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Mahler coefficients of a certain class of compatible functions on $\mathbb{Z}_p$

by

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Summary. We prove some properties of Mahler coefficients of a class of compatible functions on $\mathbb{Z}_p$. Contractive mappings and isometric transformations are included in this class.

1. Introduction. Let $\mathbb{Z}_p$ be the group of p-adic integers where every $x \in \mathbb{Z}_p$ has the p-adic representation $x = \sum_{i=0}^{\infty} x_i p^i$ with $x_i \in \{0, \dots, p-1\}$ for each $i \geq 0$. The p-adic valuation $\nu(x)$ of x is defined as the least nonnegative integer i such that $x_i > 0$. We recall that the p-adic absolute value $|x|$ of x is given by $|x| = p^{-\nu(x)}$.

Specific classes of functions are described in terms of their Mahler coefficients in many papers [4, 2, 3, 5, 6, 7, 8]. In this work we study the class of compatible functions satisfying the congruence (3.1) below. It can be seen that it includes contractive mappings as well as isometric transformations. Additional congruence properties of Mahler coefficients of these functions are described in Theorem 1.1:

THEOREM 1.1. *Let $(a_i)_i$ be the Mahler coefficients of a compatible function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ satisfying the conditions of Lemma 3.2. For any nonnegative integers n and $r < p^n$ we have*

$$a_{p^{n+1}-r-1} = 0 \pmod{p^{n+1}}.$$

Our techniques are mainly based on convolution properties and Lemma 2.3.

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2. Preliminary results. It is known that every continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ can be represented by means of its Mahler expansion [9, Theorem 1, p. 173]

$$f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i}.$$

The notion of discrete convolution on the set of positive integers and its impact on Mahler expansions of continuous functions on $\mathbb{Z}_p$ were explored in [9].

Definition 2.1 ([9, §1.4, p. 166]). Let A be a commutative ring. The *discrete convolution* of the functions $f, g : \mathbb{N} \to A$ is defined by

$$f * g(n) = \sum_{i=0}^{n-1} f(i)g(n-1-i), \qquad \forall n \geq 1,$$
$$f * g(0) = 0.$$

Lemma 2.2 ([9, Proposition 1, p. 169]). *Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = \sum_{i=0}^{n} a_i \binom{n}{i}$. Then*

$$f * 1(n) = \sum_{i=0}^{n-1} a_i \binom{n}{i+1}.$$

Lemma 2.2 can be easily proved using the hockey-stick identity

$$\sum_{x=i}^{n} \binom{x}{i} = \binom{n+1}{i+1}. \tag{2.1}$$

Formula (2.1) will be used several times throughout the paper.

The following result is easily deduced from [9, Proposition 1, p. 169, and Theorem 1, p. 173]:

Lemma 2.3. *Let $(a_i)_i$ be the Mahler coefficients of a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$. Then*

$$f * \binom{x}{r} = \sum_{i=0}^{\infty} a_i \binom{x}{i+r+1}.$$

It can also be verified by means of the following version of the Vandermonde identity:

$$\sum_{x+y=m} \binom{x}{i}\binom{y}{j} = \binom{m+1}{i+j+1}. \tag{2.2}$$

A mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called *compatible* if it preserves all the congruences of $\mathbb{Z}_p$. The following result provides necessary and sufficient conditions for compatibility in terms of Mahler coefficients.

LEMMA 2.4 ([1, Theorem 3.53, p. 78]). *A function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is compatible if and only if its Mahler coefficients $(a_i)_i$ satisfy $a_i = 0 \pmod{p^{\lfloor \log_p i \rfloor}}$ for every positive integer i.*

Sufficient conditions for isometricity of transformations in terms of their Mahler coefficients are given by

LEMMA 2.5 ([1, Theorem 4.40, p. 111]). *A function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is isometric if its Mahler coefficients $(a_i)_i$ satisfy*

(1) $a_1 \neq 0 \pmod{p}$;
(2) $a_i = 0 \pmod{p^{\lfloor \log_p i \rfloor + 1}}$ *for every positive integer $i \geq 2$.*

3. Main results. We begin with the following technical result which facilitates many parts of our proofs.

LEMMA 3.1 ([5, Lemma 2.3]). *Let l and i be positive integers such that $i < p^l$, where p is a prime number. Then*

$$\nu\left(\binom{p^l}{i}\right) = l - \nu(i).$$

LEMMA 3.2. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a compatible function satisfying*

$$\sum_{l=0}^{p-1} f(t + lp^{n-1}) - f(t) = 0 \pmod{p^n}, \qquad \forall n \geq 1. \tag{3.1}$$

Let r and n be positive integers such that $n \geq \lfloor \log_p r \rfloor + 2$. Let F_n be the 1-Lipschitz periodic function related to f by

$$F_n(t + up^{n-1}) = f(t), \qquad \forall t \in \{0, \ldots, p^{n-1} - 1\},\ u \in \mathbb{Z}_p.$$

Then

$$f * \binom{x}{r}(p^n) = F_n * \binom{x}{r}(p^n) \pmod{p^n}.$$

REMARK 3.3. It can be easily seen that every isometric function $f : \mathbb{Z}_p \to \mathbb{Z}_p$, $p \geq 3$, satisfies (3.1). Indeed, for every positive integer n we have

$$\sum_{l=0}^{p-1} f(t + lp^{n-1}) - f(t) = p^{n-1} \sum_{l=0}^{p-1} l \pmod{p^n} = 0 \pmod{p^n}.$$

Proof of Lemma 3.2. Since f is 1-Lipschitz, we can easily deduce that F_n is also 1-Lipschitz. In order to see that it is periodic, we notice that for all $x \in \mathbb{Z}_p$, there exist unique $t \in \{0, \ldots, p^{n-1} - 1\}$ and $u \in \mathbb{Z}_p$ such that $x = t + up^{n-1}$. Hence, $x + p^{n-1} = t + (u+1)p^{n-1}$ and

$$F_n(x) = F_n(t + up^{n-1}) = f(t) = F_n(t + (u+1)p^{n-1}) = F_n(x + p^{n-1}).$$

We have

$$f * \binom{x}{r}(p^n) = \sum_{i=0}^{p^n-1} f(i)\binom{p^n-1-i}{r} \tag{3.2}$$

$$= \sum_{t=0}^{p^{n-1}-1}\sum_{l=0}^{p-1} f(t+lp^{n-1})\binom{p^n-1-t-lp^{n-1}}{r}$$

$$= \sum_{t=0}^{p^{n-1}-1}\sum_{l=0}^{p-1}(f(t+lp^{n-1})-f(t))\binom{p^n-1-t-lp^{n-1}}{r}$$

$$+ \sum_{t=0}^{p^{n-1}-1}\sum_{l=0}^{p-1} f(t)\binom{p^n-1-t-lp^{n-1}}{r}$$

$$= \sum_{t=0}^{p^{n-1}-1}\sum_{l=0}^{p-1}(f(t+lp^{n-1})-f(t))\left(\binom{p^n-1-t-lp^{n-1}}{r}-\binom{p^n-1-t}{r}\right)$$

$$+ \sum_{t=0}^{p^{n-1}-1}\binom{p^n-1-t}{r}\sum_{l=0}^{p-1} f(t+lp^{n-1})-f(t)$$

$$+ \sum_{t=0}^{p^{n-1}-1}\sum_{l=0}^{p-1} f(t)\binom{p^n-1-t-lp^{n-1}}{r}.$$

Since f is compatible, we have

$$f(t+lp^{n-1})-f(t) = 0 \pmod{p^{n-1}}, \qquad \forall t \in \{0,\ldots,p^{n-1}-1\}. \tag{3.3}$$

Moreover, from Lemma 2.4 we find that $p^{\lfloor \log_p r\rfloor}\binom{x}{r}$ is 1-Lipschitz. This means that

$$p^{\lfloor \log_p r\rfloor}\left(\binom{p^n-1-t-lp^{n-1}}{r}-\binom{p^n-1-t}{r}\right) = 0 \pmod{p^{n-1}},$$

which implies that

$$\binom{p^n-1-t-lp^{n-1}}{r}-\binom{p^n-1-t}{r} = 0 \pmod{p^{n-1-\lfloor \log_p r\rfloor}} \tag{3.4}$$
$$= 0 \pmod{p}.$$

Combining (3.3) and (3.4) we can see that the first sum on the right-hand side in (3.2) vanishes modulo p^n. The same holds for the second sum due to (3.1). It follows that

$$\begin{aligned} f*\binom{x}{r}(p^n) &= \sum_{t=0}^{p^{n-1}-1}\sum_{l=0}^{p-1} f(t)\binom{p^n-1-t-lp^{n-1}}{r} \pmod{p^n} \\ &= \sum_{t=0}^{p^{n-1}-1}\sum_{l=0}^{p-1} F_n(t+lp^{n-1})\binom{p^n-1-t-lp^{n-1}}{r} \pmod{p^n} \\ &= \sum_{i=0}^{p^n-1} F_n(i)\binom{p^n-1-i}{r} \pmod{p^n} = F_n*\binom{x}{r}(p^n) \pmod{p^n}. \ \blacksquare \end{aligned}$$

THEOREM 3.4. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a function that satisfies the assumptions of Lemma* 3.2. *Then*

$$\begin{aligned} f*\binom{x}{r}(p^{n+1}) = \sum_{i=1}^{p-1}\binom{p}{i+1}\sum_{s_1=1}^{r}\sum_{s_2=1}^{r-s_1}\cdots \\ \cdots \sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})}\binom{p^n}{s_1}\cdots\binom{p^n}{s_i} f*\binom{x}{r-(s_1+\cdots+s_i)}(p^n) \\ + pf*\binom{x}{r}(p^n) \pmod{p^{n+1}} \end{aligned}$$

for any positive integers n and $r < p^n$.

Proof. Let F_n be the 1-Lipschitz periodic function introduced in Lemma 3.2. For any positive integers $l \le p$, $r < p^n$ and $i \ge lp^n$, we have

$$\begin{aligned} F_{n+1}*\binom{x}{r}(i) &= \left(F_{n+1}*\binom{x}{r-1}*1\right)(i) = \sum_{j=0}^{i-1} F_{n+1}*\binom{x}{r-1}(j) \\ &= \sum_{j=0}^{lp^n-1} F_{n+1}*\binom{x}{r-1}(j) + \sum_{j=lp^n}^{i-1} F_{n+1}*\binom{x}{r-1}(j) \\ &= F_{n+1}*\binom{x}{r}(lp^n) + \sum_{j=lp^n}^{i-1} F_{n+1}*\binom{x}{r-1}(j). \end{aligned} \tag{3.5}$$

Similarly, for every positive integer $k \le p$,

$$\begin{aligned} F_{n+1}*\binom{x}{r}(kp^n) &= \sum_{j_1=0}^{kp^n-1} F_{n+1}*\binom{x}{r-1}(j_1) = \sum_{l=0}^{k-1}\sum_{j_1=lp^n}^{(l+1)p^n-1} F_{n+1}*\binom{x}{r-1}(j_1) \\ &= \sum_{l=0}^{k-1}\sum_{j_1=lp^n}^{(l+1)p^n-1} F_{n+1}*\binom{x}{r-1}(lp^n) + \sum_{l=0}^{k-1}\sum_{j_1=lp^n}^{(l+1)p^n-1}\sum_{j_2=lp^n}^{j_1-1} F_{n+1}*\binom{x}{r-2}(j_2), \end{aligned}$$

where the last identity follows from (3.5).

Continuing this way, we can easily prove recursively on $m \in \{1, \ldots, r-1\}$ that

$$F_{n+1} * \binom{x}{r}(kp^n) = \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} F_{n+1} * \binom{x}{r-1}(lp^n) + \cdots$$
$$+ \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_m=lp^n}^{j_{m-1}-1} F_{n+1} * \binom{x}{r-m}(lp^n)$$
$$+ \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_{m+1}=lp^n}^{j_m-1} F_{n+1} * \binom{x}{r-(m+1)}(j_{m+1}).$$

Applying this formula with $m = r-1$, we get

$$F_{n+1} * \binom{x}{r}(kp^n) = \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} F_{n+1} * \binom{x}{r-1}(lp^n) \tag{3.6}$$
$$+ \sum_{m=2}^{r-1} \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_m=lp^n}^{j_{m-1}-1} F_{n+1} * \binom{x}{r-m}(lp^n)$$
$$+ \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_r=lp^n}^{j_{r-1}-1} F_{n+1} * \binom{x}{0}(j_r).$$

The first sum in (3.6) can be written in the form

$$\sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} F_{n+1} * \binom{x}{r-1}(lp^n) \tag{3.7}$$
$$= \sum_{l=0}^{k-1} F_{n+1} * \binom{x}{r-1}(lp^n) \sum_{j_1=lp^n}^{(l+1)p^n-1} 1$$
$$= p^n \sum_{l=0}^{k-1} F_{n+1} * \binom{x}{r-1}(lp^n).$$

In a similar way, for all $m \in \{2, \ldots, r-1\}$ we have

$$\sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_m=lp^n}^{j_{m-1}-1} F_{n+1} * \binom{x}{r-m}(lp^n)$$
$$= \sum_{l=0}^{k-1} F_{n+1} * \binom{x}{r-m}(lp^n) \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_{m-1}=lp^n}^{j_{m-2}-1} \sum_{j_m=lp^n}^{j_{m-1}-1} 1.$$

Moreover, we will prove by recursion on $t \in \{0, \ldots, m-2\}$ that

$$\sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_m=lp^n}^{j_{m-1}-1} 1 = \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_{m-t}=lp^n}^{j_{m-t-1}-1} \binom{j_{m-t}-lp^n}{t}. \tag{3.8}$$

Indeed, (3.8) obviously holds for $t = 0$. Moreover, for all $t \in \{0, \ldots, m-3\}$,

$$\begin{aligned}\sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_{m-t-1}=lp^n}^{j_{m-t-2}-1} \sum_{j_{m-t}=lp^n}^{j_{m-t-1}-1} \binom{j_{m-t}-lp^n}{t} \\ = \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_{m-t-1}=lp^n}^{j_{m-t-2}-1} \sum_{j_{m-t}=0}^{j_{m-t-1}-lp^n-1} \binom{j_{m-t}}{t} \\ = \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_{m-t-1}=lp^n}^{j_{m-t-2}-1} \binom{j_{m-t-1}-lp^n}{t+1},\end{aligned}$$

where the last equality follows from (2.1). Thus, we have proved by induction that (3.8) holds for all $t \in \{0, \ldots, m-2\}$. Using the same techniques we obtain, for $t = m-2$,

$$\begin{aligned}\sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_m=lp^n}^{j_{m-1}-1} 1 &= \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \binom{j_2-lp^n}{m-2} \\ &= \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=0}^{j_1-lp^n-1} \binom{j_2}{m-2} = \sum_{j_1=lp^n}^{(l+1)p^n-1} \binom{j_1-lp^n}{m-1} \\ &= \sum_{j_1=0}^{(l+1)p^n-lp^n-1} \binom{j_1}{m-1} = \binom{p^n}{m}.\end{aligned} \tag{3.9}$$

It follows that for all $m \in \{2, \ldots, r-1\}$, we have

$$\begin{aligned}\sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_m=lp^n}^{j_{m-1}-1} F_{n+1} * \binom{x}{r-m}(lp^n) \\ = \binom{p^n}{m} \sum_{l=0}^{k-1} F_{n+1} * \binom{x}{r-m}(lp^n) = \binom{p^n}{m} \sum_{l=1}^{k-1} \left(F_{n+1} * \binom{x}{r-m}\right)(lp^n).\end{aligned} \tag{3.10}$$

In order to analyze the last sum in (3.6), we first notice that for all $k \leq p$ and $i \geq kp^n$, we have

$$F_{n+1} * 1(i) = \sum_{j=0}^{i-1} F_{n+1}(j) = \sum_{l=0}^{k-1} \sum_{j=lp^n}^{(l+1)p^n-1} F_{n+1}(j) + \sum_{j=kp^n}^{i-1} F_{n+1}(j)$$
$$= \sum_{l=0}^{k-1} \sum_{j=0}^{p^n-1} F_{n+1}(j+lp^n) + \sum_{j=0}^{i-kp^n-1} F_{n+1}(j+kp^n)$$
$$= \sum_{l=0}^{k-1} \sum_{j=0}^{p^n-1} f(j) + \sum_{j=0}^{i-kp^n-1} f(j) = kf * 1(p^n) + f * 1(i-kp^n).$$

It follows that

$$\sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_r=lp^n}^{j_{r-1}-1} F_{n+1} * \binom{x}{0}(j_r) \tag{3.11}$$
$$= \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_r=lp^n}^{j_{r-1}-1} F_{n+1} * 1(j_r)$$
$$= \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_r=lp^n}^{j_{r-1}-1} lf * 1(p^n)$$
$$+ \sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_r=lp^n}^{j_{r-1}-1} f * 1(j_r - lp^n).$$

Proceeding as in (3.9) we see that

$$\sum_{l=0}^{k-1} \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_r=lp^n}^{j_{r-1}-1} lf * 1(p^n) \tag{3.12}$$
$$= \sum_{l=0}^{k-1} lf * 1(p^n) \sum_{j_1=lp^n}^{(l+1)p^n-1} \sum_{j_2=lp^n}^{j_1-1} \cdots \sum_{j_r=lp^n}^{j_{r-1}-1} 1$$
$$= \sum_{l=0}^{k-1} lf * 1(p^n) \binom{p^n}{r} = \binom{p^n}{r}\binom{k}{2} f * 1(p^n) = 0 \ (\mathrm{mod}\ p^{n+1}),$$

because f satisfies condition (3.1) and hence

$$f * 1(p^n) = \sum_{j=0}^{p^n-1} f(j) = \sum_{t=0}^{p^{n-1}-1} \sum_{l=0}^{p-1} f(t+lp^{n-1}) = 0 \ (\mathrm{mod}\ p^n),$$

and according to [5, Lemma 2.3], $\binom{p^n}{r} = 0 \ (\mathrm{mod}\ p)$ when $1 \le r < p^n$.

Proceeding as in (3.9) shows that the second part of (3.11) can be written in the form

$$(3.13)\qquad \sum_{l=0}^{k-1}\sum_{j_1=lp^n}^{(l+1)p^n-1}\sum_{j_2=lp^n}^{j_1-1}\cdots\sum_{j_r=lp^n}^{j_{r-1}-1} f*1(j_r-lp^n)$$
$$=\sum_{l=0}^{k-1}\sum_{j_1=lp^n}^{(l+1)p^n-1}\sum_{j_2=lp^n}^{j_1-1}\cdots\sum_{j_r=0}^{j_{r-1}-lp^n-1} f*1(j_r)$$
$$=\sum_{l=0}^{k-1}\sum_{j_1=lp^n}^{(l+1)p^n-1}\sum_{j_2=lp^n}^{j_1-1}\cdots\sum_{j_{r-1}=lp^n}^{j_{r-2}-1} f*\binom{x}{1}(j_{r-1}-lp^n)$$
$$\vdots$$
$$=\sum_{l=0}^{k-1}\sum_{j_1=lp^n}^{(l+1)p^n-1} f*\binom{x}{r-1}(j_1-lp^n)$$
$$=\sum_{l=0}^{k-1}\sum_{j_1=0}^{p^n-1} f*\binom{x}{r-1}(j_1)=kf*\binom{x}{r}(p^n).$$

Combining (3.6), (3.7) and (3.10)–(3.13) we conclude that

$$(3.14)\qquad F_{n+1}*\binom{x}{r}(kp^n)=\sum_{m=1}^{r-1}\binom{p^n}{m}\sum_{l=1}^{k-1}\left(F_{n+1}*\binom{x}{r-m}\right)(lp^n)$$
$$+kf*\binom{x}{r}(p^n)\pmod{p^{n+1}}.$$

Using (3.14) we will prove by induction on $k\in\{1,\dots,p\}$ that

$$(3.15)\qquad F_{n+1}*\binom{x}{r}(kp^n)=\sum_{i=1}^{k-1}\binom{k}{i+1}\sum_{s_1=1}^{r-1}\sum_{s_2=1}^{r-s_1-1}\cdots\sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})-1}$$
$$\binom{p^n}{s_1}\cdots\binom{p^n}{s_i}f*\binom{x}{r-(s_1+\cdots+s_i)}(p^n)$$
$$+kf*\binom{x}{r}(p^n)\pmod{p^{n+1}}.$$

For $k=1$, (3.15) follows immediately from (3.14). Assume (3.15) holds for every $l\in\{1,\dots,k\}$, where $k\geq 1$. We have, according to (3.14),

$$F_{n+1}*\binom{x}{r}((k+1)p^n)$$
$$=\sum_{m=1}^{r-1}\binom{p^n}{m}\sum_{l=1}^{k}\left(F_{n+1}*\binom{x}{r-m}\right)(lp^n)+(k+1)f*\binom{x}{r}(p^n)\pmod{p^{n+1}}$$

$$= \sum_{m=1}^{r-1} \binom{p^n}{m} \sum_{l=1}^{k} \sum_{i=1}^{l-1} \binom{l}{i+1} \sum_{s_1=1}^{r-m-1} \sum_{s_2=1}^{r-m-s_1-1} \cdots \sum_{s_i=1}^{r-m-(s_1+\cdots+s_{i-1})-1} \binom{p^n}{s_1} \cdots \binom{p^n}{s_i} f * \binom{x}{r-m-(s_1+\cdots+s_i)}(p^n)$$

$$+ lf * \binom{x}{r-m}(p^n) + (k+1) f * \binom{x}{r}(p^n) \pmod{p^{n+1}}$$

$$= \sum_{m=1}^{r-1} \binom{p^n}{m} \sum_{l=2}^{k} \sum_{i=1}^{l-1} \binom{l}{i+1} \sum_{s_1=1}^{r-m-1} \sum_{s_2=1}^{r-m-s_1-1} \cdots \sum_{s_i=1}^{r-m-(s_1+\cdots+s_{i-1})-1} \binom{p^n}{s_1} \cdots \binom{p^n}{s_i} f * \binom{x}{r-m-(s_1+\cdots+s_i)}(p^n)$$

$$+ \sum_{m=1}^{r-1} \binom{p^n}{m} \sum_{l=1}^{k} lf * \binom{x}{r-m}(p^n) + (k+1) f * \binom{x}{r}(p^n) \pmod{p^{n+1}}$$

$$= \sum_{m=1}^{r-1} \binom{p^n}{m} \sum_{l=2}^{k} \sum_{i=2}^{l} \binom{l}{i} \sum_{s_1=1}^{r-m-1} \sum_{s_2=1}^{r-m-s_1-1} \cdots \sum_{s_{i-1}=1}^{r-m-(s_1+\cdots+s_{i-2})-1} \binom{p^n}{s_1} \cdots \binom{p^n}{s_{i-1}} f * \binom{x}{r-m-(s_1+\cdots+s_{i-1})}(p^n)$$

$$+ \sum_{m=1}^{r-1} \binom{p^n}{m} f * \binom{x}{r-m}(p^n) \sum_{l=1}^{k} l + (k+1) f * \binom{x}{r}(p^n) \pmod{p^{n+1}}$$

$$= \sum_{i=2}^{k} \sum_{m=1}^{r-1} \binom{p^n}{m} \sum_{s_1=1}^{r-m-1} \sum_{s_2=1}^{r-m-s_1-1} \cdots \sum_{s_{i-1}=1}^{r-m-(s_1+\cdots+s_{i-2})-1} \binom{p^n}{s_1} \cdots \binom{p^n}{s_{i-1}} f * \binom{x}{r-m-(s_1+\cdots+s_{i-1})}(p^n) \sum_{l=i}^{k} \binom{l}{i}$$

$$+ \sum_{m=1}^{r-1} \binom{p^n}{m} f * \binom{x}{r-m}(p^n) \sum_{l=1}^{k} l + (k+1) f * \binom{x}{r}(p^n) \pmod{p^{n+1}}$$

$$= \sum_{i=2}^{k} \sum_{m=1}^{r-1} \binom{p^n}{m} \sum_{s_1=1}^{r-m-1} \sum_{s_2=1}^{r-m-s_1-1} \cdots \sum_{s_{i-1}=1}^{r-m-(s_1+\cdots+s_{i-2})-1} \binom{p^n}{s_1} \cdots \binom{p^n}{s_{i-1}} f * \binom{x}{r-m-(s_1+\cdots+s_{i-1})}(p^n) \binom{k+1}{i+1}$$

$$+ \sum_{m=1}^{r-1} \binom{p^n}{m} f * \binom{x}{r-m}(p^n) \binom{k+1}{2} + (k+1) f * \binom{x}{r}(p^n) \pmod{p^{n+1}}.$$

Replacing m by $\tilde{s}_1$ and s_j by $\tilde{s}_{j+1}$ for all $1 \le j \le i-1$, we obtain

$$F_{n+1} * \binom{x}{r}((k+1)p^n) = \sum_{i=2}^{k} \binom{k+1}{i+1} \sum_{\tilde{s}_1=1}^{r-1} \binom{p^n}{\tilde{s}_1} \sum_{\tilde{s}_2=1}^{r-\tilde{s}_1-1} \sum_{\tilde{s}_3=1}^{r-\tilde{s}_1-\tilde{s}_2-1} \cdots$$
$$\cdots \sum_{\tilde{s}_i=1}^{r-\tilde{s}_1-(\tilde{s}_2+\cdots+\tilde{s}_{i-1})-1} \binom{p^n}{\tilde{s}_2} \cdots \binom{p^n}{\tilde{s}_i} f * \binom{x}{r-(\tilde{s}_1+\tilde{s}_2+\cdots+\tilde{s}_i)}(p^n)$$
$$+ \binom{k+1}{2} \sum_{\tilde{s}_1=1}^{r-1} \binom{p^n}{\tilde{s}_1} f * \binom{x}{r-\tilde{s}_1}(p^n) + (k+1) f * \binom{x}{r}(p^n) \pmod{p^{n+1}}$$
$$= \sum_{i=1}^{k} \binom{k+1}{i+1} \sum_{\tilde{s}_1=1}^{r-1} \sum_{\tilde{s}_2=1}^{r-\tilde{s}_1-1} \sum_{\tilde{s}_3=1}^{r-\tilde{s}_1-\tilde{s}_2-1} \cdots$$
$$\cdots \sum_{\tilde{s}_i=1}^{r-\tilde{s}_1-(\tilde{s}_2+\cdots+\tilde{s}_{i-1})-1} \binom{p^n}{\tilde{s}_1} \cdots \binom{p^n}{\tilde{s}_i} f * \binom{x}{r-(\tilde{s}_1+\tilde{s}_2+\cdots+\tilde{s}_i)}(p^n)$$
$$+ (k+1) f * \binom{x}{r}(p^n) \pmod{p^{n+1}},$$

which finishes the proof of (3.15).

The result follows by combining (3.15) for $k = p$ with Lemma 3.2. ∎

The following result can be obtained from Theorem 3.4. It can also be proved by using Vandermonde's identity.

Corollary 3.5. *For any positive integers n and $r < p^n$ we have*

$$\binom{p^{n+1}}{r+2} = \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=1}^{r-s_1-1} \cdots$$
$$\cdots \sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})-1} \binom{p^n}{s_1} \cdots \binom{p^n}{s_i} \binom{p^n}{r-(s_1+\cdots+s_i)+2}$$
$$+ p \binom{p^n}{r+2} \pmod{p^{n+1}}.$$

Proof. Since the function $f(x) = x$ has Mahler coefficients $a_1 = 1$ and $a_i = 0$ for all $i \neq 1$, applying Theorem 3.4 and Lemma 2.3 for $f(x) = x$ gives

$$\binom{p^{n+1}}{r+2} = f * \binom{x}{r}(p^{n+1}) = \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r} \sum_{s_2=1}^{r-s_1} \cdots$$
$$\cdots \sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})} \binom{p^n}{s_1} \cdots \binom{p^n}{s_i} f * \binom{x}{r-(s_1+\cdots+s_i)}(p^n)$$
$$+ pf * \binom{x}{r}(p^n) \pmod{p^{n+1}}$$

$$= \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=1}^{r-s_1-1} \cdots \sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})-1} \binom{p^n}{s_1} \cdots$$
$$\cdots \binom{p^n}{s_i} \binom{p^n}{r-(s_1+\cdots+s_i)+2} + p\binom{p^n}{r+2} \pmod{p^{n+1}}. \blacksquare$$

Proof of Theorem 1.1. We first prove that

$$\nu(a_j) \geq \lfloor \log_p(j+1) \rfloor. \tag{3.16}$$

According to [1, Theorem 3.53, p. 78], we only need to prove formula (3.16) when $\lfloor \log_p(j+1) \rfloor = \lfloor \log_p j \rfloor + 1$, that is, for $j = p^i - 1$ with some positive integer i. We prove by induction on $i \geq 1$ that

$$\nu(a_{p^i-1}) \geq \lfloor \log_p(p^i) \rfloor = i.$$

According to Lemma 2.3 we have

$$\sum_{j=0}^{p-1} \binom{p}{j+1} a_j = f * 1(p) = \sum_{j=0}^{p-1} f(j) = 0 \pmod{p},$$

where the third equality is obtained from condition (3.1). Since

$$\binom{p}{j+1} = 0 \pmod{p}, \qquad \forall j \in \{0, \ldots, p-2\},$$

we obtain

$$a_{p-1} = -\sum_{j=0}^{p-2} \binom{p}{j+1} a_j = 0 \pmod{p}.$$

Formula (3.16) is proved for $j = p-1$. Assume it holds for all $j \leq p^i - 2$ for some positive integer i. Since f satisfies condition (3.1), from Lemma 2.3 we have

$$\sum_{j=0}^{p^i-1} \binom{p^i}{j+1} a_j = f * 1(p^i) = \sum_{j=0}^{p^i-1} f(j) = \sum_{t=0}^{p^{i-1}-1} \sum_{l=0}^{p-1} f(t+lp^{i-1}) = 0 \pmod{p^i}.$$

By our assumption and from Lemma 3.1 we get

$$\nu\left(\binom{p^i}{j+1} a_j\right) = i - \nu(j+1) + \nu(a_j)$$
$$\geq i - \nu(j+1) + \lfloor \log_p(j+1) \rfloor \geq i, \qquad \forall j \in \{0, \ldots, p^i - 2\}.$$

It follows that

$$a_{p^i-1} = -\sum_{j=0}^{p^i-2} \binom{p^i}{j+1} a_j \pmod{p^i} = 0 \pmod{p^i},$$

which finishes the proof of (3.16).

Combining Theorem 3.4 and Lemma 2.3 gives

$$\sum_{j=0}^{p^{n+1}-r-1}\binom{p^{n+1}}{j+r+1}a_j = \sum_{i=1}^{p-1}\binom{p}{i+1}\sum_{s_1=1}^{r-1}\sum_{s_2=1}^{r-s_1-1}\cdots \cdots\sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})-1}\binom{p^n}{s_1}\cdots\binom{p^n}{s_i} \times \sum_{j=0}^{p^n-r+(s_1+\cdots+s_i)-1}\binom{p^n}{j+r-(s_1+\cdots+s_i)+1}a_j + p\sum_{j=0}^{p^n-r-1}\binom{p^n}{j+r+1}a_j \pmod{p^{n+1}}. \tag{3.17}$$

We prove the following congruences:

$$\binom{p^{n+1}}{j+r+1}a_j = \sum_{i=1}^{p-1}\binom{p}{i+1}\sum_{s_1=1}^{r-1}\sum_{s_2=1}^{r-s_1-1}\cdots \cdots\sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})-1}\binom{p^n}{s_1}\cdots\binom{p^n}{s_i}\binom{p^n}{j+r-(s_1+\cdots+s_i)+1}a_j + p\binom{p^n}{j+r+1}a_j \pmod{p^{n+1}}, \quad \forall j\le p^n-r-1, \tag{3.18}$$

$$\binom{p^{n+1}}{j+r+1}a_j = 0 \pmod{p^{n+1}}, \quad \forall j\in\{p^n-r,\dots,p^{n+1}-r-1\}, \tag{3.19}$$

and

$$\binom{p}{i+1}\binom{p^n}{s_1}\cdots\binom{p^n}{s_i}\binom{p^n}{j+r-(s_1+\cdots+s_i)+1}a_j = 0 \pmod{p^{n+1}}, \quad \forall i\in\{1,\dots,p-1\}, s_1+\cdots+s_i\le r-1, \quad \forall j\in\{p^n-r,\dots,p^n-r+(s_1+\cdots+s_i)-1\}. \tag{3.20}$$

The result will follow immediately by combining (3.17)–(3.20).

In order to prove (3.18), we will use Corollary 3.5 and formula (3.16). According to Corollary 3.5, for all $j\le p^n-r-1$ we have

$$\binom{p^{n+1}}{j+r+1} = \sum_{i=1}^{p-1}\binom{p}{i+1}\sum_{s_1=1}^{j+r-2}\sum_{s_2=1}^{j+r-s_1-2}\cdots\sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2}\binom{p^n}{s_1}\cdots\binom{p^n}{s_i}\binom{p^n}{j+r-(s_1+\cdots+s_i)+1} + p\binom{p^n}{j+r+1} \pmod{p^{n+1}} \tag{3.21}$$

$$
\begin{aligned}
&= \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=1}^{j+r-s_1-2} \cdots \sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\quad + \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=r}^{j+r-2} \sum_{s_2=1}^{j+r-s_1-2} \cdots \sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\quad + p\binom{p^n}{j+r+1} \pmod{p^{n+1}} \\
&= \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=1}^{r-s_1-1} \cdots \sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\quad + \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=r-s_1}^{j+r-s_1-2} \cdots \sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\quad + \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=r}^{j+r-2} \sum_{s_2=1}^{j+r-s_1-2} \cdots \sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\quad + p\binom{p^n}{j+r+1} \pmod{p^{n+1}} \\
&\qquad\vdots \\
&= \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=1}^{r-s_1-1} \cdots \sum_{s_i=1}^{r-(s_1+\cdots+s_{i-1})-1} \cdots \\
&\quad + \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=1}^{r-s_1-1} \cdots \sum_{s_i=r-(s_1+\cdots+s_{i-1})}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\qquad\vdots \\
&\quad + \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=1}^{r-1} \sum_{s_2=r-s_1}^{r-s_1-1} \cdots \sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\quad + \sum_{i=1}^{p-1} \binom{p}{i+1} \sum_{s_1=r}^{j+r-2} \sum_{s_2=1}^{j+r-s_1-2} \cdots \sum_{s_i=1}^{j+r-(s_1+\cdots+s_{i-1})-2} \cdots \\
&\quad + p\binom{p^n}{j+r+1} \pmod{p^{n+1}}.
\end{aligned}
$$

In this formula, except the first line, the sums are taken over $s_1+\cdots+s_i \geq r$. From Lemma 3.1 and (3.16) we have

$$\begin{aligned}\nu\Big(\Big(&\binom{p^n}{j+r-(s_1+\cdots+s_i)+1}a_j\Big)\\&= n-\nu(j+r-(s_1+\cdots+s_i)+1)+\nu(a_j)\\&\geq n-\nu(j+r-(s_1+\cdots+s_i)+1)+\lfloor\log_p(j+1)\rfloor\\&\geq n-\nu(j+r-(s_1+\cdots+s_i)+1)+\lfloor\log_p(j+r-(s_1+\cdots+s_i)+1)\rfloor\\&\geq n.\end{aligned}$$

Hence, we must have

$$\nu\Big(\binom{p^n}{s_1}\binom{p^n}{j+r-(s_1+\cdots+s_i)+1}a_j\Big)\geq n+1, \qquad \forall s_1+\cdots+s_i\geq r. \tag{3.22}$$

The congruence (3.18) can be obtained by multiplying (3.21) by a_j, and then applying (3.22).

Now we prove (3.19). Let $j\in\{p^n-r,\dots,p^{n+1}-r-2\}$ and $t=\nu(j+r+1)$. Then $p^n+1\leq j+r+1\leq p^{n+1}-1$ and $t\leq n$, which leads to $j+r+1\geq p^n+p^t$. It follows that $j\geq p^n-r-1+p^t\geq p^t$. According to Lemma 2.4 this implies that $\nu(a_j)\geq t$, because f is compatible. From Lemma 3.1 we get $\nu\big(\binom{p^{n+1}}{j+r+1}a_j\big)=n+1-t+\nu(a_j)\geq n+1$. Formula (3.19) is thus proved.

Similarly, for $i\in\{1,\dots,p-1\}$, $s_1+\cdots+s_i\leq r-1$, and $j\in\{p^n-r,\dots,p^n-r+(s_1+\cdots+s_i)-1\}$, we also have $\nu(a_j)\geq t$. Here, we consider two cases: $t=\nu(j+r+1)<\nu(s_1+\cdots+s_i)$ and $t=\nu(j+r+1)\geq\nu(s_1+\cdots+s_i)$.

In the first case we have $\nu(j+r-(s_1+\cdots+s_i)+1)=t$, which implies that

$$\nu\Big(\binom{p^n}{j+r-(s_1+\cdots+s_i)+1}a_j\Big)=n-t+\nu(a_j)\geq n.$$

In this case (3.20) follows from $\nu\big(\binom{p^n}{s_1}\big)\geq 1$, because $1\leq s_1<p^n$. Now, if $t\geq\nu(s_1+\cdots+s_i)$, we get

$$\nu(a_j)\geq t\geq\nu(s_1+\cdots+s_i)\geq\min_{1\leq l\leq i}\nu(s_l),$$

hence

$$\nu\Big(\binom{p^n}{s_1}\cdots\binom{p^n}{s_i}\Big)\geq n-\min_{1\leq l\leq i}\nu(s_l)\geq n-t.$$

Therefore,

$$\nu\Big(\binom{p^n}{s_1}\cdots\binom{p^n}{s_i}a_j\Big)\geq n-t+t=n,$$

and (3.20) is obtained from $\nu\big(\binom{p^n}{j+r-(s_1+\cdots+s_i)+1}\big)\geq 1$.

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