

Critical ideals for countable compact spaces

by

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Abstract. For each countable ordinal α , we introduce an ideal conv_α and use it to characterize the class of all compact countable spaces which are homeomorphic to the space $\omega^\alpha \cdot n + 1$ with the order topology. The characterization is expressed in terms of finding a convergent subsequence defined on a set not belonging to conv_α .

1. Introduction. See Sections 2 and 3 for notions and notations used in the introduction.

For an ideal \mathcal{I} on ω , we define the class $\text{FinBW}(\mathcal{I})$ of all topological spaces X having the property that for every sequence $(x_n)_{n \in \omega}$ in X there exists $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n \in A}$ is convergent in X [11, Definition 1.1].

We see that $\text{FinBW}(\text{Fin})$ coincides with the class of all sequentially compact spaces, and one can show that $\text{FinBW}(\text{Fin} \otimes \text{Fin})$ coincides with the class of all finite spaces, whereas $\text{FinBW}(\text{BI})$ coincides with the class of all boring spaces [11, Theorem 6.5]. Using the Katětov order, one can show that the previously mentioned ideals are critical for the classes of spaces considered above, as shown in the following theorem.

THEOREM 1.1 ([11, Theorem 6.5]). *Let \mathcal{I} be an ideal on ω .*

- (1) $\mathcal{I} \leq_K \text{Fin} \Leftrightarrow \text{FinBW}(\mathcal{I})$ coincides with the class of all sequentially compact spaces.
- (2) $\text{Fin} \otimes \text{Fin} \leq_K \mathcal{I} \Leftrightarrow \text{FinBW}(\mathcal{I})$ coincides with the class of all finite spaces.

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(3) *The following conditions are equivalent:*

- (a) $\text{BI} \leq_K \mathcal{I}$ and $\text{Fin} \otimes \text{Fin} \not\leq_K \mathcal{I}$.
- (b) $\text{FinBW}(\mathcal{I})$ coincides with the class of all boring spaces.

In [14, Section 2.7], the author proved that

$$\text{conv} \not\leq_K \mathcal{I} \iff [0, 1] \in \text{FinBW}(\mathcal{I}).$$

Using the above equivalence, one can show that the ideal conv turns out to be critical for uncountable compact metric spaces as shown in the following theorem.

THEOREM 1.2. *Let \mathcal{I} be an ideal on ω . The following conditions are equivalent:*

- (1) $\text{conv} \not\leq_K \mathcal{I}$.
- (2) $\text{FinBW}(\mathcal{I})$ contains an uncountable compact metric space.
- (3) $\text{FinBW}(\mathcal{I})$ contains all compact metric spaces.

In the realm of compact countable spaces, we have the following theorem of Mazurkiewicz and Sierpiński which characterizes these spaces in terms of countable ordinals with the order topology. Note that every compact countable space is metrizable (see e.g. [16, Proposition 8.5.7]), and consequently it is sequentially compact.

THEOREM 1.3 ([13], see also [16, Theorem 8.6.10]). *A countable space X is compact if and only if X is homeomorphic to the space $\omega^\alpha \cdot n + 1$ with the order topology for some countable ordinal α and $n \in \omega$.*

Let \mathbb{K} denote the class of all countable compact spaces and \mathbb{K}_α denote the class of all compact spaces which are homeomorphic to $\omega^\beta \cdot n + 1$ for some $\beta \leq \alpha$ and $n \in \omega$. By the Mazurkiewicz–Sierpiński theorem,

$$\mathbb{K} = \bigcup_{\alpha < \omega_1} \mathbb{K}_\alpha.$$

The main objective of this paper is to define the ideals conv_α and prove the following theorem which shows that conv_α is a critical ideal for \mathbb{K}_α .

THEOREM. *Let \mathcal{I} be an ideal on ω and α be a countable ordinal. The following conditions are equivalent:*

- (1) $\text{conv}_{(1+\alpha)+1} \leq_K \mathcal{I}$ and $\text{conv}_{1+\alpha} \not\leq_K \mathcal{I}$.
- (2) $\text{FinBW}(\mathcal{I}) \cap \mathbb{K} = \mathbb{K}_\alpha$.

The next theorem (interesting in its own right) is a crucial ingredient in the proof of the theorem above.

THEOREM. *For any ideal \mathcal{I} on ω and countable ordinal α ,*

$$\text{conv}_{1+\alpha} \not\leq_K \mathcal{I} \iff \omega^\alpha + 1 \in \text{FinBW}(\mathcal{I}).$$

In this paper, we introduce and study the hierarchy of critical ideals conv_α . We analyze their mutual position under the Katětov order, showing that the critical ideals form a strictly decreasing hierarchy. The Katětov order exhibits rich structure around this hierarchy: for every pair of consecutive critical ideals, there exist continuum many pairwise incomparable ideals lying above one and not below the other. We provide a combinatorial characterization of the members of these ideals and observe that two of them coincide with ideals already examined in the existing literature. The main theorems concerning these ideals are proved together with the necessary auxiliary lemmas. We compare the critical ideals with the ideal conv and show that the entire sequence is critical for the space ω_1 in the previously defined sense, while no single ideal is critical for this space. Furthermore, we establish that the critical ideals possess the property KAT, and we determine their Borel complexity and associated cardinal characteristics in the context of ideals on countable sets.

2. Preliminaries. All topological spaces considered in the paper are assumed to be Hausdorff. A sequentially compact space X is called *boring* [11, Definition 3.21] if there exists a finite set $F \subseteq X$ such that each one-to-one convergent sequence in X converges to some point in F .

Recall that an ordinal number α is equal to the set of all ordinal numbers less than α . In particular, the smallest infinite ordinal number $\omega = \{0, 1, \dots\}$ is equal to the set of all natural numbers \mathbb{N} , and each natural number $n = \{0, \dots, n-1\}$ is equal to the set of all natural numbers less than n . Using this identification, we can for instance write $n \in k$ instead of $n < k$ and $n < \omega$ instead of $n \in \omega$ or $A \cap n$ instead of $A \cap \{0, 1, \dots, n-1\}$. Moreover, for ordinals $\alpha < \beta$, we write $[\alpha, \beta]$ to denote the set of all ordinals ξ such that $\alpha \leq \xi \leq \beta$, and similarly for (α, β) and other intervals. If A is a set of ordinals, we write $\text{ot}(A)$ to denote the *order type* of A , i.e. the unique ordinal α such that A and α are isomorphic.

An *ideal* on a set X is a nonempty family $\mathcal{I} \subseteq \mathcal{P}(X)$ which is closed under taking finite unions (i.e. if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$) and subsets (i.e. if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$). For an ideal \mathcal{I} , we write $\mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$ and call it the *coideal* of \mathcal{I} , and we write $\mathcal{I}^* = \{A \subseteq X : X \setminus A \in \mathcal{I}\}$ and call it the *dual filter* of \mathcal{I} . An ideal \mathcal{I} is *tall* if for every infinite $A \subseteq X$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$.

By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the topology of the Cantor space $\{0, 1\}^\omega$ and therefore we can assign topological complexity to ideals on ω . In particular, an ideal \mathcal{I} is Borel (F_σ , analytic, resp.) if \mathcal{I} is Borel (F_σ , analytic, resp.) as a subset of the Cantor space.

EXAMPLE 2.1.

- (1) $\text{Fin}(X)$ is the ideal of all finite subsets of X . We write Fin instead of $\text{Fin}(\omega)$.
- (2) If $\phi : \mathcal{P}(\omega) \rightarrow [0, \infty]$ is a countably additive measure such that $\phi(\omega) = \infty$, then \mathcal{I}_ϕ given by

$$A \in \mathcal{I}_\phi \iff \phi(A) < \infty$$

is a *summable ideal*. Fin is a summable ideal given by the measure $\phi(A) = |A|$ for all $A \subseteq \omega$. Each summable ideal is Σ_2^0 .

- (3) \mathcal{I}_d is the ideal of all subsets of ω of asymptotic density zero:

$$A \in \mathcal{I}_d \iff \limsup_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0.$$

It is a Π_3^0 ideal.

- (4) $\{\emptyset\} \otimes \text{Fin}$ is the ideal on $\omega \times \omega$ defined by

$$A \in \{\emptyset\} \otimes \text{Fin} \iff \forall i (|\{j : (i, j) \in A\}| < \omega).$$

It is a Π_3^0 ideal.

- (5) $\text{Fin}^2 = \text{Fin} \otimes \text{Fin}$ is the ideal on $\omega \times \omega$ defined by

$$A \in \text{Fin} \otimes \text{Fin} \iff \exists i_0 \forall i \geq i_0 (|\{j : (i, j) \in A\}| < \omega).$$

It is a Σ_4^0 ideal.

- (6) BI is the ideal on $\omega \times \omega \times \omega$ introduced in [11, Definition 4.1] and defined by

$$A \in \mathcal{BI} \iff \exists i_0 [\forall i < i_0 (\{(j, k) : (i, j, k) \in A\} \in \text{Fin} \otimes \text{Fin}) \wedge \forall i \geq i_0 (|\{(j, k) : (i, j, k) \in A\}| < \omega)].$$

It is a Σ_4^0 ideal.

The vertical section of a set $A \subseteq X \times Y$ at a point $x \in X$ is defined by $A_{(x)} = \{y \in Y : (x, y) \in A\}$. For ideals \mathcal{I} and \mathcal{J} on X and Y respectively, we define the following ideal (called the *Fubini product* of \mathcal{I} and \mathcal{J}):

$$\mathcal{I} \otimes \mathcal{J} = \{A \subseteq X \times Y : \{x \in X : A_{(x)} \notin \mathcal{J}\} \in \mathcal{I}\}.$$

Using the notion of the Fubini product, one can see that (see [11])

$$\text{BI} = (\{\emptyset\} \otimes \text{Fin}^2) \cap (\text{Fin} \otimes \text{Fin}(\omega^2)).$$

An ideal \mathcal{I} is a *P-ideal* if for every sequence $(A_n)_{n \in \omega}$ of elements of \mathcal{I} there is $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all $n \in \omega$. The ideals Fin , \mathcal{I}_d , $\{\emptyset\} \otimes \text{Fin}$ and all summable ideals are P-ideals, whereas Fin^2 and BI are not P-ideals.

Let \mathcal{I} and \mathcal{J} be ideals on X and Y respectively. We say that \mathcal{I} and \mathcal{J} are *isomorphic* (written $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $f : X \rightarrow Y$ such that $A \in \mathcal{I} \iff f[A] \in \mathcal{J}$ for every $A \subseteq X$. We say that \mathcal{J} is *below* \mathcal{I} in the *Katětov order* (written $\mathcal{J} \leq_K \mathcal{I}$) if there is a function $f : X \rightarrow Y$ such

that $f^{-1}[A] \in \mathcal{I}$ for every $A \in \mathcal{J}$. We say that the ideals \mathcal{I} and \mathcal{J} are \leq_K -equivalent if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$.

A relationship between Katětov order and classes $\text{FinBW}(\mathcal{I})$ is expressed in the following theorem which will be used in other sections.

THEOREM 2.2 ([11, Corollary 10.2]). *If $\mathcal{I} \leq_K \mathcal{J}$, then $\text{FinBW}(\mathcal{J}) \subseteq \text{FinBW}(\mathcal{I})$.*

3. Critical ideals. In this section, we define the critical ideals conv_α which characterize the classes of compact spaces described in the introduction. We also provide the combinatorial characterization of members of these ideals. Finally, we show that two of the newly defined ideals have already been studied in the existing literature.

For a topological space X , we write $c(X)$ to denote the set of all convergent sequences in X . For a subset $D \subseteq X$ of a topological space X , we write $\text{conv}(D)$ to denote the ideal on D consisting of all subsets of D which can be covered by ranges of finitely many sequences in D which are convergent in X , i.e. $A \in \text{conv}(D)$ if and only if $A \subseteq D$ and there exist $k \in \omega$ and sequences $(d_n^{(i)})_{n \in \omega} \in c(X) \cap D^\omega$ for $i < k$ such that

$$A \subseteq \bigcup_{i < k} \{d_n^{(i)} : n \in \omega\}.$$

A point $p \in X$ is an *accumulation* (or *limit*) *point* of a set $A \subseteq X$ in a topological space X if $p \in \text{cl}(A \setminus \{p\})$. We denote by A^d the *derived set* of A , i.e. the set of all accumulation points of A .

LEMMA 3.1. *Let X be a sequentially compact space. Let $D \subseteq X$ be a countable infinite subset of X .*

(1) *For every $A \subseteq D$,*

$$A \in \text{conv}(D) \iff A^d \text{ is finite.}$$

(2) *For every infinite set $A \subseteq D$ there exists an infinite set $B \subseteq A$ such that $B \in \text{conv}(D)$. In particular, the ideal $\text{conv}(D)$ is tall.*

Proof. ((1), \Rightarrow) Take any $A \in \text{conv}(D)$. There exists $k \in \omega$ and sequences $(d_n^{(i)})_{n \in \omega} \in c(X) \cap D^\omega$ for $i < k$ such that $A \subseteq \bigcup_{i < k} \{d_n^{(i)} : n \in \omega\}$. The sequence $(d_n^{(i)})_{n \in \omega}$ is convergent for each i , so let $\{d_i : i < k\}$ be the set of their limits. Then $A^d \subseteq (\bigcup_{i < k} \{d_n^{(i)} : n \in \omega\})^d \subseteq \{d_i : i < k\}$ which is finite.

((1), \Leftarrow) Assume that A is infinite, A^d is finite and let k be the cardinality of the set. We will prove the statement by induction on $k \in \omega$.

For $k = 1$, let $A^d = \{d\}$. Since A is countably infinite, we can enumerate its elements injectively as $\{a_n : n \in \omega\}$. We claim that the sequence $(a_n)_{n \in \omega}$ is convergent to d . Let U be an open neighborhood of d and suppose for the sake of contradiction that there exist infinitely many elements of the

set $\{a_n : n \in \omega\}$ that are not in U . Since X is sequentially compact, there exists an infinite set $C \subseteq \omega$ such that $(a_n)_{n \in C}$ is convergent to a point p that is different from d , so $p \in A^d$, which contradicts our initial assumption. Consequently, we can cover A with a convergent sequence, so $A \in \text{conv}(D)$.

Suppose the statement is true for k . Take $A^d = \{d_i : i < k + 1\}$. Since X is Hausdorff we can find a collection $\{U_i \subseteq X : d_i \in U_i\}$ of open sets that are pairwise disjoint. Define a partition of A by $B_1 = A \cap U_k$ and $B_2 = A \setminus U_k$.

We claim $B_1^d \subseteq \{d_k\}$ and $B_2^d \subseteq \{d_i : i < k\}$. Take $i < k$. Then $B_1 \cap U_i \subseteq U_k \cap U_i = \emptyset$, so $d_i \notin B_1^d$. Similarly, $B_2 \cap U_k = \emptyset$, so $d_k \notin B_2^d$.

So by the induction hypothesis $B_1 \in \text{conv}(D)$ and $B_2 \in \text{conv}(D)$ and therefore $A \in \text{conv}(D)$.

(2) Take an infinite set $A \subseteq D$. Take distinct points $a_n \in A$ for $n \in \omega$. Since X is sequentially compact, there is a convergent subsequence $(a_{k_n})_{n \in \omega}$. Then $B = \{a_{k_n} : n \in \omega\}$ is an infinite subset of A such that $B \in \text{conv}(D)$. ■

DEFINITION 3.2.

(1) If $X = [0, 1]$ with the Euclidean topology, we define the ideal

$$\text{conv} = \text{conv}(\mathbb{Q} \cap [0, 1]).$$

(2) For a countable ordinal $\alpha \geq 2$ and the space $X = \omega^\alpha + 1$ with the order topology, we define the ideal

$$\text{conv}_\alpha = \text{conv}(\omega^\alpha + 1).$$

In the rest of the paper, we write “ α ” to mean “a countable ordinal $\alpha \geq 2$ ”.

Below we show that the Borel complexity of conv_α is at most Σ_4^0 , and in Corollary 7.4 we will show that this complexity is not lower.

PROPOSITION 3.3. $\text{conv}_\alpha \in \Sigma_4^0$ for every α .

Proof. We use a standard quantifier-counting argument. Let $\{U_n : n \in \omega\}$ be a countable basis for the topology of $X = \omega^\alpha + 1$. By Lemma 3.1, $A \in \text{conv}_\alpha \Leftrightarrow |A^d| < \omega$, hence

$$A \in \text{conv}_\alpha$$

$$\iff \exists F \in [X]^{<\omega} \forall \beta \in X \setminus F \exists n \in \omega (\beta \in U_n \wedge \forall \gamma \in U_n (\gamma \notin A)).$$

Since the sets $\{A \subseteq X : \gamma \notin A\}$ are closed in X for every γ , we can count quantifiers to deduce that conv_α is $F_{\sigma\delta\sigma}$. ■

The following *purely combinatorial* characterization of members of the topologically defined ideal conv_α will be used repeatedly in the following sections.

PROPOSITION 3.4. Let $A \subseteq \omega^\alpha + 1$. The following conditions are equivalent:

- (1) $A \in \text{conv}_\alpha$.
(2) For every increasing sequence $(\lambda_n)_{n < \omega}$ in ω^α , the set $A \cap [\lambda_n, \lambda_{n+1}]$ is finite for all but finitely many n .

Proof. (1) \Rightarrow (2) Take $A \in \text{conv}_\alpha$ and suppose for the sake of contradiction that there exists an increasing sequence $(\lambda_n)_{n \in \omega}$ in ω^α and an infinite set $B \subseteq \omega$ such that the set $A \cap [\lambda_n, \lambda_{n+1}]$ is infinite for every $n \in B$. Since $\omega^\alpha + 1$ is sequentially compact, we can find accumulation points of A in $(\lambda_n, \lambda_{n+1}]$ for every $n \in B$. Then A^d is infinite, so $A \notin \text{conv}_\alpha$ by Lemma 3.1, a contradiction.

(2) \Rightarrow (1) Take $A \notin \text{conv}_\alpha$. By Lemma 3.1, A^d is infinite, so we can pick a strictly increasing sequence $(\lambda_n)_{n \in \omega}$ in A^d . Since successor ordinals are isolated points, λ_n is a limit ordinal for each n . Then $(\xi, \lambda_{n+1}]$ is a neighborhood of λ_{n+1} for every $\xi \in (\lambda_n, \lambda_{n+1})$, and consequently $A \cap (\lambda_n, \lambda_{n+1}]$ is infinite for every $n \in \omega$. ■

In the case of successor ordinals, we can obtain the following extension of the above characterization.

PROPOSITION 3.5. *Let $A \subseteq \omega^{\alpha+1} + 1$. The following conditions are equivalent:*

- (1) $A \in \text{conv}_{\alpha+1}$.
(2) For every increasing sequence $(\lambda_n)_{n < \omega}$ in $\omega^{\alpha+1}$, the set $A \cap [\lambda_n, \lambda_{n+1}]$ is finite for all but finitely many n .
(3) (a) $A \cap (\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)]$ is finite for all but finitely many n ,
(b) $A \cap (\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)] \in \text{conv}((\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)])$ for all n .

Proof. (1) \Leftrightarrow (2) This is proved in Proposition 3.4.

(2) \Rightarrow (3a) Take $\lambda_n = \omega^\alpha \cdot n$ and apply item (2).

(2) \Rightarrow (3b) Let $n \in \omega$. Then we obtain (3b) by applying (2) to every increasing sequence $(\lambda_k)_{k \in \omega}$ in $(\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)]$.

(3) \Rightarrow (1) Let n_0 be such that $A \cap (\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)]$ is finite for every $n \geq n_0$. Then

$$A^d \subseteq \bigcup_{n < n_0} (A \cap (\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)])^d \cup \{\omega^{\alpha+1}\}.$$

Since $A \cap (\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)] \in \text{conv}((\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)])$ for every $n < n_0$, we deduce that A^d is finite, hence $A \in \text{conv}_{\alpha+1}$. ■

Since $\omega^\alpha + 1$ and $(\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)]$ are order isomorphic, we see that the ideals conv_α and $\text{conv}((\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)])$ are isomorphic for every n . Then we can use item (3) of the above proposition to obtain an ideal isomorphic to $\text{conv}_{\alpha+1}$.

COROLLARY 3.6. *The ideal $\text{conv}_{\alpha+1}$ is isomorphic to the ideal*

$$(\text{Fin} \otimes \text{Fin}(\omega^\alpha + 1)) \cap (\{\emptyset\} \otimes \text{conv}_\alpha).$$

We finish this section with a proposition which shows that the results from Section 4 extend Theorem 1.1.

PROPOSITION 3.7. *The ideals conv_2 and $\text{Fin} \otimes \text{Fin}$ (conv_3 and BI , resp.) are isomorphic.*

Proof. Apply Corollary 3.6 and the fact that $\text{BI} = (\{\emptyset\} \otimes \text{Fin}^2) \cap (\text{Fin} \otimes \text{Fin}(\omega^2))$. ■

4. Proofs of the main theorems. In this section we provide proofs of the main theorems together with the necessary lemmata.

LEMMA 4.1. *Let \mathcal{I} be an ideal on ω and α be a countable limit ordinal. If $f : \omega \rightarrow \alpha$ is a function such that $f^{-1}(\xi) \in \mathcal{I}$ for each $\xi \in \alpha$, then for any set $B \subseteq \omega$ such that $B \notin \mathcal{I}$ there exists an increasing sequence $(\lambda_n)_{n \in \omega}$ in α such that*

- (1) $B \cap f^{-1}[[\lambda_n, \lambda_{n+1}]] \in \mathcal{I}$ for each n ,
- (2) $B \cap f^{-1}[[\lambda_0, \lambda]] \notin \mathcal{I}$, where $\lambda = \sup_n \lambda_n$.

Proof. We prove the lemma by induction on α . For $\alpha = \omega$, the sequence $\lambda_n = n$ works. Assume the lemma holds for any limit $\beta < \alpha$. Since α is a limit ordinal, there is an increasing sequence $(\alpha_n)_{n < \omega}$ in α such that $\alpha_0 = 0$ and $\alpha = \sup_n \alpha_n$. If $B \cap f^{-1}[[0, \alpha_n]] \in \mathcal{I}$ for each $n < \omega$, the sequence $\lambda_n = \alpha_n$ works. Now, assume there is $n < \omega$ with $B \cap f^{-1}[[0, \alpha_n]] \notin \mathcal{I}$. Let β be the minimal limit ordinal such that $\beta \leq \alpha_n$ and $B \cap f^{-1}[[0, \beta]] \notin \mathcal{I}$. Applying the inductive hypothesis, we obtain an increasing sequence $(\lambda_n)_{n \in \omega}$ in β with the required properties. ■

THEOREM 4.2. *For any ideal \mathcal{I} on ω ,*

$$\text{conv}_{1+\alpha} \not\leq_K \mathcal{I} \iff \omega^\alpha + 1 \in \text{FinBW}(\mathcal{I}).$$

In particular, if α is infinite, then $\text{conv}_\alpha \not\leq_K \mathcal{I} \iff \omega^\alpha + 1 \in \text{FinBW}(\mathcal{I})$.

Proof. (\Rightarrow) Let $f : \omega \rightarrow \omega^\alpha + 1$. We need to find $D \notin \mathcal{I}$ such that the subsequence $f \upharpoonright D$ is convergent.

If there exists $\xi \in \omega^\alpha + 1$ such that $D = f^{-1}(\xi) \notin \mathcal{I}$, then $f \upharpoonright D$ is convergent to ξ , so we are done. Now, suppose $f^{-1}(\xi) \in \mathcal{I}$ for each $\xi \in \omega^\alpha + 1$.

We define $A_\xi = f^{-1}(\xi)$ for each $\xi \in \omega^\alpha + 1$.

Let $g : \omega \rightarrow \omega^{1+\alpha} + 1$ be a function such that

- (1) $g[A_\xi] \subseteq [\omega \cdot \xi, \omega \cdot (\xi + 1))$ for each $\xi < \omega^\alpha$,
- (2) $g[A_{\omega^\alpha}] \subseteq \{\omega^{1+\alpha}\}$,
- (3) $g \upharpoonright (\omega \setminus A_{\omega^\alpha})$ is injective.

Since $\text{conv}_{1+\alpha} \not\leq_K \mathcal{I}$, there exists a set $B \notin \mathcal{I}$ such that $g[B] \in \text{conv}_{1+\alpha}$. By Lemma 4.1, there exists an increasing sequence $(\lambda_n)_{n \in \omega}$ in ω^α such that $B \cap f^{-1}[[\lambda_n, \lambda_{n+1}]] \in \mathcal{I}$ for each n and $C = B \cap f^{-1}[[\lambda_0, \lambda]] \notin \mathcal{I}$, where

$\lambda = \sup_n \lambda_n$. Since $(\lambda_n)_{n \in \omega}$ is an increasing sequence in ω^α , the sequence $(\omega \cdot \lambda_n)_{n \in \omega}$ is increasing in $\omega^{1+\alpha}$, so by Proposition 3.4, there is $n_0 \in \omega$ such that $g[B] \cap [\omega \cdot \lambda_n, \omega \cdot \lambda_{n+1})$ is finite for each $n \geq n_0$. Since g is injective on $\omega \setminus A_{\omega^\alpha}$, the set

$$B \cap g^{-1}[[\omega \cdot \lambda_n, \omega \cdot \lambda_{n+1})] = B \cap \bigcup_{\lambda_n \leq \xi < \lambda_{n+1}} A_\xi$$

is also finite for each $n \geq n_0$.

Let

$$D = C \setminus \bigcup_{n < n_0} \left(\bigcup_{\lambda_n \leq \xi < \lambda_{n+1}} A_\xi \right).$$

Since $C \notin \mathcal{I}$ and for each n

$$C \cap \bigcup_{\lambda_n \leq \xi < \lambda_{n+1}} A_\xi = B \cap f^{-1}[[\lambda_n, \lambda_{n+1})] \in \mathcal{I},$$

we conclude that $D \notin \mathcal{I}$. Once we show that $f \upharpoonright D$ is convergent to λ , the proof will be finished.

Let U be an open neighborhood of λ . Then there exists some k_0 such that $(\lambda_{k_0}, \lambda] \subseteq U$. Notice that

$$\begin{aligned} \{n \in D : f(n) \notin U\} &\subseteq \{n \in D : f(n) \leq \lambda_{k_0}\} \\ &\subseteq D \cap \bigcup_{n < k_0} \left(\bigcup_{\lambda_n \leq \xi < \lambda_{n+1}} A_\xi \right) = \bigcup_{n < k_0} \left(D \cap \bigcup_{\lambda_n \leq \xi < \lambda_{n+1}} A_\xi \right). \end{aligned}$$

Since the set $D \cap \bigcup_{\lambda_n \leq \xi < \lambda_{n+1}} A_\xi$ is finite for each n , we see that $\{n \in D : f(n) \notin U\}$ is finite as well.

(\Leftarrow) Suppose that $\text{conv}_{1+\alpha} \leq_K \mathcal{I}$ and let $f : \omega \rightarrow \omega^{1+\alpha} + 1$ be a witnessing function, meaning that $B \notin \mathcal{I}$ implies that $f[B] \notin \text{conv}_{1+\alpha}$. Let $g : \omega^{1+\alpha} + 1 \rightarrow \omega^\alpha + 1$ be a function such that $g(\omega^{\alpha+1}) = \omega^\alpha$ and

$$g^{-1}(\xi) = [\omega \cdot \xi, \omega \cdot (\xi + 1)) \quad \text{for each } \xi \in \omega^\alpha.$$

Let $h = g \circ f$. Then $h : \omega \rightarrow \omega^\alpha + 1$, so once we show that the subsequence $h \upharpoonright A$ is not convergent for any $A \notin \mathcal{I}$, the proof will be finished. Suppose for the sake of contradiction that there exists some $A \notin \mathcal{I}$ such that $h \upharpoonright A$ is convergent to some $\beta \in \omega^\alpha + 1$.

CASE (1): β is a successor ordinal or $\beta = 0$. Let $U = \{\beta\}$. As U is a neighborhood of β , there is a finite set $K \subseteq \omega$ such that $h[A \setminus K] = \{\beta\}$. Then $f[A \setminus K] \subseteq g^{-1}(\beta) = [\omega \cdot \beta, \omega \cdot \beta + \omega)$. But then $f[A \setminus K]^d \subseteq [\omega \cdot \beta, \omega \cdot \beta + \omega)^d = \{\omega \cdot \beta + \omega\}$. Then $f[A] \in \text{conv}_{\alpha+1}$ (by Lemma 3.1), a contradiction with f being the witness for $\text{conv}_{1+\alpha} \leq_K \mathcal{I}$.

CASE (2): β is a nonzero limit ordinal. Let $(\lambda_n)_{n < \omega}$ be any increasing sequence in β such that $\beta = \sup_n \lambda_n$. We claim that the set $f[A] \cap [\omega \cdot \lambda_n, \omega \cdot \lambda_{n+1})$ is finite for all n .

Suppose for the sake of contradiction that $f[A] \cap [\omega \cdot \lambda_n, \omega \cdot \lambda_{n+1})$ is infinite for some n . Then the set $A \cap f^{-1}[[\omega \cdot \lambda_n, \omega \cdot \lambda_{n+1})]$ is infinite. The set $V = (\lambda_{n+1}, \beta)$ is an open neighborhood of β , but

$$A \cap f^{-1}[[\omega \cdot \lambda_n, \omega \cdot \lambda_{n+1})] \subseteq \{i \in A : h(i) \notin V\},$$

which leads to a contradiction with β being the limit of the subsequence $h \upharpoonright A$.

If we show that $f[A]^d$ is finite, then $f[A] \in \text{conv}_{\alpha+1}$ (by Proposition 3.1), which will lead to a contradiction with f being the witness for $\text{conv}_{\alpha+1} \leq_K \mathcal{I}$.

Since

$$\begin{aligned} f[A]^d &= (f[A] \cap [0, \omega \cdot \beta))^d \cup (f[A] \cap [\omega \cdot \beta, \omega \cdot (\beta + 1)))^d \\ &\quad \cup (f[A] \cap [\omega \cdot (\beta + 1), \omega^{1+\alpha}))^d, \end{aligned}$$

it is enough to show that the three sets from the above union are finite.

First, we show that $(f[A] \cap [0, \omega \cdot \beta))^d \subseteq \{\omega \cdot \beta\}$. Take any $\gamma \in [0, \omega \cdot \beta)$. Then $\gamma < \omega \cdot \lambda_k$ for some k . Let $U = [0, \omega \cdot \lambda_k)$. Then U is an open set containing γ . Moreover, $f[A] \cap U = \bigcup_{n < k} f[A] \cap [\omega \cdot \lambda_n, \omega \cdot \lambda_{n+1})$, which is finite. Hence γ is not an accumulation point of $f[A]$.

Second, we observe that

$$(f[A] \cap [\omega \cdot \beta, \omega \cdot (\beta + 1)))^d \subseteq [\omega \cdot \beta, \omega \cdot (\beta + 1))^d = \{\omega \cdot (\beta + 1)\}.$$

Third, we show that $(f[A] \cap [\omega \cdot (\beta + 1), \omega^{1+\alpha}))^d = \emptyset$. Let $K = \{n \in A : h(n) > \beta\}$. Since $U = [0, \beta)$ is an open neighborhood of β , K must be finite, so the set $f[A] \cap [\omega \cdot (\beta + 1), \omega^{1+\alpha})$ is finite as well, and consequently its derived set is empty. ■

The *disjoint union of topological spaces* X_i for $i < n$ ($n < \omega$) is a topological space on the set

$$\bigsqcup_{i < n} X_i = \bigcup_{i < n} X_i \times \{i\}$$

with open sets of the form

$$\bigcup_{i < n} U_i \times \{i\} \quad \text{for any open sets } U_i \text{ in } X_i.$$

LEMMA 4.3. *If $X \in \text{FinBW}(\mathcal{I})$, then $\bigsqcup_{i < n} X \in \text{FinBW}(\mathcal{I})$ for any $n < \omega$.*

Proof. Take any sequence $f : \omega \rightarrow \bigsqcup_{i < n} X$ and let $\pi : \bigsqcup_{i < n} X \rightarrow X$ be the projection onto the first coordinate. Then there exists $A \notin \mathcal{I}$ such that the subsequence $(\pi \circ f) \upharpoonright A$ is convergent to some $p \in X$. Let us observe that $f^{-1}[X \times \{i\}] \cap A \notin \mathcal{I}$ for some $i < n$. Let $C = f^{-1}[X \times \{i\}] \cap A \notin \mathcal{I}$. We claim that the subsequence $f \upharpoonright C$ is convergent to (p, i) . Take any open neighborhood U of (p, i) . Then $\pi[U]$ is an open neighborhood of $\pi(p, i) = p$,

so there exists some finite set K such that $(\pi \circ f)[A \setminus K] \subseteq \pi[U]$, and it follows that $f[C \setminus K] \subseteq U$. ■

THEOREM 4.4. *Let \mathcal{I} be an ideal on ω .*

- (1) $\text{conv}_{1+\alpha} \not\leq_K \mathcal{I} \Rightarrow \mathbb{K}_\alpha \subseteq \text{FinBW}(\mathcal{I})$.
- (2) $\text{conv}_{(1+\alpha)+1} \leq_K \mathcal{I} \Rightarrow \text{FinBW}(\mathcal{I}) \cap \mathbb{K} \subseteq \mathbb{K}_\alpha$.
- (3) $\text{conv}_{(1+\alpha)+1} \leq_K \mathcal{I}$ and $\text{conv}_{1+\alpha} \not\leq_K \mathcal{I} \Leftrightarrow \text{FinBW}(\mathcal{I}) \cap \mathbb{K} = \mathbb{K}_\alpha$.

In particular, if α is infinite, then

- $\text{conv}_\alpha \not\leq_K \mathcal{I} \Rightarrow \mathbb{K}_\alpha \subseteq \text{FinBW}(\mathcal{I})$.
- $\text{conv}_{\alpha+1} \leq_K \mathcal{I} \Rightarrow \text{FinBW}(\mathcal{I}) \cap \mathbb{K} \subseteq \mathbb{K}_\alpha$.
- $\text{conv}_{\alpha+1} \leq_K \mathcal{I}$ and $\text{conv}_\alpha \not\leq_K \mathcal{I} \Leftrightarrow \text{FinBW}(\mathcal{I}) \cap \mathbb{K} = \mathbb{K}_\alpha$.

Proof. (1) If $\text{conv}_{1+\alpha} \not\leq_K \mathcal{I}$ then by Theorem 4.2, $\omega^\alpha + 1 \in \text{FinBW}(\mathcal{I})$ which implies that $\omega^\beta + 1 \in \text{FinBW}(\mathcal{I})$ for any $\beta \leq \alpha$. Since $\omega^\beta \cdot n + 1$ is homeomorphic to a disjoint union of finitely many copies of the space $\omega^\beta + 1$, we can apply Lemma 4.3 to obtain $\omega^\beta \cdot n + 1 \in \text{FinBW}(\mathcal{I})$ for any $\beta \leq \alpha$ and $n \in \omega$. Consequently, $\mathbb{K}_\alpha \subseteq \text{FinBW}(\mathcal{I})$.

(2) If $\text{conv}_{(1+\alpha)+1} \leq_K \mathcal{I}$ then by Theorem 4.2, $\omega^{\alpha+1} + 1 \notin \text{FinBW}(\mathcal{I})$, which implies that $\omega^\beta \cdot n + 1 \notin \text{FinBW}(\mathcal{I})$ for each $\beta \geq \alpha + 1$ and $n \in \omega \setminus \{0\}$, which leads to $\text{FinBW}(\mathcal{I}) \cap \mathbb{K} \subseteq \mathbb{K}_\alpha$ by Theorem 1.3.

(3) follows from items (1) and (2), and Theorem 4.2. ■

5. Katětov order among the critical ideals. In this section, we show that the critical ideals form a strictly decreasing hierarchy with respect to the Katětov order. Moreover, we demonstrate that the Katětov order exhibits rich structure around this hierarchy, namely, for every pair of consecutive critical ideals, there exist continuum many pairwise incomparable ideals lying above one and not below the other.

PROPOSITION 5.1. *If $\alpha < \beta$, then $\text{conv}_\beta \leq_K \text{conv}_\alpha$.*

Proof. It is enough to notice that the function $f : \omega^\alpha + 1 \rightarrow \omega^\beta + 1$ given by $f(\xi) = \xi$ is a witness for $\text{conv}_\beta \leq_K \text{conv}_\alpha$. ■

LEMMA 5.2. *Let $A, B \subseteq \omega^\alpha$. If the order types of A and B are smaller than ω^α , then the order type of $A \cup B$ is smaller than ω^α , and consequently $\omega^\alpha \setminus (A \cup B) \neq \emptyset$.*

Proof. Suppose for the sake of contradiction that the order type of $A \cup B$ is ω^α . Since ω^α is an indecomposable ordinal, we obtain (see e.g. [10, Exercise 5 in Chapter I]) that the order type of A is ω^α or the order type of B is ω^α , a contradiction. ■

DEFINITION 5.3. For every α and every ideal \mathcal{I} on ω , we define $\text{conv}_{\alpha+1}^{\mathcal{I}}$ to be the ideal on $\omega^{\alpha+1} + 1$ given by

$$A \in \text{conv}_{\alpha+1}^{\mathcal{I}} \iff \{i \in \omega : |A \cap (\omega^\alpha \cdot i, \omega^\alpha \cdot (i+1))| = \omega\} \in \mathcal{I} \text{ and} \\ A \cap (\omega^\alpha \cdot i, \omega^\alpha \cdot (i+1)) \in \text{conv}((\omega^\alpha \cdot i, \omega^\alpha \cdot (i+1))) \text{ for each } i \in \omega.$$

The following straightforward proposition reveals a relationship between the ideals $\text{conv}_{\alpha+1}$ and $\text{conv}_{\alpha+1}^{\mathcal{I}}$, and provides a counterpart of Corollary 3.6.

PROPOSITION 5.4.

- (1) $\text{conv}_{\alpha+1} = \text{conv}_{\alpha+1}^{\text{Fin}} \subseteq \text{conv}_{\alpha+1}^{\mathcal{I}}$.
- (2) $\text{conv}_{\alpha+1}^{\mathcal{I}}$ is isomorphic to the ideals $(\mathcal{I} \otimes \text{Fin}(\omega^\alpha + 1)) \cap (\{\emptyset\} \otimes \text{conv}_\alpha)$.

LEMMA 5.5. Let \mathcal{I} be an ideal on ω . If a function $f : \omega^{\alpha+1} + 1 \rightarrow \omega^\alpha + 1$ and an increasing sequence $(\lambda_n)_{n < \omega}$ are such that $\lambda_0 = 0$, $\sup\{\lambda_n : n < \omega\} = \omega^\alpha$ and the order type of the set $f^{-1}[[\lambda_n, \lambda_{n+1})]$ is smaller than ω^α for each $n < \omega$, then there exists $A \notin \text{conv}_{\alpha+1}^{\mathcal{I}}$ such that $f[A] \in \text{conv}_\alpha$.

Proof. First, we notice that by Lemma 5.2,

$$[\omega^\alpha \cdot k, \omega^\alpha \cdot (k+1)) \setminus \bigcup_{i \in F} f^{-1}[[\lambda_i, \lambda_{i+1})] \neq \emptyset$$

for every finite set $F \subseteq \omega$ and $k \in \omega$. Now, we take any function $g : \omega \rightarrow \omega$ such that $g^{-1}(n)$ is infinite for each $n \in \omega$, and inductively pick elements $a_n \in \omega^{\alpha+1}$ for $n \in \omega$ in such a way that

$$a_n \in [\omega^\alpha \cdot g(n), \omega^\alpha \cdot (g(n)+1)) \setminus \bigcup \{f^{-1}[[\lambda_i, \lambda_{i+1})] : \exists j < n (f(a_j) \in [\lambda_i, \lambda_{i+1}))\}.$$

Then the set $A = \{a_n : n \in \omega\}$ has the following properties:

- (1) $A \cap [\omega^\alpha \cdot k, \omega^\alpha \cdot (k+1))$ is infinite for each $k \in \omega$,
- (2) for each $n \in \omega$ there is $i \in \omega$ such that $f(a_n) \in [\lambda_i, \lambda_{i+1})$,
- (3) $|f[A] \cap [\lambda_i, \lambda_{i+1})| \leq 1$ for each $i \in \omega$.

By property (1), we get $[\omega^\alpha \cdot k, \omega^\alpha \cdot (k+1)) \cap A^d \neq \emptyset$ for each $k \in \omega$, and consequently $A \notin \text{conv}_{\alpha+1}^{\mathcal{I}}$. By properties (2) and (3), we find that $(f[A])^d = \{\omega^\alpha\}$, and consequently $f[A] \in \text{conv}_\alpha$. ■

THEOREM 5.6. $\text{conv}_\alpha \not\leq_K \text{conv}_{\alpha+1}^{\mathcal{I}}$ for every α and every ideal \mathcal{I} . In particular, $\text{conv}_\alpha \not\leq_K \text{conv}_{\alpha+1}$ for every α .

Proof. We proceed by induction on α . First, we consider the successor case. We assume that $\text{conv}_\alpha \not\leq_K \text{conv}_{\alpha+1}^{\mathcal{I}}$ for every ideal \mathcal{I} on ω and want to show that $\text{conv}_{\alpha+1} \not\leq_K \text{conv}_{\alpha+2}^{\mathcal{I}}$ for every ideal \mathcal{I} on ω . Let $\lambda_n = \omega^\alpha \cdot n$ for each $n \in \omega$. Then $(\lambda_n)_{n < \omega}$ is an increasing sequence in $\omega^{\alpha+1}$ such that $\lambda_0 = 0$ and $\sup\{\lambda_n : n < \omega\} = \omega^{\alpha+1}$. Suppose for the sake of contradiction that $\text{conv}_{\alpha+1} \leq_K \text{conv}_{\alpha+2}^{\mathcal{I}}$ for some ideal \mathcal{I} on ω , and let $f : \omega^{\alpha+2} + 1 \rightarrow \omega^{\alpha+1} + 1$ be a witness for this. We have two cases.

CASE (1): *The order type of $f^{-1}[[\lambda_n, \lambda_{n+1}]]$ is smaller than $\omega^{\alpha+1}$ for each $n \in \omega$.* In this case, we can use Lemma 5.5, to obtain a set $A \notin \text{conv}_{\alpha+2}^{\mathcal{I}}$ such that $f[A] \in \text{conv}_{\alpha+1}$. But this contradicts the fact that f is a witness for $\text{conv}_{\alpha+1} \leq_K \text{conv}_{\alpha+2}^{\mathcal{I}}$.

CASE (2): *The order type of $f^{-1}[[\lambda_n, \lambda_{n+1}]]$ is greater than or equal to $\omega^{\alpha+1}$ for some $n \in \omega$.* Let $A \subseteq f^{-1}[[\lambda_n, \lambda_{n+1}]]$ be a set which has the order type equal to $\omega^{\alpha+1}$. Then the order type of $f[A]$ is at most ω^α , since

$$\text{ot}(f[A]) \leq \text{ot}([\lambda_n, \lambda_{n+1}]) = \text{ot}([\omega^\alpha \cdot n, \omega^\alpha \cdot (n+1)]) = \omega^\alpha.$$

Then using the function $f \upharpoonright A$, we can obtain a witness for $\text{conv}_\alpha \leq_K \text{conv}_{\alpha+1}$, a contradiction with the inductive hypothesis.

Finally, we consider the limit case. We assume that α is a limit ordinal and $\text{conv}_\beta \not\leq_K \text{conv}_{\beta+1}^{\mathcal{I}}$ for each $\beta < \alpha$ and each ideal \mathcal{I} on ω . Let $(\lambda_n)_{n < \omega}$ be an increasing sequence in ω^α such that $\lambda_0 = 0$ and $\sup\{\lambda_n : n < \omega\} = \omega^\alpha$. Suppose for the sake of contradiction that $\text{conv}_\alpha \leq_K \text{conv}_{\alpha+1}^{\mathcal{I}}$ for some ideal \mathcal{I} on ω , and let $f : \omega^{\alpha+1} + 1 \rightarrow \omega^\alpha + 1$ be a witness for this. We have two cases.

CASE (1): *The order type of $f^{-1}[[\lambda_n, \lambda_{n+1}]]$ is smaller than ω^α for each $n \in \omega$.* In this case, we can use Lemma 5.5, to obtain a set $A \notin \text{conv}_{\alpha+1}^{\mathcal{I}}$ such that $f[A] \in \text{conv}_\alpha$. But this contradicts the fact that f is a witness for $\text{conv}_\alpha \leq_K \text{conv}_{\alpha+1}^{\mathcal{I}}$.

CASE (2): *The order type of $f^{-1}[[\lambda_n, \lambda_{n+1}]]$ is greater than or equal to ω^α for some $n \in \omega$.* Let $A \subseteq f^{-1}[[\lambda_n, \lambda_{n+1}]]$ be a set which has the order type equal to ω^α . Then the order type of $f[A]$ is smaller than ω^α since

$$\text{ot}(f[A]) \leq \text{ot}([\lambda_n, \lambda_{n+1}]) \leq \text{ot}(\lambda_{n+1}) = \lambda_{n+1} < \omega^\alpha.$$

Let $\beta < \alpha$ be such that $\lambda_{n+1} < \omega^\beta$. Then using the function $f \upharpoonright A$, we can obtain a witness for $\text{conv}_\beta \leq_K \text{conv}_\alpha$. Since $\beta + 1 < \alpha$, we get $\text{conv}_\alpha \leq_K \text{conv}_{\beta+1}$ by Proposition 5.1. Consequently, $\text{conv}_\beta \leq_K \text{conv}_{\beta+1}$, a contradiction with the inductive hypothesis. ■

COROLLARY 5.7. *If $\alpha < \beta$, then $\text{conv}_\beta \leq_K \text{conv}_\alpha$ and $\text{conv}_\alpha \not\leq_K \text{conv}_\beta$.*

Proof. It follows from Proposition 5.1 and Theorem 5.6. ■

LEMMA 5.8. *Let \mathcal{I}_0 and \mathcal{I}_1 be ideals on ω such that $\mathcal{I}_0 \not\leq_K \mathcal{I}_1$ and \mathcal{I}_1 is summable and tall. Let \mathcal{I} be a tall ideal on a countable set X such that*

$$\mathcal{I} \not\leq_K ((\mathcal{I}_1 \upharpoonright A) \otimes \text{Fin}(X)) \cap (\{\emptyset\} \otimes \mathcal{I})$$

for every $A \notin \mathcal{I}_1$. Assume also that $\text{Fin} \subseteq \mathcal{I}_0, \mathcal{I}_1$, $\text{Fin}(X) \subseteq \mathcal{I}$ and $X \notin \mathcal{I}$. Then

$$(\mathcal{I}_0 \otimes \text{Fin}(X)) \cap (\{\emptyset\} \otimes \mathcal{I}) \not\leq_K (\mathcal{I}_1 \otimes \text{Fin}(X)) \cap (\{\emptyset\} \otimes \mathcal{I}).$$

Proof. Denote $\mathcal{J}_i = (\mathcal{I}_i \otimes \text{Fin}(X)) \cap (\{\emptyset\} \otimes \mathcal{I})$ for $i = 0, 1$. Let $f : \omega \times X \rightarrow \omega \times X$ be arbitrary. We show that f is not a Katětov reduction for $\mathcal{J}_0 \leq_K \mathcal{J}_1$. If there is $n \in \omega$ such that $f[\{n\} \times X] \cap (\{k\} \times X)$ is finite for all $k \in \omega$, then $C = \{n\} \times X \notin \mathcal{J}_1$, but $f[C] \in \mathcal{J}_0$, so we are done. Hence, assume that

$$E_n = \{k \in \omega : f[\{n\} \times X] \cap (\{k\} \times X) \text{ is infinite}\}$$

is nonempty for all $n \in \omega$ and put $T = \{n \in \omega : E_n \text{ is infinite}\}$. Since $\mathcal{I}_0 \not\leq_K \mathcal{I}_1$, either $\mathcal{I}_0 \not\leq_K \mathcal{I}_1 \upharpoonright T$ or $\mathcal{I}_0 \not\leq_K \mathcal{I}_1 \upharpoonright (\omega \setminus T)$.

Assume first that $\mathcal{I}_0 \not\leq_K \mathcal{I}_1 \upharpoonright T$. In particular, T is infinite in this case. For each $n \in T$ inductively find $g(n) \in \omega$ and $A_n \subseteq X$ such that

- (a) $g(n) \in E_n$,
- (b) $g(n) < g(n+1)$,
- (c) $f^{-1}[\{g(n)\} \times A_n] \cap (\{n\} \times X)$ is infinite,
- (d) $A_n \in \mathcal{I}$

(note that (c) is possible to obtain because (a) implies that $f[\{n\} \times X] \cap (\{g(n)\} \times X)$ is infinite, while (d) is possible to obtain as \mathcal{I} is tall). Since $g : T \rightarrow \omega$ and $\mathcal{I}_0 \not\leq_K \mathcal{I}_1 \upharpoonright T$, there is $B \subseteq T$ such that $g[B] \in \mathcal{I}_0$, but $B \notin \mathcal{I}_1$. Then

$$C = \bigcup_{n \in B} \{g(n)\} \times A_n \in \mathcal{J}_0$$

by $g[B] \in \mathcal{I}_0$ and items (b) and (d), while $f^{-1}[C] \notin \mathcal{I}_1 \otimes \text{Fin}(X)$ by $B \notin \mathcal{I}_1$ and item (c), so also $f^{-1}[C] \notin \mathcal{J}_1$.

Assume now that $\mathcal{I}_0 \not\leq_K \mathcal{I}_1 \upharpoonright (\omega \setminus T)$. For each $n \in \omega \setminus T$ define

$$B_n = (f^{-1}[f[\{n\} \times X] \setminus (E_n \times X)])_{(n)}.$$

For every $n \in \omega \setminus T$ the set $f[\{n\} \times X] \setminus (E_n \times X)$ belongs to \mathcal{J}_0 , so if $B_n \notin \mathcal{I}$ for some n , then we are done (as $\mathcal{J}_1 \not\supseteq \{n\} \times B_n \subseteq f^{-1}[f[\{n\} \times X] \setminus (E_n \times X)]$). Hence, assume that $B_n \in \mathcal{I}$ for every $n \in \omega \setminus T$.

Define $H_k = \{n \in \omega \setminus T : \max E_n \leq k\}$ for every $k \in \omega$.

Consider first the case that $H_k \notin \mathcal{I}_1 \upharpoonright (\omega \setminus T)$ for some k . Observe that $\mathcal{I} \upharpoonright (\omega \setminus B_n)$ is \leq_K -equivalent to \mathcal{I} , for every $n \notin T$ (as $B_n \in \mathcal{I}$), so

$$((\mathcal{I}_1 \upharpoonright H_k) \otimes \text{Fin}(X)) \cap \left((\{\emptyset\} \otimes \mathcal{I}) \upharpoonright \left(\bigcup_{n \in H_k} \{n\} \times B_n \right) \right)$$

is \leq_K -equivalent to $((\mathcal{I}_1 \upharpoonright H_k) \otimes \text{Fin}(X)) \cap ((\{\emptyset\} \otimes \mathcal{I}) \upharpoonright (H_k \times X))$. Since $\mathcal{I} \not\leq_K ((\mathcal{I}_1 \upharpoonright H_k) \otimes \text{Fin}(X)) \cap (\{\emptyset\} \otimes \mathcal{I})$ (by the assumptions of this lemma) and $\mathcal{J}_0 \upharpoonright ((k+1) \times X)$ is \leq_K -equivalent to \mathcal{I} , we can find $A \in \mathcal{J}_0 \upharpoonright ((k+1) \times X)$ such that $f^{-1}[A] \notin \mathcal{J}_1$.

Assume from now on that $H_k \in \mathcal{I}_1 \upharpoonright (\omega \setminus T)$ for all k . Let ϕ be a measure on $\omega \setminus T$ such that $\mathcal{I}_1 \upharpoonright (\omega \setminus T) = \{A \subseteq \omega \setminus T : \phi(A) < \infty\}$. For each k find a

finite set $G_k \subseteq H_k \setminus H_{k-1}$ such that

$$\phi(G_k) \geq \phi(H_k \setminus H_{k-1}) - 1/2^{k+1}.$$

Define $G = \bigcup_{k \in \omega} G_k$ and observe that $\phi((\omega \setminus T) \setminus G) \leq 1$. Indeed, if $\phi((\omega \setminus T) \setminus G) > 1$, then there should exist some finite $F \subseteq (\omega \setminus T) \setminus G$ such that $\phi(F) > 1$. Since $\omega \setminus T = \bigcup_{k \in \omega} H_k$ and $H_k \subseteq H_{k+1}$, there is m such that $F \subseteq H_m \setminus G = H_m \setminus \bigcup_{k \leq m} G_k$. But we have

$$\begin{aligned} \phi(H_m) &\geq \phi(F) + \phi\left(\bigcup_{k \leq m} G_k\right) = \phi(F) + \sum_{k \leq m} \phi(G_k) \\ &\geq \phi(F) + \sum_{k \leq m} \left(\phi(H_k \setminus H_{k-1}) - 1/2^{k+1}\right) \\ &= \phi(F) - \frac{2^{m+1} - 1}{2^{m+1}} + \phi\left(\bigcup_{k \leq m} H_k \setminus H_{k-1}\right) \\ &= \phi(F) - \frac{2^{m+1} - 1}{2^{m+1}} + \phi(H_m), \end{aligned}$$

which gives us $\phi(F) \leq (2^{m+1} - 1)/2^{m+1} < 1$, a contradiction.

Since $(\omega \setminus T) \setminus G \in \mathcal{I}_1$, the restrictions $\mathcal{I}_1 \upharpoonright (\omega \setminus T)$ and $\mathcal{I}_1 \upharpoonright G$ are isomorphic (by [12, Proposition 1.2] and the fact that \mathcal{I}_1 is tall). Let $h : G \rightarrow \omega$ be given by $h \upharpoonright G_k = k$. Since $\mathcal{I}_0 \not\leq \mathcal{I}_1 \upharpoonright (\omega \setminus T)$, there is $B \in \mathcal{I}_0$ such that $h^{-1}[B] \subseteq G$ and $h^{-1}[B] \notin \mathcal{I}_1$.

For every $k \in B$ and $n \in G_k$, using the tallness of \mathcal{I} , we can find some $A_n \in \mathcal{I}$ such that $(f^{-1}[\{k\} \times A_n])_{(n)}$ is infinite and $\{k\} \times A_n \subseteq f[\{n\} \times X]$ (this is possible as $n \in G_k \subseteq H_k \setminus H_{k-1}$ means that $k \in E_n$). Define

$$C = \bigcup_{k \in B} \bigcup_{n \in G_k} \{k\} \times A_n = \bigcup_{k \in B} \left(\{k\} \times \bigcup_{n \in G_k} A_n \right).$$

Then $C \in \mathcal{J}_0$, but $f^{-1}[C] \notin \mathcal{J}_1$. This finishes the proof. ■

THEOREM 5.9. *For every α , there are 2^ω many pairwise \leq_K -incomparable Π_2^0 ideals that are above $\text{conv}_{\alpha+1}$, but not above conv_α in the Katětov order.*

Proof. By [6, Theorem 1] (see also [15, Corollary 3.6]), there is a family $\{\mathcal{I}_\beta : \beta < 2^\omega\}$ of tall summable ideals such that $\mathcal{I}_\beta \not\leq_K \mathcal{I}_\gamma$ for all distinct $\beta, \gamma < 2^\omega$.

We claim that the family $\{\text{conv}_{\alpha+1}^{\mathcal{I}_\beta} : \beta < 2^\omega\}$ of ideals is as desired.

For every $\beta < 2^\omega$ the ideal $\text{conv}_{\alpha+1}^{\mathcal{I}_\beta}$ is \leq_K -above $\text{conv}_{\alpha+1}$ (as $\text{conv}_{\alpha+1} \subseteq \text{conv}_{\alpha+1}^{\mathcal{I}_\beta}$), but not \leq_K -above conv_α (by Theorem 5.6). Moreover, $\text{conv}_{\alpha+1}^{\mathcal{I}_\beta} \not\leq_K \text{conv}_{\alpha+1}^{\mathcal{I}_\gamma}$ for all distinct $\beta, \gamma < 2^\omega$ (by Lemma 5.8, which can be applied thanks to Theorem 5.6 and the fact that conv_α is tall).

Finally, observe that each $\text{conv}_{\alpha+1}^{\mathcal{I}_\beta}$ is Π_5^0 , since it is isomorphic to $(\{\emptyset\} \otimes \text{conv}_\alpha) \cap (\mathcal{I}_\beta \otimes \text{Fin}(\omega^\alpha + 1))$ and $\{\emptyset\} \otimes \text{conv}_\alpha$ is Π_5^0 (by [14, Proposition 1.6.16] and the fact that conv_α is Σ_4^0 by Proposition 3.3 and $\{\emptyset\}$ is Π_1^0), while $\mathcal{I}_\beta \otimes \text{Fin}(\omega^\alpha + 1)$ is Σ_4^0 (by [14, Proposition 1.6.16] and the fact that \mathcal{I}_β and $\text{Fin}(\omega^\alpha + 1)$ are Σ_2^0). ■

6. Critical ideals versus the ideal conv and the space ω_1 . In this section, we compare the critical ideals to the ideal conv and show that the whole sequence is critical for the space ω_1 in the previously defined sense, while no single ideal is critical for it.

PROPOSITION 6.1. *We have $\text{conv} \leq_K \mathcal{I}_d$.*

Proof. In [5, Example 3], the author showed that $[0, 1] \notin \text{FinBW}(\mathcal{I}_d)$, so $\text{conv} \leq_K \mathcal{I}_d$ by Theorem 1.2. ■

Below, we consider ω_1 as a topological space with the order topology.

PROPOSITION 6.2. *For every α , $\omega_1 \in \text{FinBW}(\text{conv}) \setminus \text{FinBW}(\text{conv}_\alpha)$.*

Proof. In [11, Proposition 11.1(a)], the author showed that $\omega_1 \in \text{FinBW}(\mathcal{I}_d)$. Consequently, $\omega_1 \in \text{FinBW}(\text{conv})$ by Proposition 6.1 and Theorem 2.2. On the other hand, the sequence $f : \omega^\alpha + 1 \rightarrow \omega_1$ given by $f(\xi) = \xi$ is a witness for $\omega_1 \notin \text{FinBW}(\text{conv}_\alpha)$. ■

COROLLARY 6.3. *For every α , $\text{conv} \leq_K \text{conv}_\alpha$ and $\text{conv}_\alpha \not\leq_K \text{conv}$.*

Proof. Using Proposition 6.2 and Theorem 2.2, we obtain $\text{conv}_\alpha \not\leq_K \text{conv}$. Now, suppose for the sake of contradiction that $\text{conv} \not\leq_K \text{conv}_\alpha$ for some α . Then $[0, 1] \in \text{FinBW}(\text{conv}_\alpha)$ by Theorem 1.2. Since the space $\omega^{\alpha+1} + 1$ is homeomorphic to a closed subset of $[0, 1]$, we obtain $\omega^{\alpha+1} + 1 \in \text{FinBW}(\text{conv}_\alpha)$. Then by Theorem 4.2, we obtain $\text{conv}_{1+(\alpha+1)} \not\leq_K \text{conv}_\alpha$, which contradicts Corollary 5.7. ■

COROLLARY 6.4.

(1) *For any ideal \mathcal{I} ,*

$$\omega_1 \in \text{FinBW}(\mathcal{I}) \iff \text{conv}_\alpha \not\leq_K \mathcal{I} \text{ for any } \alpha.$$

(2) *There is no single ideal \mathcal{I}_{ω_1} such that for any ideal \mathcal{I} ,*

$$\omega_1 \in \text{FinBW}(\mathcal{I}) \iff \mathcal{I}_{\omega_1} \not\leq_K \mathcal{I}.$$

Proof. (1, \Rightarrow) Suppose that $\text{conv}_\alpha \leq_K \mathcal{I}$ for some α . By Theorem 2.2, $\omega_1 \in \text{FinBW}(\mathcal{I}) \subseteq \text{FinBW}(\text{conv}_\alpha)$, a contradiction with Proposition 6.2.

(1, \Leftarrow) Suppose that $\omega_1 \notin \text{FinBW}(\mathcal{I})$. Then there is $f : \omega \rightarrow \omega_1$ such that $f \upharpoonright A$ is not convergent for any $A \notin \mathcal{I}$. Since ω_1 has uncountable cofinality, there is a countable α such that $f \upharpoonright \omega \subseteq \omega^\alpha + 1$. Then $f : \omega \rightarrow \omega^\alpha + 1$, so f is a witness for $\omega^\alpha + 1 \notin \text{FinBW}(\mathcal{I})$. Thus $\text{conv}_{1+\alpha} \leq_K \mathcal{I}$ by Theorem 4.2.

(2) Suppose for the sake of contradiction that there is an ideal \mathcal{I}_{ω_1} such that $\omega_1 \in \text{FinBW}(\mathcal{I}) \Leftrightarrow \mathcal{I}_{\omega_1} \not\leq_K \mathcal{I}$ for any ideal \mathcal{I} . Then $\omega_1 \notin \text{FinBW}(\mathcal{I}_{\omega_1})$, so by item (1) we find α_0 with $\text{conv}_{\alpha_0} \leq_K \mathcal{I}_{\omega_1}$. We claim that $\mathcal{I}_{\omega_1} \not\leq_K \text{conv}_{\alpha_0+1}$. Indeed, otherwise we would obtain $\mathcal{I}_{\omega_1} \leq_K \text{conv}_{\alpha_0+1} \leq_K \text{conv}_{\alpha_0} \leq_K \mathcal{I}_{\omega_1}$, so $\text{conv}_{\alpha_0} \leq_K \text{conv}_{\alpha_0+1}$, a contradiction with Theorem 5.6. Now, since $\mathcal{I}_{\omega_1} \not\leq_K \text{conv}_{\alpha_0+1}$, we can use our assumption to obtain $\omega_1 \in \text{FinBW}(\text{conv}_{\alpha_0+1})$. Then by item (1) with $\alpha = \alpha_0 + 1$ we get $\text{conv}_{\alpha_0+1} \not\leq_K \text{conv}_{\alpha_0+1}$, a contradiction. ■

LEMMA 6.5. *If \mathcal{I} and \mathcal{J} are tall ideals, \mathcal{J} is a P-ideal and $\mathcal{J} \not\leq_K \mathcal{I}$ then $\mathcal{J} \not\leq_K \mathcal{I} \otimes \text{Fin}$.*

Proof. Let $f : \omega \times \omega \rightarrow \omega$ be arbitrary. We need to find $A \in \mathcal{J}$ such that $f^{-1}[A] \notin \mathcal{I} \otimes \text{Fin}$. Define

$$T = \{n \in \omega : f \upharpoonright (\{n\} \times \omega) \text{ is finite-to-one}\}.$$

There are two possibilities: either $T \in \mathcal{I}$ or $T \notin \mathcal{I}$.

If $T \in \mathcal{I}$, for each $n \in \omega \setminus T$ find an infinite $A_n \subseteq \{n\} \times \omega$ and $g(n) \in \omega$ such that $f[A_n] = \{g(n)\}$. Since \mathcal{I} and $\mathcal{I} \upharpoonright (\omega \setminus T)$ are isomorphic (by [12, Proposition 1.2] as $T \in \mathcal{I}$ and \mathcal{I} is tall) and $\mathcal{J} \not\leq_K \mathcal{I}$, there is $A \in \mathcal{J}$ such that $g^{-1}[A] \notin \mathcal{I} \upharpoonright (\omega \setminus T)$. Then

$$f^{-1}[A] \supseteq \bigcup_{n \in g^{-1}[A]} A_n \notin \mathcal{I} \otimes \text{Fin},$$

so A is the required set.

If $T \notin \mathcal{I}$, using tallness of \mathcal{J} , for each $n \in T$ find an infinite $B_n \in \mathcal{J}$ such that $B_n \subseteq f \upharpoonright (\{n\} \times \omega)$. Since \mathcal{J} is a P-ideal, there is $A \in \mathcal{J}$ such that $B_n \setminus A \in \text{Fin}$ for all $n \in \omega$. In particular,

$$f^{-1}[A] \cap (\{n\} \times \omega) \supseteq f^{-1}[A \cap B_n] \cap (\{n\} \times \omega) \notin \text{Fin}$$

for all $n \in T$. Hence, $f^{-1}[A] \notin \mathcal{I} \otimes \text{Fin}$. ■

THEOREM 6.6. *There are 2^ω many pairwise \leq_K -incomparable Σ_4^0 ideals that are above conv , but not above any conv_α in the Katětov order.*

Proof. By [6, Theorem 1] (see also [Corollary 3.6]), there is a family $\{\mathcal{I}_\beta : \beta < 2^\omega\}$ of ideals such that each \mathcal{I}_β is a tall summable ideal and $\mathcal{I}_\beta \not\leq_K \mathcal{I}_\gamma$ for all distinct $\beta, \gamma < 2^\omega$.

Define $\mathcal{J}_\beta = (\mathcal{I}_\beta \otimes \text{Fin}) \cap (\{\emptyset\} \otimes \mathcal{I}_d)$ for all $\beta < 2^\omega$.

Now we show that $\text{conv}_\alpha \not\leq_K \mathcal{J}_\beta$ for all $\beta < 2^\omega$ and all α . Indeed, if some \mathcal{J}_β would be \leq_K -above some conv_α , then also $\mathcal{J}_\beta \upharpoonright (\{\emptyset\} \times \omega)$ would be \leq_K -above conv_α . However, $\mathcal{J}_\beta \upharpoonright (\{\emptyset\} \times \omega)$ is isomorphic to \mathcal{I}_d , so it would contradict Corollary 6.4 and [11, Proposition 11.1(a)] (where it is shown that $\omega_1 \in \text{FinBW}(\mathcal{I}_d)$). Hence, $\text{conv}_\alpha \not\leq_K \mathcal{J}_\beta$.

To show that $\text{conv} \leq_K \mathcal{J}_\beta$ for all $\beta < 2^\omega$, let $(I_n)_{n \in \omega}$ be a sequence of pairwise disjoint closed subintervals of $[0, 1]$ with rational endpoints and such that $(I_n)_{n \in \omega}$ converges to 0 (that is, if $x_n \in I_n$ for all n , then $\lim_n x_n = 0$). For each $n \in \omega$, since $\text{conv} \upharpoonright (I_n \cap \mathbb{Q})$ and conv are isomorphic and $\text{conv} \leq_K \mathcal{I}_d$ (by Proposition 6.1), there is $g_n : \omega \rightarrow I_n \cap \mathbb{Q}$ witnessing $\text{conv} \upharpoonright (I_n \cap \mathbb{Q}) \leq_K \mathcal{I}_d$. Let $f : \omega \times \omega \rightarrow [0, 1] \cap \mathbb{Q}$ be given by $f(n, k) = g_n(k)$. Then f witnesses $\text{conv} \leq_K \mathcal{J}_\beta$.

Now we prove that $\mathcal{J}_\beta \not\leq_K \mathcal{J}_\gamma$ for all distinct $\beta, \gamma < 2^\omega$. This will follow from Lemma 5.8 once we show that all its assumptions are met. It is known that \mathcal{I}_d is a tall P-ideal and $\mathcal{I}_d \not\leq_K \mathcal{K}$ for every summable ideal \mathcal{K} (by Proposition 6.1 and [9, Corollary 3.14] where the authors proved that $\text{conv} \not\leq_K \mathcal{I}$ for any F_σ ideal \mathcal{I}). Hence, $\mathcal{I}_d \not\leq_K \mathcal{K} \otimes \text{Fin}$ for every summable ideal \mathcal{K} (by Lemma 6.5). Since $\mathcal{I}_\gamma \upharpoonright A$ is a summable ideal, for every $A \notin \mathcal{I}_\gamma$, we find that $\mathcal{I}_d \not\leq_K ((\mathcal{I}_\gamma \upharpoonright A) \otimes \text{Fin}) \cap (\{\emptyset\} \otimes \mathcal{I}_d)$ (since $((\mathcal{I}_\gamma \upharpoonright A) \otimes \text{Fin}) \cap (\{\emptyset\} \otimes \mathcal{I}) \subseteq (\mathcal{I}_\gamma \upharpoonright A) \otimes \text{Fin}$).

Finally, observe that $\{\emptyset\} \otimes \mathcal{I}_d$ is Π_3^0 by [14, Proposition 1.6.16] and the fact that \mathcal{I}_d is Π_3^0 and $\{\emptyset\}$ is Π_1^0 , while $\mathcal{I}_\beta \otimes \text{Fin}$ is Σ_4^0 by [14, Proposition 1.6.16] and the fact that \mathcal{I}_β and Fin are Σ_2^0 . ■

7. Additional properties of critical ideals. In this section, we establish that the critical ideals have the property KAT, and we compute their Borel complexity and cardinal characteristics in the context of ideals on countable sets.

7.1. Critical ideals have the property KAT. We say that an ideal \mathcal{I} on X contains an isomorphic copy of an ideal \mathcal{J} on Y (written $\mathcal{J} \sqsubseteq \mathcal{I}$) if there is a bijection $f : X \rightarrow Y$ such that $f^{-1}[A] \in \mathcal{I}$ for every $A \in \mathcal{J}$. In [1, Lemma 3.3] it is shown that, for the case of tall ideals, the definition of \sqsubseteq can be relaxed to require only that the function be one-to-one, instead of bijective.

PROPOSITION 7.1. $\text{conv}_\alpha \sqsubseteq \text{conv}_\alpha \otimes \{\emptyset\}$ for every α .

Proof. Let $f : (\omega^\alpha + 1) \times \omega \rightarrow \omega^\alpha + 1$ be a one-to-one function such that $f[\{\omega^\alpha\} \times \omega] = [0, \omega)$ and for each $\xi \in \omega^\alpha$ we have

$$f[\{\xi\} \times \omega] = [\omega \cdot (1 + \xi), \omega \cdot (1 + \xi + 1)).$$

We claim that f is a witness for $\text{conv}_\alpha \sqsubseteq \text{conv}_\alpha \otimes \{\emptyset\}$. Since the ideal conv_α is tall (by Lemma 3.1(2)) and the function f is one-to-one, we only need to show that $f^{-1}[A] \in \text{conv}_\alpha \otimes \{\emptyset\}$ for every $A \in \text{conv}_\alpha$. Take any $A \in \text{conv}_\alpha$ and suppose for the sake of contradiction that $f^{-1}[A] \notin \text{conv}_\alpha \otimes \{\emptyset\}$. Then

$$B = \{\xi \in \omega^\alpha + 1 : f^{-1}[A] \cap (\{\xi\} \times \omega) \neq \emptyset\} \notin \text{conv}_\alpha.$$

By Proposition 3.4, there exists an increasing sequence $(\lambda_n)_{n \in \omega}$ in ω^α and an infinite set $C \subseteq \omega$ such that the intersection $B \cap [\lambda_n, \lambda_{n+1})$ is infinite for each $n \in C$. Then for each $n \in C$ we can find $\tau_n \in [\omega \cdot (1 + \lambda_n), \omega \cdot (1 + \lambda_{n+1} + 1)) \cap A^d$. Consequently, A^d is infinite, so $A \notin \text{conv}_\alpha$, a contradiction. ■

We say that an ideal \mathcal{J} on Y is below an ideal \mathcal{I} on X in the *Katětov–Blass order* (written $\mathcal{J} \leq_{KB} \mathcal{I}$) if there is a finite-to-one function $f : X \rightarrow Y$ such that $f^{-1}[A] \in \mathcal{I}$ for every $A \in \mathcal{I}$. An ideal \mathcal{J} has the *property Kat* (see [1]) if for any ideal \mathcal{I} ,

$$\mathcal{J} \leq_K \mathcal{I} \iff \mathcal{J} \sqsubseteq \mathcal{I}.$$

The following corollary shows that the ideals conv_α have the property *Kat*.

COROLLARY 7.2. *For any ideal \mathcal{I} ,*

$$\text{conv}_\alpha \leq_K \mathcal{I} \iff \text{conv}_\alpha \leq_{KB} \mathcal{I} \iff \text{conv}_\alpha \sqsubseteq \mathcal{I}.$$

Proof. The implication “ $\text{conv}_\alpha \leq_K \mathcal{I} \Rightarrow \text{conv}_\alpha \sqsubseteq \mathcal{I}$ ” follows from Proposition 7.1 and [1, Theorem 3.4], whereas the implications “ $\text{conv}_\alpha \sqsubseteq \mathcal{I} \Rightarrow \text{conv}_\alpha \leq_{KB} \mathcal{I}$ ” and “ $\text{conv}_\alpha \leq_{KB} \mathcal{I} \Rightarrow \text{conv}_\alpha \leq_K \mathcal{I}$ ” are obvious. ■

Using Proposition 3.7, we can see that the above corollary extends [4, Theorem 6.2] and [11, Proposition 4.4], where the authors proved the above corollary for $\text{Fin} \otimes \text{Fin} \approx \text{conv}_2$ and $\text{BI} \approx \text{conv}_3$, respectively.

7.2. Borel complexity and P^- property. An ideal \mathcal{I} on X is P^- (*hereditary weak P*) if for every partition \mathcal{A} of any set $C \in \mathcal{I}^+$ into sets from \mathcal{I} there exists $B \in \mathcal{I}^+$ such that $B \subseteq C$ and $B \cap A$ is finite for each $A \in \mathcal{A}$ (see [9, p. 2030] and [11, Definition 4.8], resp.). It is known that conv is not a P^- ideal (see e.g. [11, proof of Proposition 4.10(b)]), however, there are P^- ideals which are above conv in the Katětov order (for instance, \mathcal{I}_d is P^- by [11, Proposition 4.9] and $\text{conv} \leq_K \mathcal{I}_d$ by Proposition 6.1). Every ideal conv_α is above the ideal conv in the Katětov order, but there is no P^- ideal above any conv_α ideal as shown by the following proposition.

PROPOSITION 7.3. *If \mathcal{I} is a P^- ideal, then $\text{conv}_\alpha \not\leq_K \mathcal{I}$ for every α . In particular, conv_α is not P^- for any α .*

Proof. If \mathcal{I} is a P^- ideal, then $\omega_1 \in \text{FinBW}(\mathcal{I})$ by [11, Proposition 6.1]. Thus, Corollary 6.4 finishes the proof. ■

COROLLARY 7.4. *$\text{conv}_\alpha \in \Sigma_4^0 \setminus \Pi_4^0$ for every α .*

Proof. In [11, Proposition 4.9], the author proved that every Π_4^0 ideal is P^- , so conv_α is not Π_4^0 by Proposition 7.3. On the other hand, conv_α is Σ_4^0 by Proposition 3.3. ■

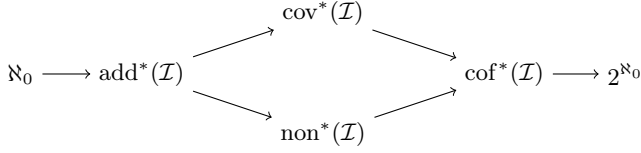


Fig. 1. Relationships between cardinal characteristics for ideals on ω ($\kappa \rightarrow \lambda$ means $\kappa \leq \lambda$ in this diagram).

7.3. Cardinal characteristics of critical ideals. Some properties of ideals can be described by cardinal characteristics associated with them. There are four well known cardinal characteristics called additivity, covering, uniformity, and cofinality defined in the following way for an ideal \mathcal{I} on X (see e.g. [2]): $\text{add}(\mathcal{I}) = \min \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\}$, $\text{cov}(\mathcal{I}) = \min \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\}$, $\text{non}(\mathcal{I}) = \min \{|\mathcal{A}| : A \notin \mathcal{I}\}$, and $\text{cof}(\mathcal{I}) = \min \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I} \exists A \in \mathcal{A} (B \subseteq A)\}$. These characteristics are useful in the case of ideals on an uncountable set X (for instance in the case of the σ -ideal of all meager sets and the σ -ideal of all Lebesgue null sets).

Apart from the cofinality, the other cardinal characteristics are trivial, since they are all equal to \aleph_0 .

Fortunately, Hernández and Hrušák introduced in [7] (see also [8]) certain versions of these characteristics more suitable for tall ideals on countable sets ($A \subseteq^* B$ means that $A \setminus B$ is finite):

$$\begin{aligned}
\text{add}^*(\mathcal{I}) &= \min \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \neg \exists B \in \mathcal{I} \forall A \in \mathcal{A} (A \subseteq^* B)\}, \\
\text{cov}^*(\mathcal{I}) &= \min \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in [\omega]^{\aleph_0} \exists A \in \mathcal{A} (|A \cap B| = \aleph_0)\}, \\
\text{non}^*(\mathcal{I}) &= \min \{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \wedge \forall B \in \mathcal{I} \exists A \in \mathcal{A} (|A \cap B| < \aleph_0)\}, \\
\text{cof}^*(\mathcal{I}) &= \min \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I} \exists A \in \mathcal{A} (B \subseteq^* A)\}.
\end{aligned}$$

There are some inequalities that hold among these characteristics for all ideals (see also Figure 1): $\aleph_0 \leq \text{add}^*(\mathcal{I}) \leq \text{cov}^*(\mathcal{I}) \leq \text{cof}^*(\mathcal{I}) \leq 2^{\aleph_0}$ and $\aleph_0 \leq \text{add}^*(\mathcal{I}) \leq \text{non}^*(\mathcal{I}) \leq \text{cof}^*(\mathcal{I}) \leq 2^{\aleph_0}$ (see e.g. [8, p. 578]).

It is known (see e.g. [8]) that $\text{add}^*(\text{conv}) = \text{add}^*(\text{Fin} \otimes \text{Fin}) = \aleph_0$, $\text{non}^*(\text{conv}) = \text{non}^*(\text{Fin} \otimes \text{Fin}) = \aleph_0$, $\text{cov}^*(\text{conv}) = \text{cof}^*(\text{conv}) = 2^{\aleph_0}$, $\text{cov}^*(\text{Fin} \otimes \text{Fin}) = \mathfrak{b}$ and $\text{cof}^*(\text{Fin} \otimes \text{Fin}) = \mathfrak{d}$, where \mathfrak{b} is the *bounding number*, i.e. the smallest cardinality of any unbounded family in the poset (ω^ω, \leq^*) and \mathfrak{d} is the *dominating number*, i.e. the smallest cardinality of any dominating family in the poset (ω^ω, \leq^*) (for more on these cardinals see for instance [3]). Moreover, if $\mathcal{I} \leq_K \mathcal{J}$, then $\text{cov}^*(\mathcal{I}) \geq \text{cov}^*(\mathcal{J})$ and $\text{non}^*(\mathcal{I}) \leq \text{non}^*(\mathcal{J})$ [7, Proposition 3.1].

The following theorem provides the values of the above mentioned characteristics for the critical ideals considered in the paper.

PROPOSITION 7.5. *For every α ,*

$$\text{non}^*(\text{conv}_\alpha) = \text{add}^*(\text{conv}_\alpha) = \aleph_0, \quad \text{cov}^*(\text{conv}_\alpha) = \mathfrak{b}, \quad \text{cof}^*(\text{conv}_\alpha) = \mathfrak{d}.$$

Proof. ($\text{non}^*(\text{conv}_\alpha) = \aleph_0$) Since $\text{conv}_\alpha \leq_K \text{conv}_2 \leq_K \text{Fin} \otimes \text{Fin}$, we obtain $\aleph_0 \leq \text{non}^*(\text{conv}_\alpha) \leq \text{non}^*(\text{Fin} \otimes \text{Fin}) = \aleph_0$.

($\text{add}^*(\text{conv}_\alpha) = \aleph_0$) Since $\aleph_0 \leq \text{add}^*(\text{conv}_\alpha) \leq \text{non}^*(\text{conv}_\alpha) = \aleph_0$, it follows that $\text{add}^*(\text{conv}_\alpha) = \aleph_0$.

($\text{cov}^*(\text{conv}_\alpha) = \mathfrak{b}$) Since $\text{conv}_\alpha \leq_K \text{conv}_2 \leq_K \text{Fin} \otimes \text{Fin}$, it follows that $\text{cov}^*(\text{conv}_\alpha) \geq \text{cov}^*(\text{Fin} \otimes \text{Fin}) = \mathfrak{b}$. We will show the reverse inequality by induction on α . Assume the statement holds for any $\beta < \alpha$.

Successor case: $\alpha = \gamma + 1$. Notice that for any infinite set $X \in \text{conv}_{\gamma+1}$, either there is some n such that the set $X \cap \omega^\gamma \cdot n$ is infinite, or the set $X \cap [\omega^\gamma \cdot n, \omega^\gamma \cdot (n+1)]$ is finite for each n . Since conv_γ is isomorphic to $\text{conv}(\omega^\gamma \cdot n)$, by inductive hypothesis we obtain a witness, say $\mathcal{A}_n \subseteq \text{conv}(\omega^\gamma \cdot n)$, for $\text{cov}^*(\text{conv}(\omega^\gamma \cdot n)) = \mathfrak{b}$ for each $n \geq 1$. Since $\text{cov}^*(\text{Fin} \otimes \text{Fin}) = \mathfrak{b}$, there exists a family $\mathcal{B} \subseteq \mathcal{P}(\omega^{\gamma+1} + 1)$ with the following properties:

- $|\mathcal{B}| = \mathfrak{b}$,
- if $F \in \mathcal{B}$ then the set $F \cap [\omega^\gamma \cdot n, \omega^\gamma \cdot (n+1))$ is finite for each n ,
- for any infinite $A \subseteq \omega^{\gamma+1} + 1$, if $A \cap [\omega^\gamma \cdot n, \omega^\gamma \cdot (n+1))$ is finite for each n then there exists some $F \in \mathcal{B}$ such that $A \cap F$ is infinite.

Let $\mathcal{C} = \mathcal{B} \cup \bigcup_{n < \omega} \mathcal{A}_n$. Then \mathcal{C} is a subfamily of $\text{conv}_{\gamma+1}$ of cardinality \mathfrak{b} having the property that for any infinite $D \in \text{conv}_{\gamma+1}$ there exists some $C \in \mathcal{C}$ such that the set $D \cap C$ is infinite, so it follows that $\text{cov}^*(\text{conv}_{\gamma+1}) \leq \mathfrak{b}$.

Limit case: α is a limit ordinal. Let $(\alpha_n)_{n < \omega}$ be a strictly increasing sequence that is cofinal in α with $\alpha_0 = 0$. Notice that for any infinite set $X \in \text{conv}_\alpha$, either there is some n such that $X \cap \omega^{\alpha_n}$ is infinite, or $X \cap [\omega^{\alpha_n}, \omega^{\alpha_{n+1}}]$ is finite for each n . By inductive hypothesis, we can find a witness, say $\mathcal{A}_n \subseteq \text{conv}_{\alpha_n}$, for $\text{cov}^*(\text{conv}_{\alpha_n}) = \mathfrak{b}$ for each $n \geq 1$. Since $\text{cov}^*(\{\text{Fin} \otimes \text{Fin}\}) = \mathfrak{b}$, there exists a family $\mathcal{B} \subseteq \mathcal{P}(\omega^\alpha + 1)$ with the following properties:

- $|\mathcal{B}| = \mathfrak{b}$,
- if $F \in \mathcal{B}$ then $F \cap [\omega^{\alpha_n}, \omega^{\alpha_{n+1}}]$ is finite for each n ,
- for any infinite $A \subseteq \omega^\alpha + 1$, if the set $A \cap [\omega^{\alpha_n}, \omega^{\alpha_{n+1}}]$ is finite for each n , then there exists some $F \in \mathcal{B}$ such that $A \cap F$ is infinite.

Analogously to the previous case, the family $\mathcal{C} = \mathcal{B} \cup \bigcup_{n < \omega} \mathcal{A}_n$ is a witness for the required inequality.

($\text{cof}^*(\text{conv}_\alpha) = \mathfrak{d}$) First, we prove that $\text{cof}^*(\text{conv}_\alpha) \leq \mathfrak{d}$ by induction on α . Assume the statement holds for any $\beta < \alpha$.

Successor case: $\alpha = \gamma + 1$. Notice that for any set $A \in \text{conv}_{\gamma+1}$ there is some n such that $A \cap \omega^\gamma \cdot n \in \text{conv}(\omega^\gamma \cdot n)$ and for any $k \geq n$ the set $A \cap [\omega^\gamma \cdot k, \omega^\gamma \cdot (k+1))$ is finite. Since conv_γ is isomorphic to $\text{conv}(\omega^\gamma \cdot n)$, by inductive hypothesis we obtain a witness, say $\mathcal{A}_n \subseteq \text{conv}(\omega^\gamma \cdot n)$, for $\text{cof}^*(\text{conv}(\omega^\gamma \cdot n)) \leq \mathfrak{d}$ for each $n \geq 1$. Since $\text{cof}^*(\text{Fin} \otimes \text{Fin}) = \mathfrak{d}$, there exists

a family $\mathcal{B} \subseteq \mathcal{P}(\omega^{\gamma+1} + 1)$ with the following properties:

- $|\mathcal{B}| = \mathfrak{d}$,
- if $F \in \mathcal{B}$ then $F \cap [\omega^\gamma \cdot n, \omega^\gamma \cdot (n+1))$ is finite for each n ,
- for any $A \subseteq \omega^{\gamma+1} + 1$, if $A \cap [\omega^\gamma \cdot n, \omega^\gamma \cdot (n+1))$ is finite for each n then there exists some $F \in \mathcal{B}$ such that $A \subseteq^* F$.

Let $\mathcal{C} = \{A \cup B : A \in \bigcup_{n \in \omega} \mathcal{A}_n, B \in \mathcal{B}\}$. Then \mathcal{C} is a subfamily of $\text{conv}_{\gamma+1}$ such that $|\mathcal{C}| = \mathfrak{d}$ and for any $D \in \text{conv}_{\gamma+1}$ there exists some $C \in \mathcal{C}$ with $D \subseteq^* C$, so it follows that $\text{cof}^*(\text{conv}_{\gamma+1}) \leq \mathfrak{d}$.

Limit case: α is a limit ordinal. Let $(\alpha_n)_{n < \omega}$ be a strictly increasing sequence that is cofinal in α with $\alpha_0 = 0$. Then conv_α equals

$$\bigcup_{n \in \omega} \{A \subseteq \omega^\alpha + 1 : A \cap \omega^{\alpha_n} \in \text{conv}_{\alpha_n} \wedge \forall k \geq n \ A \cap [\omega^{\alpha_k}, \omega^{\alpha_{k+1}}) \in \text{Fin}\}.$$

Let $\mathcal{A}_n \subseteq \text{conv}_{\alpha_n}$ be a witness for $\text{cof}^*(\text{conv}_{\alpha_n}) = \mathfrak{d}$. Since $\text{cof}^*(\text{Fin} \otimes \text{Fin}) = \mathfrak{d}$, there exists a family $\mathcal{B} \subseteq \mathcal{P}(\omega^\alpha + 1)$ with the following properties:

- $|\mathcal{B}| = \mathfrak{d}$,
- if $F \in \mathcal{B}$ then $F \cap [\omega^{\alpha_n}, \omega^{\alpha_{n+1}})$ is finite for each n ,
- for any $A \subseteq \omega^\alpha + 1$, if $A \cap [\omega^{\alpha_n}, \omega^{\alpha_{n+1}})$ is finite for each n , then there exists some $F \in \mathcal{B}$ such that $A \subseteq^* F$.

Analogously to the previous case, the family $\mathcal{C} = \{A \cup B : A \in \bigcup_{n \in \omega} \mathcal{A}_n, B \in \mathcal{B}\}$ is a witness for the required inequality.

Now, we prove that $\mathfrak{d} \leq \text{cof}^*(\text{conv}_\alpha)$. Let \mathcal{A} be the witness for $\text{cof}^*(\text{conv}_\alpha)$. For any $g \in \omega^\omega$ we define the set $B_g = \bigcup_{n < \omega} [\omega \cdot n, \omega \cdot n + g(n)]$ and for any $A \in \text{conv}_\alpha$ we define $f_A = \{(n, \max\{k : \omega \cdot n + k \in A\}) : n \in \omega\}$, with the convention that $\max \emptyset = 0$ and $\max(C) = 0$ for any infinite set C . We will show that $\mathcal{F} = \{f_A : A \in \mathcal{A}\}$ is a dominating family in ω^ω . Take any $g \in \omega^\omega$. Since the derivative of the set B_g is contained in $\{\omega^2\}$, it follows that $B_g \in \text{conv}_\alpha$, so there exists some set $A \in \mathcal{A}$ such that $B_g \subseteq^* A$; then $g(n) \leq f_A(n)$ for all but finitely many n . Hence $\mathfrak{d} \leq |\mathcal{F}| = |\mathcal{A}| = \text{cof}^*(\text{conv}_\alpha)$. ■

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