4.29 the isomorphism of the Reeb graphs is induced by a self-homeomorphism of Σ_g that maps one system to the other and gives a strong equivalence between the epimorphisms.

REMARK 4.30. Theorem 3.1 for non-orientable surfaces of even genus shows that the analogue of Theorem 4.29 does not hold for them in general. In fact, we may construct two simple Morse functions on S_{2g} whose Reeb graphs are isomorphic, but their Reeb epimorphisms are not strongly equivalent. Thus we must endow Reeb graphs with additional information.

Lychak–Prishlyak [19] equipped the Reeb graphs of simple Morse functions on non-orientable surfaces with signs $+$ or $-$ near vertices of degree 3, which come from the compatibility of orientations when attaching handles in the corresponding critical levels. More precisely, each sign is assigned to a pair of incident edges (one incoming and one outgoing) at a vertex v of degree 3. For the procedure of assignment of signs we refer the reader to [19]. Two *Reeb graphs with signs* are called *equivalent* if they are isomorphic and it is possible to obtain identical signs by the following operation: for a given edge, reverse all signs assigned to it.

THEOREM 4.31 (Lychak–Prishlyak [19]). *Two simple Morse functions on a closed non-orientable surface are topologically conjugate if and only if their Reeb graphs with signs are equivalent.*

LEMMA 4.32. *Let Γ be a graph with good orientation whose vertices have degrees* 1 *or* 3. *Then there are exactly* 2^r *equivalence classes of graphs with signs, where* $r = \beta_1(\Gamma)$.

Proof. First, look at the case of the *canonical graph*, presented in Figure 8(a). It is an easy exercise to show that any such graph with signs is equivalent to a configuration of the form shown in the figure, where in the r places of "?" we can put arbitrary signs. Moreover, all such 2^r configurations are pairwise non-equivalent. The same can be shown for the initial graph with configurations of signs as in Figure 8(b).

Now, note that by [6, 23] there is a sequence of modifications of Reeb graphs which transform Γ to the canonical graph. It is left to the reader to check that for graphs with some vertices of degree 1 or 3, these modifications do not change the number of non-equivalent configurations of signs. ∎

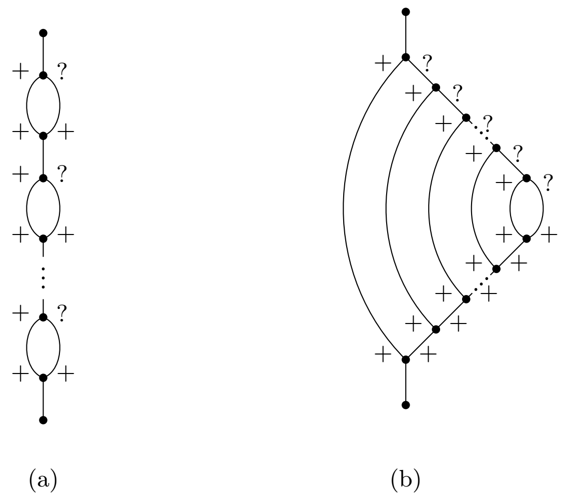

Fig. 8. The canonical and initial graphs with signs.

REMARK 4.33. For a graph with some vertices of degree 2 the number of non-equivalent configurations of signs may vary depending on the position of these vertices in the graph. Moreover, note that the Reeb graph of a simple Morse function on a non-orientable surface of odd genus always has a vertex of degree 2. The same is true if a surface is non-orientable of even genus $2g$ and the Reeb graph has cycle rank smaller than g. In view of Lemma 4.21 it is only reasonable to consider the case of simple Morse functions on a non-orientable surface of genus $2g$ whose Reeb graphs have cycle rank g.

Theorem 4.34. *Let $f_1, f_2\colon S_{2g} \to \mathbb{R}$ be simple Morse functions on a closed non-orientable surface of genus $2g$ such that $\beta_1(\mathcal{R}(f_1)) = g = \beta_1(\mathcal{R}(f_2))$. Then they are topologically conjugate if and only if their Reeb graphs are isomorphic and their Reeb epimorphisms are strongly equivalent.*

Proof. For a manifold M and a graph Γ, denote by $\mathcal{M}(M,\Gamma)$ the set of all simple Morse functions f on M whose Reeb graphs are isomorphic to Γ, and by $\mathcal{M}(M,\Gamma)/\mathrm{t.c.}$ the set of their topological conjugacy classes. Moreover, let $\mathrm{Signs}(\Gamma)$ be the set of equivalence classes of configurations of signs in Γ. By Theorem 4.31 the natural map $\mathcal{M}(S_{2g},\Gamma)/\mathrm{t.c.} \to \mathrm{Signs}(\Gamma)$ associating with a function the configuration of signs in its Reeb graph as in [19] is injective. Now, suppose Γ has some vertices of degrees 1 and 3 and $\beta_1(\Gamma) = g$. Then for any configuration of signs in Γ except the one with pluses only, we can produce a simple Morse function on S_{2g} which realizes it (see [22] for the procedure). The configuration of signs with pluses only leads to a function on an orientable surface. By Lemma 4.32 the set $\mathrm{Signs}(\Gamma)$ has 2^r elements, so $\mathcal{M}(S_{2g},\Gamma)/\mathrm{t.c.}$ has $2^r - 1$ elements.

Now, by Theorem 4.17 and Corollary 4.23 the map from $\mathcal{M}(S_{2g},\Gamma)/\mathrm{t.c.}$ to the set of strong equivalence classes of Reeb epimorphisms of functions from $\mathcal{M}(S_{2g},\Gamma)$ is surjective. Since the latter set also has $2^r - 1$ elements by Theorem 3.1, the map is a bijection and the theorem is proved. ∎

5. Final remarks. The approach we presented could be used to study C^1-functions with finitely many critical points, since it assigns to such a function a combinatorial invariant (the Reeb graph) or a non-commutative algebraic invariant (the Reeb epimorphism). Conversely, it gives a geometric description of the set of homomorphisms from the fundamental group of a compact manifold into a finitely generated free group. As of now, the following problems seem to be the next natural steps in such studies.

1. One of the problems we wish to focus on in future work is extendability of independent systems of hypersurfaces (cf. Section 4.5). For an independent and regular system $\mathcal{N}$ let us denote by $E(\mathcal{N})$ the maximum size of an independent and regular system which extends $\mathcal{N}$, and by $F(\varphi_{\mathcal{N}})$ the maximum rank of a free group onto which there is an epimorphism which $\varphi_{\mathcal{N}}$ factorizes through. It is clear that $E(\mathcal{N}) \le F(\varphi_{\mathcal{N}})$. We would like to know for which closed manifolds M equality holds for any independent system of hypersurfaces in M. Using Theorem 2.10 it can be shown that $\mathcal{N}$ is framed cobordant to a system which can be extended to size $F(\varphi_{\mathcal{N}})$, but we do not know whether $\mathcal{N}$ itself can be extended. In particular, we wish to investigate the case when all numbers $F(\varphi_{\mathcal{N}})$ are equal to $\mathrm{corank}(\pi_1(M))$. This problem can also be seen more algebraically. By Proposition 4.24 the extendability of

$\mathcal{N}$ to the size of $\operatorname{corank}(\pi_1(M))$ is equivalent to the equality

$$\operatorname{corank}(\pi_1(M)) = \operatorname{corank}\big(\pi_1(M)/\langle\pi_1(\mathcal{N})\rangle^{\pi_1(M)}\big).$$

Moreover, since $E(\mathcal{N}) = \mathcal{R}(M|N)+r$, where r is the size of $\mathcal{N}$, this problem is related to the computability of the corank and Reeb number.

2. The last section shows that the relations between conjugacy of simple Morse functions and strong equivalence of Reeb epimorphisms are quite complicated even for surfaces. These relations deserve a careful investigation for higher-dimensional manifolds, especially for 3-manifolds. However, in contrast to surfaces, the problem of conjugacy of Morse functions on manifolds of dimension at least 3 is much more difficult due to their more complicated surgery description (see [26]). One approach is to equip Reeb graphs with labels on edges, which correspond to diffeomorphism types of submanifolds corresponding to them. Isomorphism of Reeb graphs which preserves these labels is the first necessary condition for conjugacy of functions. In particular, for orientable 3-manifolds regular level sets are orientable surfaces, so we equip edges only with non-negative integers corresponding to their genera. Thus the problem is how many strong equivalence classes of Reeb epimorphisms of simple Morse functions with fixed Reeb graph with labels are there and how many of them correspond to a single conjugacy class of these functions. In particular, we ask whether there are finitely many such classes.

It should be noted that there are some results regarding genera of regular level sets of Morse functions on closed orientable 3-manifolds. For example, O. Saeki [29, Theorem 6.5] showed that there is a Morse function $f\colon M \to \mathbb{R}$ with maximum genus among labels in $\mathcal{R}(f)$ at most 1 if and only if M is the connected sum of copies of S^3, $\mathrm{S}^2\times\mathrm{S}^1$ and lens spaces. Moreover, N. Kitazawa [17, Theorem 1] described a realization of such a labelled graph as the Reeb graph of a Morse function on a 3-manifold.

3. The structure of the set $\operatorname{Hom}(G, F_r)$ for G a finitely generated group has been intensively studied by several authors by the use of Makanin–Razborov diagrams theory (see [21, 27]). It led to the solution of the Tarski problem on the existence of solution to a system of equations in a free finitely generated group and culminated in Sela's works (cf. [31] and more recent articles of that author; see also [1] for a nice and relatively uncomplicated introduction to this theory). The Makanin–Razborov diagram of a group G consists of its quotients and quotient maps in such a way that every homomorphism $\varphi\colon G \to F_r$ can be *M-R factorized* through some branch of quotients $G \xrightarrow{q_1} L_1 \xrightarrow{q_2} \cdots \xrightarrow{q_k} L_k$, where L_k is a free group. This means that there is an element $\psi \in \operatorname{Hom}(L_k, F_r)$ and modular automorphisms

$\alpha \in \mathrm{Mod}(G)$ and $\alpha_i \in \mathrm{Mod}(L_i)$ for $1 \leq i < k$ such that

$$\varphi = \psi \circ q_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1 \circ q_1 \circ \alpha.$$

The equivalence and strong equivalence relations in $\mathrm{Hom}(G, F_r)$ cannot be derived from Makanin–Razborov diagrams in general. However, these notions are closely related for groups with branches of length 1 in their M-R diagrams, e.g. for surface groups.

The computation of the set $\mathrm{Epi}(\pi_1(M), F_r)$ of epimorphisms $\pi_1(M) \to F_r$ and the set $\mathrm{Epi}(\pi_1(M), F_r)/\simeq$ of their strong equivalence classes may also rely on calculating framed cobordism classes of independent systems of hypersurfaces. By Theorem 2.10 there is a bijection $\mathrm{Epi}(\pi_1(M), F_r) \cong \mathcal{H}_r^{\mathrm{fr}}(M)$, and $\overline{\overline{\Theta}}\colon \mathcal{H}_r^{\mathrm{fr}}(M)/\mathrm{Diff}_\bullet(M) \to \mathrm{Epi}(\pi_1(M), F_r)/\simeq$ is surjective. Moreover, in some cases it is bijective, e.g. for hyperbolic manifolds. More generally, there are bijections

$$\mathrm{Hom}(\pi_1(M), F_r) \cong [M, \textstyle\bigvee^r \mathrm{S}^1] \cong \varOmega_r^{\mathrm{fr},1}(M)$$

between the sets of all homomorphisms $\pi_1(M) \to F_r$, the set of homotopy classes of maps $M \to \bigvee^r \mathrm{S}^1$ and the set $\varOmega_r^{fr,1}(M)$ of all framed cobordism classes of systems of hypersurfaces of size r in M (1 stands for codimension 1). Consequently, any description of the latter set would give information about the corresponding set of homomorphisms.

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