

OPEN RETRACTIONS OF INDECOMPOSABLE CONTINUA, II

BY

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Abstract. This paper is a sequel to [Colloq. Math. 148 (2017), 191–194]. Given an arbitrary continuum Z , we consider the class of all indecomposable continua that contain Z as an open retract with Cantor set fibers and are closures of countable unions of topological copies of Z . We prove that this class admits no common model. Furthermore, when Z is tree-like, we show that there exists a subclass of such continua that admits no common model and whose members are all tree-like.

1. Introduction. In this paper, all spaces are metrizable and every map is continuous. A *continuum* means a nonempty compact connected metric space. If X is a space and $A \subset X$, then we denote the interior, closure, and boundary of A in X by $\text{Int}_X(A)$, $\text{Cl}_X(A)$, and $\text{Bd}_X(A)$, respectively.

An *arc* is a continuum that is homeomorphic to the closed interval $[0, 1]$, and a *tree* is a 1-dimensional polyhedron that contains no simple closed curves. A continuum M is called a *common model* for a class \mathcal{C} of continua if for every $X \in \mathcal{C}$, there exists a continuous surjection $f : M \rightarrow X$. For example, by the Hahn–Mazurkiewicz theorem (see [8, Section VIII.2]), the closed interval $[0, 1]$ is a common model for the class of all Peano continua. Similarly, the pseudo-arc is a common model for the class of all arc-like continua (see [2, 5, 7]).

On the other hand, Waraszkiewicz [9] showed that there exists an uncountable family of compactifications of the half-open interval $(0, 1]$, each having the circle as its remainder, with no common model for the entire family. Likewise, in [6, Section 20], a corresponding uncountable family was constructed whose remainders are all simple triods, again with no common model. In particular, that result implies there is no common model for the class of all triod-like continua. In [6, Section 20], numerous results concerning common models for various classes of continua are presented. The reader is referred to [4] for relatively recent literature on this topic.

2020 *Mathematics Subject Classification*: Primary 54F15; Secondary 54C10.

Key words and phrases: common model, indecomposable continuum, open map, Cantor set.

Received 16 July 2025; revised 10 October 2025.

Published online 11 December 2025.

Given an arbitrary continuum Z , we study the class of continua that contain Z as an open retract with Cantor set fibers and are closures of countable unions of topological copies of Z . In [3], it is shown that this class is nonempty. In the present paper, we prove that it admits no common model. Furthermore, when Z is tree-like, we show that there exists a subclass of such continua that admits no common model and whose members are all tree-like.

2. Preliminaries. In this section, we present some basic definitions and provide a slight improvement of a result from [6, Section 20]. This improved result will be used in the subsequent section.

DEFINITION 2.1. A continuum X is said to be *arc-like* (respectively, *tree-like*) if for every $\varepsilon > 0$ there exists a continuous surjection $f : X \rightarrow A$ onto an arc A (respectively, onto a tree T) such that for every point y in A (respectively, in T), the diameter of the fiber $f^{-1}(y)$ is less than ε .

DEFINITION 2.2 (Mirror-glued continuum). Let $X', X'' \subset I^\infty \times [-1, 1]$ be subcontinua and let $A' \subset X', A'' \subset X''$ be closed subsets. Define $X := X' \cup X''$, and suppose

- (1) $X' \subset I^\infty \times [0, 1]$;
- (2) $X' \cap (I^\infty \times \{0\}) \neq \emptyset$;
- (3) $X'' = \{(x, t) \in I^\infty \times [-1, 0] : (x, -t) \in X'\}$;
- (4) $X' \cap (I^\infty \times \{1\}) = A', X'' \cap (I^\infty \times \{-1\}) = A''$.

Then we call X the (X', A') - (X'', A'') *mirror-glued continuum*.

DEFINITION 2.3 (Tri-fold symmetric extension). Let X be a mirror-glued continuum as in Definition 2.2. Regard X as a subset of $I^\infty \times \mathbb{R}$ and define

$$X^{+1} := \{(x, t) \in I^\infty \times \mathbb{R} : (x, t - 1) \in X\}.$$

Then set

$$\begin{aligned} T_1^X &= \{(x, t, 0) : (x, t) \in X^{+1}\}, \\ T_2^X &= \{(x, -t, 0) : (x, t) \in X^{+1}\}, \\ T_3^X &= \{(x, 0, t) : (x, t) \in X^{+1}\}. \end{aligned}$$

We define the *tri-fold symmetric extension* of X by

$$Y_X := T_1^X \cup T_2^X \cup T_3^X.$$

Moreover, we define the map $f^* : Y_X \rightarrow T_1^X$ as follows:

$$f^*(x, y, z) = \begin{cases} (x, y, 0) & \text{if } (x, y, z) \in T_1^X, \\ (x, -y, 0) & \text{if } (x, y, z) \in T_2^X, \\ (x, z, 0) & \text{if } (x, y, z) \in T_3^X. \end{cases}$$

DEFINITION 2.4 (Meandering continuum with pattern \mathbf{m}). Let X be the mirror-glued continuum built from (X', A') and (X'', A'') , and let Y_X be its tri-fold symmetric extension. Define the ambient space

$$Q := I^\infty \times \mathbb{R} \times \mathbb{R},$$

with natural projections

$$p : Q \times [0, 3] \rightarrow Q, \quad q : Q \times [0, 3] \rightarrow [0, 3].$$

We call $\mathbf{m} = (m(i))_{i=0}^\infty \subset \{1, 2, 3\}$ a *meandering pattern* if $m(2i) = m(2i + 1)$ for each $i \geq 0$.

We construct a continuum $M_X^{\mathbf{m}}$ as follows. There exist

- topological copies $\{Z_i\}_{i=0}^\infty$ of X , and
- a strictly decreasing sequence $(d_i)_{i=1}^\infty \subset (0, 1]$ with $\lim_{i \rightarrow \infty} d_i = 0$

such that if A'_i and A''_i denote the images of A' and A'' in each copy Z_i , then

- (1) $Z_0 \subset q^{-1}([1, 3])$, with $Z_0 \cap q^{-1}(3) = A'_0$ and $Z_0 \cap q^{-1}(1) = A''_0$;
- (2) for each $i \geq 1$, $Z_i \subset q^{-1}([d_{i+1}, d_i])$;
- (3) for each $i \geq 0$, $Z_i \cap Z_{i+1} = A''_i = A'_{i+1}$;
- (4) for each $i \geq 0$, $p|_{Z_i} : Z_i \rightarrow T_{m(i)}^X$ is a homeomorphism;
- (5) the union

$$M_X^{\mathbf{m}} := (Y_X \times \{0\}) \cup \bigcup_{i=0}^{\infty} Z_i$$

is a continuum.

We call $M_X^{\mathbf{m}}$ the *meandering continuum with pattern \mathbf{m}* associated with X , and denote the collection of all such continua by \mathbb{M}_X . For each such continuum $M_X^{\mathbf{m}}$ and each $i \geq 1$, we define $F : M_X^{\mathbf{m}} \rightarrow F(M_X^{\mathbf{m}}) \subset T_1^X \times [0, 3]$ by

$$F(x, y, z, t) = (f^*(x, y, z), t).$$

The following result is easy to prove. Hence, we omit the proof.

LEMMA 2.5. *Let X be a topological space, and suppose $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Let $f : X \rightarrow Y$ be a function into a topological space Y . If for each $\lambda \in \Lambda$, the restriction $f|_{X_\lambda} : X_\lambda \rightarrow Y$ is an open map (with respect to the subspace topologies), then f is an open map.*

LEMMA 2.6. *Let X be the mirror-glued continuum built from (X', A') and (X'', A'') , and let Y_X be its tri-fold symmetric extension. Consider $M_X^{\mathbf{m}}$ to be the meandering continuum with pattern \mathbf{m} associated with X , let F be as in Definition 2.4 and let $\pi : F(M_X^{\mathbf{m}}) \rightarrow T_1^X$ be the natural projection, then $\pi \circ F$ is an open map.*

Proof. Let $M_X^{\mathbf{m}} := (Y_X \times \{0\}) \cup \bigcup_{i=0}^{\infty} Z_i$. Then we can easily see that $(\pi \circ F)|_{Z_i} : Z_i \rightarrow T_i^X$ is a homeomorphism for each $i \geq 0$. Also, it is easy to see that $(\pi \circ F)|_{T_j^X \times \{0\}} : T_j^X \times \{0\} \rightarrow T_1^X$ is a homeomorphism for each $j \in \{1, 2, 3\}$. Therefore, using Lemma 2.5 finishes the proof. ■

LEMMA 2.7. *Let $J = [0, 1]$ be the mirror-glued continuum with*

$$J' = [0, 1], \quad J'' = [0, 1], \quad A' = \{1\}, \quad A'' = \{0\}.$$

Fix a meandering pattern $\mathbf{m} = (m(i))_{i=0}^{\infty} \subset \{1, 2, 3\}$, and let $M_J^{\mathbf{m}} \in \mathbb{M}_J$ be the corresponding continuum. Then for any mirror-glued continuum $X = (X', A')$ - (X'', A'') , there exist

$$M_X^{\mathbf{m}} \in \mathbb{M}_X \quad \text{and} \quad f : M_X^{\mathbf{m}} \rightarrow M_J^{\mathbf{m}}$$

a continuous surjection.

Sketch of proof. Let $f_0 : X \rightarrow J$ be a surjection with $f_0(A') = \{1\}$, $f_0(A'') = \{0\}$. Write

$$M_J^{\mathbf{m}} = (Y_J \times \{0\}) \cup \bigcup_{i=0}^{\infty} J_i, \quad M_X^{\mathbf{m}} = (Y_X \times \{0\}) \cup \bigcup_{i=0}^{\infty} Z_i,$$

where $J_i \cong J$ and $Z_i \cong X$ lie in the same slices $q^{-1}([d_{i+1}, d_i])$, glued along the corresponding A', A'' , and with $p(Z_i) = T_{m(i)}^X$. Using f_0 , define surjections $f_i : Z_i \rightarrow J_i$ such that on glued boundaries we have $f_i|_{A'_i} = f_{i+1}|_{A'_{i+1}}$. Extending these consistently along the tri-fold core yields a continuous surjection $f : M_X^{\mathbf{m}} \rightarrow M_J^{\mathbf{m}}$. ■

COROLLARY 2.8. *For each mirror-glued continuum*

$$X = (X', A')$$
- (X'', A'') ,

the family \mathbb{M}_X of meandering continua has no common model.

Proof. By [6, (20.4)], \mathbb{M}_J has no common model. From Lemma 2.7, for each pattern \mathbf{m} there are $M_J^{\mathbf{m}} \in \mathbb{M}_J$ and $M_X^{\mathbf{m}} \in \mathbb{M}_X$ with a continuous surjection $f : M_X^{\mathbf{m}} \rightarrow M_J^{\mathbf{m}}$. Hence any common model for \mathbb{M}_X would induce one for \mathbb{M}_J , a contradiction. ■

3. Indecomposable continua. The main aim of this section is to prove Theorem 3.3. For this, we need the following result.

THEOREM 3.1 (see [3, Theorem 1.1]). *For each continuum X , there exists an indecomposable continuum Y containing X and an open retraction $r : Y \rightarrow X$ such that each fiber $r^{-1}(x)$ is homeomorphic to the Cantor set. Moreover, Y is homeomorphic to the closure of a countable union of topological copies of X in Y .*

REMARK 3.2. In [3, Theorem 1.1], it is stated that Y is homeomorphic to the closure, in some space, of a countable union of topological copies of X . However, it is easy to see that this closure can be taken in Y itself.

THEOREM 3.3. *Let Z be a continuum. Define \mathbb{IC}_Z to be the family of indecomposable continua Y satisfying:*

- (I) $Z \subset Y$.
- (II) *There exists an open retraction $r : Y \rightarrow Z$ such that each fiber $r^{-1}(z)$ is homeomorphic to the Cantor set.*
- (III) *There exists a sequence $\{Z'_i\}_{i=1}^\infty$ of subcontinua of Y , each homeomorphic to Z , such that $\text{Cl}_Y(\bigcup_{i=1}^\infty Z'_i) = Y$.*

Then the following hold:

- (1) *The family \mathbb{IC}_Z does not admit a common model.*
- (2) *If Z is tree-like, then there exists a subfamily \mathbb{IC}'_Z of \mathbb{IC}_Z such that \mathbb{IC}'_Z does not have a common model, and each member of \mathbb{IC}'_Z is tree-like.*

Proof. First, we prove (1). Assume, for contradiction, that \mathbb{IC}_Z has a common model C_Z .

Let X', X'' be topological copies of Z such that $|X' \cap X''| = 1$, and let $A' \subset X', A'' \subset X''$. Set $X := X' \cup X''$, and assume that (X, X', X'', A', A'') satisfies the conditions of Definition 2.2, so that X is the (X', A') - (X'', A'') mirror-glued continuum.

By Corollary 2.8, the family \mathbb{M}_X does not admit a common model.

Take any $M_X^{\mathfrak{m}} = (Y_X \times \{0\}) \cup \bigcup_{i=0}^\infty Z_i \in \mathbb{M}_X$, where $Y_X = T_1^X \cup T_2^X \cup T_3^X$. Define $F : M_X^{\mathfrak{m}} \rightarrow F(M_X^{\mathfrak{m}}) \subset T_1^X \times [0, 3]$ as in Definition 2.4. Let $\pi : F(M_X^{\mathfrak{m}}) \rightarrow T_1^X$ be the projection, and define $u : T_1^X \rightarrow X'$ as the retraction folding T_1^X onto X' .

By Lemma 2.6, $\pi \circ F : M_X^{\mathfrak{m}} \rightarrow T_1^X$ is an open map. Hence, we see that each of $\pi \circ F$ and u is an open map with totally disconnected fibers. Therefore, so is $r_M := u \circ \pi \circ F : M_X^{\mathfrak{m}} \rightarrow X' \cong Z$.

By Theorem 3.1, there exist an indecomposable continuum $Y \supseteq M_X^{\mathfrak{m}}$ and an open retraction $r_Y : Y \rightarrow M_X^{\mathfrak{m}}$ with Cantor set fibers (this Y is actually constructed using an inverse limit as in the proof of (2); while this fact will be used in the proof of (2), we do not use it here). Moreover, by Theorem 3.1 and the construction of $M_X^{\mathfrak{m}}$, we can see that there exists a sequence $\{Z'_i\}_{i=1}^\infty$ of subcontinua of Y , each homeomorphic to Z , such that $\text{Cl}_Y(\bigcup_{i=1}^\infty Z'_i) = Y$.

Define the composition $r := r_M \circ r_Y : Y \rightarrow Z$. Then r is an open retraction with each fiber homeomorphic to the Cantor set, so $Y \in \mathbb{IC}_Z$.

Since C_Z is a common model for \mathbb{IC}_Z , there exists a continuous surjection $s : C_Z \rightarrow Y$, and hence $r \circ s : C_Z \rightarrow M_X^{\mathfrak{m}}$ is also a continuous surjection. This implies that \mathbb{M}_X admits a common model, contradicting Corollary 2.8.

Therefore, $\mathbb{I}\mathbb{C}_Z$ cannot have a common model. This completes the proof of (1).

Next, we prove (2). In the proof of (1), if Z is tree-like and $|X' \cap X''| = 1$, then we can easily see that X is tree-like. Hence, for each $i \in \{1, 2, 3\}$, the continuum T_i^X is tree-like. It is also easy to see that Y_X is tree-like and $M_X^{\mathbf{m}}$ is tree-like.

In the proof of [3, Theorem 1.1], for each $i \geq 1$, there exists a family $\{G_n^i\}_{n=1}^\infty$ of topological copies of X_i satisfying:

- $X_{i+1} = \text{Cl}_{X_{i+1}}(\bigcup_{n=1}^\infty G_n^i)$;
- $G_n^i \cap G_m^i \neq \emptyset$ if and only if $|n - m| \leq 1$;
- $|G_n^i \cap G_{n+1}^i| = 1$ for each $n \geq 1$;
- for each $\varepsilon > 0$, there exist $n_0 \geq 1$ and a retraction $r_{n_0} : X_{i+1} \rightarrow \bigcup_{n=1}^{n_0} G_n^i$ such that for each $y \in \bigcup_{n=1}^{n_0} G_n^i$, the diameter of $r_{n_0}^{-1}(y)$ is smaller than ε .

Therefore, in the proof of [3, Theorem 1.1], by replacing X with $M_X^{\mathbf{m}}$ and using the facts above, it follows that X_i is tree-like for each $i \geq 1$. Consequently, the inverse limit $Y = \varprojlim \{X_i, f_i\}$ is tree-like. We denote by $\mathbb{I}\mathbb{C}'_Z$ the collection of all continua Y constructed using the above method by varying the meandering pattern \mathbf{m} . From the above discussion, every element of $\mathbb{I}\mathbb{C}'_Z$ is tree-like. Moreover, by the argument in (1), we have $\mathbb{I}\mathbb{C}'_Z \subset \mathbb{I}\mathbb{C}_Z$. It also follows that $\mathbb{I}\mathbb{C}'_Z$ does not admit a common model. ■

REMARK 3.4. By the results of [2, 5, 7], the “tree-like” condition in part (2) of Theorem 3.3 cannot be strengthened to “arc-like” even if Z is an arc.

Acknowledgements. The authors thank the referee for carefully reviewing the paper.

Funding. The first author was partially supported by JSPS KAKENHI Grant number 21K03249.

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