

Embeddings in Besov–Triebel–Lizorkin-type algebra and Sobolev spaces of Morrey scale

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Abstract. We deal with some embedding properties between the homogeneous Besov–Triebel–Lizorkin-type spaces of bounded functions and the homogeneous Sobolev spaces built on the Morrey scale.

1. Introduction. The homogeneous Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel–Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are defined as the sets of all distributions f satisfying $\|[f]_\infty\| < \infty$, where $[f]_\infty := \{f + u : u \in \mathcal{P}_\infty\}$, where \mathcal{P}_∞ is the set of all polynomials on \mathbb{R}^n and $\|\cdot\|$ is the corresponding (Besov or Triebel–Lizorkin) quasi-norm. In a recent paper [5], we proved that the subspace

$$(1.1) \quad \mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n) := \{f \in L_\infty(\mathbb{R}^n) : [f]_\infty \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)\}$$

endowed with the quasi-norm

$$\|f\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} := \|f\|_{L_\infty(\mathbb{R}^n)} + \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

is a *unital quasi-Banach algebra* for the pointwise product if $\tau \geq 0$ and $s > \max(0, n/p - n)$, and similarly for $\mathcal{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ after interchanging the roles of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, i.e.,

$$\mathcal{F}_{p,q}^{s,\tau}(\mathbb{R}^n) := \{f \in L_\infty(\mathbb{R}^n) : [f]_\infty \in \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)\}$$

endowed with the quasi-norm

$$\|f\|_{\mathcal{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} := \|f\|_{L_\infty(\mathbb{R}^n)} + \|[f]_\infty\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

We use $\mathcal{E}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote either $\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\mathcal{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, whenever it is not necessary to distinguish between B -spaces and F -spaces.

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In this work, we establish embeddings between $\mathcal{E}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and homogeneous Sobolev spaces built on the Morrey scale $\dot{W}^m(\mathcal{M}_v^w)(\mathbb{R}^n)$. For embeddings of type $\mathcal{E} \subset \dot{W}$, we use distributions vanishing at infinity, while for converse embeddings we employ *realizations* to handle the presence of polynomials in \dot{W} .

The motivation for this paper is to make these spaces more accessible for applications in areas such as PDEs, including the Navier–Stokes equations, and in the study of composition operators; see, for instance, [1, Proposition 2.4], where the space $\mathcal{B}_{2,2}^{s,0}(\mathbb{R})$ naturally appears. In this context, when $\tau = 0$, the homogeneous Besov–Triebel–Lizorkin-type spaces coincide with the classical homogeneous Besov–Triebel–Lizorkin spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$, which already have numerous well-known applications; see, e.g., [1, 2, 12, 15]. Moreover, their algebra versions $\mathcal{B}_{p,q}^s(\mathbb{R}^n)$ and $\mathcal{F}_{p,q}^s(\mathbb{R}^n)$, defined in the same way as in (1.1), play a crucial role in the study of composition operators on Besov and Triebel–Lizorkin spaces; see, e.g., [10, 11, 17, 19, 20]. We also mention that embeddings between homogeneous Besov–Triebel–Lizorkin spaces and Morrey spaces can be found in [14, Section 5.3], which further supports their relevance for PDE applications.

The paper is organized as follows: In Section 2, we recall the definitions of the relevant function spaces. In Section 3, we state our main results. Section 4 contains the proofs. The final section discusses potential extensions.

Notation.

- If $X(\mathbb{R}^n)$ is a function space defined on \mathbb{R}^n , we simply write X .
- We denote by \mathbb{N} the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- For $k \in \mathbb{Z}$ and $\eta \in \mathbb{Z}^n$, $P_{k,\eta}$ denotes the dyadic cube $2^{-k}([0, 1[^{n+\eta})$.
- The symbol \hookrightarrow indicates a continuous embedding.
- If $0 < p \leq \infty$, then $\|\cdot\|_{L_p(\Omega)}$ denotes the quasi-norm of $L_p(\Omega)$; if $\Omega = \mathbb{R}^n$ we simply write $\|\cdot\|_p$.
- For $m \in \mathbb{N}_0 \cup \{\infty\}$, we denote by \mathcal{P}_m the set of all polynomials on \mathbb{R}^n of degree $< m$; we have $\mathcal{P}_0 := \{0\}$, $\mathcal{P}_1 := \{c : c \in \mathbb{C}\}$, etc.
- If $m \in \mathbb{N}$ and E is any distribution space on \mathbb{R}^n , we denote by $\dot{W}^m(E)$ the homogeneous Sobolev space based on E , i.e., the set of all distributions $f \in E$ such that

$$\|f\|_{\dot{W}^m(E)} := \sum_{|\alpha|=m} \|f^{(\alpha)}\|_E < \infty;$$

we have $\|f\|_{\dot{W}^m(E)} = \|f + u\|_{\dot{W}^m(E)}$ for all $u \in \mathcal{P}_m$.

- If $f \in L_1$, then $\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$ and $\check{f}(x) := (2\pi)^{-n} \hat{f}(-x)$ are the Fourier transform and its inverse, respectively; they extend to any $f \in \mathcal{S}'$ in the usual way.

- \mathcal{S}_∞ denotes the set of all functions φ in the Schwartz space \mathcal{S} such that $\langle u, \varphi \rangle = 0$ for all $u \in \mathcal{P}_\infty$, and \mathcal{S}'_∞ denotes its topological dual, *the space of distributions modulo polynomials*. If $f \in \mathcal{S}'$, the mapping that takes any $[f]_\infty$ to the restriction of f to \mathcal{S}_∞ yields an isomorphism from $\mathcal{S}'/\mathcal{P}_\infty$ onto \mathcal{S}'_∞ .
- The constants c, c_1, \dots are strictly positive, depend only on the fixed parameters n, s, p, q, \dots , and their values may vary from line to line.
- We use the letters B and F to indicate the Besov and Triebel–Lizorkin spaces, respectively.
- Throughout the paper, the parameters s, p, q, τ satisfy $s \in \mathbb{R}$, $p \in]0, \infty[$ in B -spaces, $p \in]0, \infty[$ in F -spaces, $q \in]0, \infty[$ and $\tau \geq 0$, unless otherwise stated.

2. Some function spaces

2.1. The Besov–Triebel–Lizorkin-type spaces. We fix, once and for all, a standard cut-off function ρ , that is, a radial C^∞ function such that $0 \leq \rho \leq 1$, $\rho(\xi) = 1$ for $|\xi| \leq 1$, $\rho(\xi) = 0$ for $|\xi| \geq 3/2$. We define $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$; then $\text{supp } \gamma$ is contained in the annulus $1/2 \leq |\xi| \leq 3/2$ and

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1 \quad (\xi \neq 0).$$

Let us define the operators Q_j , $j \in \mathbb{Z}$, on \mathcal{S}' by $\widehat{Q_j f}(\xi) := \gamma(2^{-j}\xi)\hat{f}(\xi)$. To extend the definition of Q_j on \mathcal{S}'_∞ , we adopt the following convention:

- If $f \in \mathcal{S}'_\infty$, we define $Q_j f := Q_j g$ for any $g \in \mathcal{S}'$ such that $[g]_\infty = f$.

Then Q_j is well-defined on \mathcal{S}'_∞ since $Q_j f = 0$ ($\forall j \in \mathbb{Z}$) if and only if $f \in \mathcal{P}_\infty$. The distribution $\widehat{Q_j f}$ is of compact support, so $Q_j f$ is an analytic function of exponential type, by the Paley–Wiener theorem, and $f = \sum_{j \in \mathbb{Z}} Q_j f$ in \mathcal{S}'_∞ for all $f \in \mathcal{S}'_\infty$ (the *Littlewood–Paley decomposition*).

DEFINITION 2.1.

- (i) The *homogeneous Besov-type space* $\dot{B}_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left(\sum_{j \geq k} (2^{js} \|Q_j f\|_{L_p(P_{k,\eta})})^q \right)^{1/q} < \infty.$$

- (ii) The *homogeneous Triebel–Lizorkin-type space* $\dot{F}_{p,q}^{s,\tau}$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left\| \left(\sum_{j \geq k} (2^{js} |Q_j f|)^q \right)^{1/q} \right\|_{L_p(P_{k,\eta})} < \infty.$$

We make the usual modifications in the case $p = \infty$ and/or $q = \infty$.

We need to define the homogeneous Besov and Triebel–Lizorkin spaces.

DEFINITION 2.2.

- (i) The *homogeneous Besov space* $\dot{B}_{p,q}^s$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}_{p,q}^s} := \left(\sum_{j \in \mathbb{Z}} (2^{js} \|Q_j f\|_p)^q \right)^{1/q} < \infty.$$

- (ii) The *homogeneous Triebel–Lizorkin space* $\dot{F}_{p,q}^s$ is the set of all $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}_{p,q}^s} := \left\| \left(\sum_{j \in \mathbb{Z}} (2^{js} |Q_j f|)^q \right)^{1/q} \right\|_p < \infty.$$

We make the usual modifications if $p = \infty$ and/or $q = \infty$.

For simplicity, we use $\dot{A}_{p,q}^{s,\tau}$ to denote either $\dot{B}_{p,q}^{s,\tau}$ or $\dot{F}_{p,q}^{s,\tau}$, and similarly $\dot{A}_{p,q}^s$ to denote either $\dot{B}_{p,q}^s$ or $\dot{F}_{p,q}^s$.

REMARK 2.3. The space $\dot{A}_{p,q}^{s,\tau}$ (respectively $\dot{A}_{p,q}^s$) is a quasi-Banach space with respect to the quasi-norm $\|\cdot\|_{\dot{A}_{p,q}^{s,\tau}}$ (respectively $\|\cdot\|_{\dot{A}_{p,q}^s}$). Indeed, as observed previously, \mathcal{P}_∞ is the kernel of Q_j , hence $\|f\|_{\dot{A}_{p,q}^{s,\tau}} = 0$ implies $f = 0$ in \mathcal{S}'_∞ . The same argument applies to $\|f\|_{\dot{A}_{p,q}^s} = 0$.

REMARK 2.4. We have the following elementary observation: $\mathcal{E}_{p,q}^{s,\tau} \cap \mathcal{P}_m = \mathcal{P}_1$ for all $m \in \mathbb{N}$, in other words $\mathcal{P}_m \not\subseteq \mathcal{E}_{p,q}^{s,\tau}$ if $m \geq 2$.

The definition of $\dot{A}_{p,q}^{s,\tau}$ is independent of the choice of the function ρ (see, e.g., [31]). Also, we can extend this definition to any $\tau < 0$, but in that case $\dot{A}_{p,q}^{s,\tau} = \{0\}$ in \mathcal{S}'_∞ . In particular, $\dot{A}_{p,q}^{s,0} = \dot{A}_{p,q}^s$, and we have various embeddings:

- (P1) $\mathcal{S}_\infty \hookrightarrow \dot{A}_{p,q}^{s,\tau} \hookrightarrow \mathcal{S}'_\infty$,
(P2) $\dot{B}_{p,\min(p,q)}^{s,\tau} \hookrightarrow \dot{F}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p,\max(p,q)}^{s,\tau}$ and $\dot{A}_{p,q_1}^{s,\tau} \hookrightarrow \dot{A}_{p,q_2}^{s,\tau}$ ($q_1 \leq q_2$),
(P3) $\dot{A}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{\infty,\infty}^{s+n\tau-n/p} = \dot{F}_{\infty,\infty}^{s+n\tau-n/p}$ (see [30]),
(P4) $\dot{B}_{p_1,q}^{s_1,\tau} \hookrightarrow \dot{B}_{p_2,q}^{s_2,\tau}$ and $\dot{F}_{p_1,q}^{s_1,\tau} \hookrightarrow \dot{F}_{p_2,q}^{s_2,\tau}$ if $0 < p_1 < p_2 \leq \infty$ ($p_2 < \infty$ in F -spaces), $s_1 - n/p_1 = s_2 - n/p_2$, and $0 < r \leq \infty$ (see [29, Proposition 3.3]).

We also have the following two properties:

- (P5) Let $m \in \mathbb{N}$. Then $f \in \dot{A}_{p,q}^{s,\tau}$ if and only if $f^{(\alpha)} \in \dot{A}_{p,q}^{s-m,\tau}$ for all α with $|\alpha| = m$. Moreover, $\sum_{|\alpha|=m} \|f^{(\alpha)}\|_{\dot{A}_{p,q}^{s-m,\tau}}$ is an equivalent quasi-norm in $\dot{A}_{p,q}^{s,\tau}$ (see [27, Theorem 1.5]).

(P6) There exist constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_{\dot{A}_{p,q}^{s,\tau}} \leq \lambda^{n/p-s-n\tau} \|f(\lambda(\cdot))\|_{\dot{A}_{p,q}^{s,\tau}} \leq c_2 \|f\|_{\dot{A}_{p,q}^{s,\tau}}$$

for all $f \in \dot{A}_{p,q}^{s,\tau}$ and all $\lambda > 0$. The expression

$$\lambda^{n/p-s-n\tau} \|f(\lambda(\cdot))\|_{\dot{A}_{p,q}^{s,\tau}}$$

defines an equivalent quasi-norm in $\dot{A}_{p,q}^{s,\tau}$ (see, e.g., [4, Proposition 2], [31, Remark 8.1]).

For $\mathcal{E}_{p,q}^{s,\tau}$ we have the following particular embedding, where C_{ub} denotes the Banach space of bounded and uniformly continuous functions endowed with the supremum norm:

PROPOSITION 2.5. *If $s > n/p - n\tau$, then we have the embeddings*

$$\mathcal{S}_\infty \hookrightarrow \mathcal{E}_{p,q}^{s,\tau} \hookrightarrow B_{\infty,\infty}^{s+n\tau-n/p} \hookrightarrow C_{ub}.$$

Proof. We provide the proof for B -spaces; the argument for F -spaces is completely analogous. We first introduce an auxiliary set: Let $\dot{\mathcal{B}}_{p,q}^s$ be the set of all $f \in \mathcal{S}'$ such that $\|[f]_\infty\|_{\dot{\mathcal{B}}_{p,q}^s} < \infty$. By (P3) we have $\mathcal{B}_{p,q}^{s,\tau} \hookrightarrow L_\infty \cap \dot{\mathcal{B}}_{\infty,\infty}^{s+n\tau-n/p}$. But it is well-known that $B_{\infty,\infty}^{s+n\tau-n/p} = L_\infty \cap \dot{\mathcal{B}}_{\infty,\infty}^{s+n\tau-n/p}$ (see, e.g., [26, Theorem 2.3.3]), so we have the second embedding. The last one is well-known (see, e.g., [23, Theorem 2.2.4/1]). ■

2.2. The Morrey spaces

DEFINITION 2.6. Let $0 < p \leq w \leq \infty$. The *Morrey space* \mathcal{M}_p^w is the set of all $f \in L_{p,\text{loc}}$ such that

$$\|f\|_{\mathcal{M}_p^w} := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn(1/p-1/w)} \|f\|_{L_p(P_{k,\eta})} < \infty.$$

In this definition, the dyadic cubes $P_{k,\eta}$ can be replaced by balls. We have e.g.

$$\mathcal{M}_p^p = L_p, \quad \mathcal{M}_p^\infty = L_\infty, \quad \mathcal{M}_p^w \hookrightarrow \mathcal{M}_r^w \text{ if } p \geq r.$$

In particular, $\|f(\lambda(\cdot))\|_{\mathcal{M}_p^w} \leq c\lambda^{-n/w} \|f\|_{\mathcal{M}_p^w}$ for all $f \in \mathcal{M}_p^w$ and all $\lambda > 0$ (see, e.g., [24, (6.71)]). For further properties of these spaces, we refer the reader to [24, Section 6.1.5] and [31, Section 1.4.3].

2.3. Realization spaces. Let us introduce the space $\dot{W}^m(L_\infty) \cap \dot{W}^m(\mathcal{M}_p^w)$, endowed with its natural quasi-seminorm

$$\|f\|_* := \|f\|_{\dot{W}^m(L_\infty)} + \|f\|_{\dot{W}^m(\mathcal{M}_p^w)}.$$

Since $\dot{W}^m(L_\infty) \subset C^{m-1}$ for $m \geq 1$ (see, e.g., [6, p. 861]), it follows that $\dot{W}^m(L_\infty) \cap \dot{W}^m(\mathcal{M}_p^w)$ is embedded into $L_{1,\text{loc}}$. Moreover, since $\|u\|_* = 0$ for

all $u \in \mathcal{P}_m$, the presence of polynomials naturally leads to introducing the definition of realization spaces:

DEFINITION 2.7. A subspace E of $\dot{W}^m(L_\infty)$ is said to be a *realization* if

$$(2.1) \quad \dot{W}^m(L_\infty) = E \oplus \mathcal{P}_m;$$

such a subspace is endowed with the norm $\|f\|_E := \|f\|_{\dot{W}^m(L_\infty)}$.

If the condition (2.1) holds, we obtain a linear mapping

$$\sigma : \dot{W}^m(L_\infty)/\mathcal{P}_m \rightarrow \mathcal{S}'$$

such that

$$\forall g \in \dot{W}^m(L_\infty)/\mathcal{P}_m : \quad [\sigma(g)]_m = g,$$

and $\sigma(\dot{W}^m(L_\infty)/\mathcal{P}_m) \subset E$, where $[g]_m := \{g+u : u \in \mathcal{P}_m\}$. If σ is continuous from $\dot{W}^m(L_\infty)/\mathcal{P}_m$ into \mathcal{S}' , then it is a realization in the sense of [6, 7]. For such a mapping, the following assertion was provided in [6, proof of Theorem 1]:

$$\forall f \in \dot{W}^m(L_\infty) : \quad \tilde{\sigma}([f]_m)(x) := f(x) - \sum_{|\alpha| < m} f^{(\alpha)}(0) \frac{x^\alpha}{\alpha!},$$

which clearly satisfies $[\tilde{\sigma}([f]_m)]_m = [f]_m$, and is continuous from the space $\dot{W}^m(L_\infty)/\mathcal{P}_m$ into \mathcal{S}' since

$$|\tilde{\sigma}([f]_m)(x)| \leq c|x|^m \|f\|_{\dot{W}^m(L_\infty)} \quad (\forall f \in \dot{W}^m(L_\infty), \forall x \in \mathbb{R}^n)$$

(see again [6, (11), p. 863]). We can associate to it the range

$$E_{\tilde{\sigma}} := \{f \in \dot{W}^m(L_\infty) : f^{(\alpha)}(0) = 0, \forall |\alpha| < m\},$$

endowed with the norm $\|f\|_{E_{\tilde{\sigma}}} := \|f\|_{\dot{W}^m(L_\infty)}$, continuously embedded into \mathcal{S}' and satisfying $E_{\tilde{\sigma}} \cap \mathcal{P}_m = \{0\}$ (an easy proof).

REMARK 2.8. The space $E \cap \dot{W}^m(\mathcal{M}_p^w)$ is the space of tempered distributions endowed with the quasi-norm $\|f\| := \|f\|_*$ in the sense of realizations. For further properties of realization spaces, we refer, for example, to [6, 7, 8, 9, 13, 14] and [18, Section 3].

2.4. Distributions vanishing at infinity. Sometimes, the difficulty of handling distributions modulo polynomials requires considering convergence in \mathcal{S}' in the weak sense, that is, the following notion:

DEFINITION 2.9. We say that a distribution $f \in \mathcal{S}'$ *vanishes at infinity* if $\lim_{\lambda \downarrow 0} f(\cdot/\lambda) = 0$ in \mathcal{S}' . The set of all such distributions is denoted by \tilde{C}_0 .

We give some examples of such distributions: if $f \in L_p$ ($1 \leq p < \infty$) then $f \in \tilde{C}_0$; if $f \in L_\infty$ or $f \in \tilde{C}_0$ then $\partial_\ell f \in \tilde{C}_0$ ($\ell = 1, \dots, n$). Also, by an easy computation we have $\tilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$. Another example is the following useful assertion proved in [7, Proposition 4.2]:

LEMMA 2.10. *Let Ω be a compact subset of $\mathbb{R}^n \setminus \{0\}$. Then every bounded function f such that $\text{supp } \hat{f} \subset \Omega$ belongs to \tilde{C}_0 .*

Other examples can be found in, e.g., [3, 7, 16, 21].

3. Statement of the main results. The following result concerns the embedding of $\mathcal{E}_{p,q}^{s,\tau}$ into $\dot{W}^m(\mathcal{M}_v^w)$.

THEOREM 3.1. *Let $0 < p < \infty$, $0 \leq \tau < 1/p$ and $m \in \mathbb{N}$. Let s be such that*

$$(3.1) \quad \max\left(m, \frac{n}{p} - n\tau\right) < s < \frac{m}{\tau p} + \frac{n}{p} - n\tau.$$

For all real numbers $0 < v \leq \infty$ satisfying

$$(3.2) \quad \tau \leq \frac{1}{v} \quad \text{and} \quad \frac{1}{n} \left(m + \frac{n}{p} - s\right) < \frac{1}{v} < \frac{m + n\tau}{p(s + n\tau)},$$

we have the continuous embedding

$$\mathcal{E}_{p,q}^{s,\tau} \hookrightarrow \dot{W}^m(\mathcal{M}_v^w) \quad \text{where} \quad 1/w := 1/v - \tau.$$

REMARK 3.2. The preceding theorem covers some previous results. Namely, if $\tau = 0$ and $m = 1$, then under the assumptions of Theorem 3.1, we have:

- (for $n = 1$) $\|f'\|_{L_v(\mathbb{R})} \leq c\|f\|_{\mathcal{F}_{p,q}^s(\mathbb{R})}$, proved by the \pm -method of interpolation of Gustafsson and Peetre in [10, Proposition 13],
- $\|\partial_\ell f\|_v \leq c\|f\|_{\mathcal{B}_{p,q}^s}$ for $\ell = 1, \dots, n$ (cf. [17, Proposition 3.23]).

REMARK 3.3. For clarity, we add a few comments. In the estimate $\|f^{(\alpha)}\|_{\mathcal{M}_v^w} \leq c\|f\|_{\mathcal{E}_{p,q}^{s,\tau}}$ ($|\alpha| = m$, $f \in \mathcal{E}_{p,q}^{s,\tau}$), replacing f by $f(\lambda(\cdot))$ with any $\lambda > 0$, under the assumptions of Theorem 3.1 we obtain

$$(3.3) \quad \lambda^{m-n/w} \|f^{(\alpha)}\|_{\mathcal{M}_v^w} \leq c_1 \lambda^{s+n\tau-n/p} \| [f]_\infty \|_{\dot{A}_{p,q}^{s,\tau}} + c_2 \|f\|_\infty.$$

The constant functions belong to $\mathcal{E}_{p,q}^{s,\tau}$, and the right-hand side of (3.3) reduces to a positive constant if $f(x) \equiv c$ for some c . Thus, excluding the constant functions, we discuss the *optimality* of the assumptions of Theorem 3.1 by contradicting (3.1):

- Assume that $m - n/w < 0$; by letting $\lambda \rightarrow 0+$, the left-hand side of (3.3) makes sense if $\|f^{(\alpha)}\|_{\mathcal{M}_v^w} = 0$, i.e. $f \in \mathcal{P}_m$, but this is impossible: see Remark 2.4. Note that $1/v < (p(s + n\tau))^{-1}(m + n\tau)$ implies $m - n/w > 0$.
- Assume that $0 < m + n/p - s - n/v =: K$. The estimate (3.3) can be written as

$$\lambda^K \|f^{(\alpha)}\|_{\mathcal{M}_v^w} \leq c_1 \| [f]_\infty \|_{\dot{A}_{p,q}^{s,\tau}} + c_2 \lambda^{-(s+n\tau-n/p)} \|f\|_\infty;$$

letting $\lambda \rightarrow \infty$ shows that this inequality is impossible as above.

Owing to the last remark, a natural question arises: Does Theorem 3.1 still hold if $m - n/w = 0$ (recall $1/w := 1/v - \tau$) or $m - n/v + n/p - s = 0$? The following statement gives the answer:

THEOREM 3.4. *Let $m \in \mathbb{N}$ and $0 < v < \infty$ be such that*

$$\tau \leq \frac{1}{v} \quad \text{and} \quad \frac{m}{n} = \frac{1}{v} - \tau.$$

Let $0 < p < v$. Put $s := m - n/v + n/p$. Then the following hold:

- (i) $\mathcal{F}_{p,q}^{s,\tau} \hookrightarrow \dot{W}^m(\mathcal{M}_v^{n/m})$.
- (ii) *If either $q < v$, or $q \geq v$ and $\tau > 0$, then $\mathcal{B}_{p,q}^{s,\tau} \hookrightarrow \dot{W}^m(\mathcal{M}_v^{n/m})$.*

REMARK 3.5. In Theorem 3.4, if in addition $m = 0$, i.e. $w = \infty$, the conclusion becomes of type $\mathcal{E}_{p,q}^{s,\tau} \hookrightarrow L_\infty$, which is satisfied from the definition. On the other hand, an interesting application of this theorem is in the case of classical homogeneous Besov–Triebel–Lizorkin spaces and homogeneous Sobolev spaces: for $m \in \mathbb{N}$ and $p < n/m$ one has $\mathcal{F}_{p,q}^{n/p} \hookrightarrow \dot{W}^m(L_{n/m})$ and $\mathcal{B}_{p,q}^{n/p} \hookrightarrow \dot{W}^m(L_{n/m})$, with $q < n/m$ in B -spaces.

To see the converse of the embedding given in Theorem 3.1, we should observe that $\dot{W}^m(\mathcal{M}_p^w)$ contains \mathcal{P}_m , so we turn to realization spaces in the sense of Section 2.3.

THEOREM 3.6. *Let $m \in \mathbb{N}$ and $0 < p \leq w \leq \infty$ ($p < \infty$ in F -spaces). Let τ be a number such that $1/p - \min(1, p)/w < \tau \leq 1/p$. Let E be the space defined in (2.1). Then*

$$E \cap \dot{W}^m(\mathcal{M}_p^w) \hookrightarrow \mathcal{E}_{p,\infty}^{m,\tau}.$$

Moreover, there exists a constant $c > 0$ such that

$$(3.4) \quad \|f\|_{\mathcal{E}_{p,\infty}^{m,\tau}} \leq c \|f\|_{\dot{W}^m(L_\infty)}^{1-\beta} \|f\|_{\dot{W}^m(\mathcal{M}_p^w)}^\beta, \quad \beta := w(1/p - \tau),$$

for all $f \in E \cap \dot{W}^m(\mathcal{M}_p^w)$.

REMARK 3.7. In Theorem 3.6 we cannot replace the space $E \cap \dot{W}^m(\mathcal{M}_p^w)$ by $\dot{W}^m(L_\infty) \cap \dot{W}^m(\mathcal{M}_p^w)$. Indeed, if we take, for instance, $u_m(x) := x_1^{m-1}$, we have $\|u_m\|_{\mathcal{E}_{p,\infty}^{m,\tau}} = \infty$ if $m \geq 2$ and $\|u_1\|_{\mathcal{E}_{p,\infty}^{1,\tau}} = 1$, while the right-hand side of (3.4) is equal to 0.

4. Proofs

Proof of Theorem 3.1. Let $f \in \mathcal{E}_{p,q}^{s,\tau}$. For all $|\alpha| = m$, we will prove that

$$(4.1) \quad \|f^{(\alpha)}\|_{\mathcal{M}_v^w} \leq c \|f\|_{\mathcal{E}_{p,q}^{s,\tau}},$$

where the constant $c > 0$ is independent of f .

Set $f_1 := \sum_{l>0} Q_l f$ and $f_2 := \sum_{l \leq 0} Q_l f$. Notice that f_1 is a distribution, while f_2 is a distribution modulo polynomials. Thus, we are going to estimate

$\|f_i^{(\alpha)}\|_{\mathcal{M}_v^w}$ ($|\alpha| = m$ and $i = 1, 2$), and find conditions on the parameters s , p , v and w . On the other hand, we will make repeated use of the well-known Bernstein inequality (see, e.g., [25, Remark 1.3.2/1]):

LEMMA 4.1. *If $0 < u \leq r \leq \infty$ and $\beta \in \mathbb{N}_0^n$, then there exists a constant $c := c(n, u, r, \beta) > 0$ such that the inequality*

$$\|g^{(\beta)}\|_r \leq cR^{|\beta|+n(1/u-1/r)}\|g\|_u$$

holds for all $R > 0$ and all $g \in L_u \cap \mathcal{S}'$ satisfying $\text{supp } \hat{g} \subset \{\xi : |\xi| \leq R\}$.

We also use the following estimate: For any sequence $(g_l)_{l \in \Lambda}$ of functions in \mathcal{M}_v^w ($0 < v \leq w \leq \infty$), we have

$$(4.2) \quad \left\| \sum_{l \in \Lambda} g_l \right\|_{\mathcal{M}_v^w} \leq \left(\sum_{l \in \Lambda} \|g_l\|_{\mathcal{M}_v^w}^t \right)^{1/t}, \quad \text{where } t := \min(v, 1).$$

Indeed, using the elementary estimate

$$(4.3) \quad \sum_l a_l \leq \left(\sum_l a_l^d \right)^{1/d} \quad \text{for all } a_l \geq 0 \text{ and } 0 < d \leq 1,$$

and the Minkowski inequality in $L_v(P_{k,\eta}; \ell_t(\Lambda))$, we have

$$\begin{aligned} \left(\int_{P_{k,\eta}} \left| \sum_{l \in \Lambda} g_l(x) \right|^v dx \right)^{1/v} &\leq \left(\int_{P_{k,\eta}} \left(\sum_{l \in \Lambda} |g_l(x)|^t \right)^{v/t} dx \right)^{1/v} \\ &\leq \left(\sum_{l \in \Lambda} \left(\int_{P_{k,\eta}} |g_l(x)|^v dx \right)^{t/v} \right)^{1/t} \\ &\leq 2^{-kn(1/v-1/w)} \left(\sum_{l \in \Lambda} \|g_l\|_{\mathcal{M}_v^w}^t \right)^{1/t}; \end{aligned}$$

then we divide by $2^{-kn(1/v-1/w)}$ and take the supremum over all $k \in \mathbb{Z}$ and all $\eta \in \mathbb{Z}^n$.

STEP 1: *B-spaces.*

SUBSTEP 1.1: *Estimate of $f_1^{(\alpha)}$.* We first introduce a function g such that $[g]_\infty \in \dot{B}_{r,q}^{s,\mu}$, where $0 < r \leq w \leq \infty$ and $\mu := 1/r - 1/w$.

Let $k \leq l$. We have

$$(4.4) \quad \begin{aligned} 2^{kn\mu} \|Q_l g\|_{L_r(P_{k,\eta})} &\leq 2^{-sl} 2^{kn\mu} \left(\sum_{j \geq l} 2^{jsq} \|Q_j g\|_{L_r(P_{k,\eta})}^q \right)^{1/q} \\ &\leq 2^{-sl} 2^{kn\mu} \left(\sum_{j \geq k} 2^{jsq} \|Q_j g\|_{L_r(P_{k,\eta})}^q \right)^{1/q} \leq 2^{-sl} \|[g]_\infty\|_{\dot{B}_{r,q}^{s,\mu}}, \end{aligned}$$

where we have used $\sum_{j \geq l} \dots \leq \sum_{j \geq k} \dots$

Let $k > l$. Applying (P3), we get

$$(4.5) \quad \|Q_l g\|_\infty \leq c_1 2^{l(n/r - n\mu - s)} \| [g]_\infty \|_{\dot{B}_{r,q}^{s,\mu}} \quad (\forall l \in \mathbb{Z}).$$

Then

$$\|Q_l g\|_{L_r(P_{k,\eta})} \leq 2^{-kn/r} \|Q_l g\|_\infty \leq c_1 2^{(l-k)(n/r - n\mu)} 2^{-kn\mu - ls} \| [g]_\infty \|_{\dot{B}_{r,q}^{s,\mu}},$$

since $1/r - \mu = 1/w \geq 0$, i.e., $2^{(l-k)(n/r - n\mu)} \leq 1$. Thus

$$(4.6) \quad 2^{kn\mu} \|Q_l g\|_{L_r(P_{k,\eta})} \leq c_1 2^{-ls} \| [g]_\infty \|_{\dot{B}_{r,q}^{s,\mu}} \quad (\forall k > l).$$

Now, from (4.4) and (4.6) we have

$$\|Q_l g\|_{\mathcal{M}_r^w} \leq c 2^{-ls} \| [g]_\infty \|_{\dot{B}_{r,q}^{s,\mu}};$$

for $w = \infty$ ($\mathcal{M}_r^\infty = L_\infty$), the last inequality becomes

$$\|Q_l g\|_\infty \leq c 2^{-ls} \| [g]_\infty \|_{\dot{B}_{r,q}^{s,\mu}}.$$

Then, by the assumption $s > 0$, we can use (4.2) with $g_l := Q_l g$ and $t := \min(r, 1)$ to obtain

$$\left\| \sum_{l>0} Q_l g \right\|_{\mathcal{M}_r^w} \leq c \| [g]_\infty \|_{\dot{B}_{r,q}^{s,\mu}}$$

and

$$\sum_{l>0} \|Q_l g\|_\infty \leq c \| [g]_\infty \|_{\dot{B}_{r,q}^{s,1/r}} \quad \text{for } w = \infty.$$

Using these inequalities with $f^{(\alpha)}$, v , $\tau = 1/v - 1/w \geq 0$ and $s_1 := s - n/p + n/v - m > 0$ instead of g , r , μ and s , respectively, we need the conditions

$$(4.7) \quad \frac{1}{\omega} := \frac{1}{v} - \tau \geq 0, \quad v \leq w \quad \text{and} \quad \frac{n}{v} > m + \frac{n}{p} - s.$$

We assume that

$$(4.8) \quad p < v,$$

and use the embedding $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{v,q}^{s_1+m,\tau}$. We then need (cf. (4.7)) the condition

$$(4.9) \quad \frac{n}{p} > m + \frac{n}{p} - s \iff s > m.$$

All these facts give

$$\left\| \sum_{l>0} (Q_l f)^{(\alpha)} \right\|_{\mathcal{M}_v^w} \leq c_1 \| [f^{(\alpha)}]_\infty \|_{\dot{B}_{v,q}^{s_1,\tau}} \leq c_2 \| [f]_\infty \|_{\dot{B}_{v,q}^{s_1+m,\tau}} \leq c_3 \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}.$$

Then the series $\sum_{l>0} (Q_l f)^{(\alpha)}$ converges in \mathcal{M}_v^w . If we denote its sum by g_α , we have

$$(4.10) \quad \|g_\alpha\|_{\mathcal{M}_v^w} \leq c \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}},$$

and for $w = \infty$,

$$(4.11) \quad \sum_{l>0} \|(Q_l f)^{(\alpha)}\|_{\infty} \leq c \| [f]_{\infty} \|_{\dot{B}_{p,q}^{s,1/v}}.$$

SUBSTEP 1.2: *Estimate of $f_2^{(\alpha)}$* . By translation invariance, we have, for a fixed $l \leq 0$,

$$(4.12) \quad \|(Q_l f)^{(\alpha)}\|_{\mathcal{M}_v^w} = \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{n\tau(k+l)} \|(Q_l f)^{(\alpha)}\|_{L_v(P_{k+l,\eta})}.$$

Then

$$\|(Q_l f)^{(\alpha)}\|_{\mathcal{M}_v^w} \leq I_l + J_l,$$

where I_l and J_l are defined similarly to the right-hand side of (4.12) by taking $\sup_{k \leq 0} \cdots$ and $\sup_{k \geq 0} \cdots$ instead of $\sup_{k \in \mathbb{Z}} \cdots$, respectively; for $w = \infty$ one has

$$(4.13) \quad \|(Q_l f)^{(\alpha)}\|_{\infty} = \|(Q_l f)^{(\alpha)}\|_{\mathcal{M}_v^{\infty}} \leq I_l + J_l.$$

On the other hand, since $p < v$ (see (4.8)), by the Bernstein inequality and the inequality $\|Q_l f\|_{\infty} \leq \|\tilde{\gamma}\|_1 \|f\|_{\infty}$ we get

$$(4.14) \quad \begin{aligned} \|(Q_l f)^{(\alpha)}\|_{L_v(P_{k+l,\eta})} &\leq \|(Q_l f)^{(\alpha)}\|_{\infty}^{1-p/v} \|(Q_l f)^{(\alpha)}\|_{L_p(P_{k+l,\eta})}^{p/v} \\ &\leq c 2^{lm(1-p/v)} \|f\|_{\infty}^{1-p/v} \|(Q_l f)^{(\alpha)}\|_{L_p(P_{k+l,\eta})}^{p/v}. \end{aligned}$$

Estimation of I_l : In (4.14), by using $\sum_{j \geq l} \cdots \leq \sum_{j \geq l+k} \cdots$ for any $k \leq 0$, we get

$$\begin{aligned} &\|(Q_l f)^{(\alpha)}\|_{L_v(P_{k+l,\eta})} \\ &\leq c_1 2^{l(m-sp/v)} \|f\|_{\infty}^{1-p/v} \left[\left(\sum_{j \geq l} 2^{jq(s-m)} \|(Q_j f)^{(\alpha)}\|_{L_p(P_{k+l,\eta})}^q \right)^{1/q} \right]^{p/v} \\ &\leq c_1 2^{l(m-sp/v)} \|f\|_{\infty}^{1-p/v} \left[\left(\sum_{j \geq l+k} 2^{jq(s-m)} \|(Q_j f)^{(\alpha)}\|_{L_p(P_{k+l,\eta})}^q \right)^{1/q} \right]^{p/v} \\ &\leq c_2 2^{-kn\tau p/v} 2^{l(m-sp/v-n\tau p/v)} \|f\|_{\infty}^{1-p/v} \| [f^{(\alpha)}]_{\infty} \|_{\dot{B}_{p,q}^{s-m,\tau}}^{p/v} \\ &= c_2 2^{-kn\tau} 2^{kn\tau(1-p/v)} 2^{l(m-sp/v-n\tau p/v)} \|f\|_{\infty}^{1-p/v} \| [f^{(\alpha)}]_{\infty} \|_{\dot{B}_{p,q}^{s-m,\tau}}^{p/v}. \end{aligned}$$

So, as $k \leq 0$ and $p < v$ (i.e., $2^{kn\tau(1-p/v)} \leq 1$), we obtain

$$(4.15) \quad \begin{aligned} I_l &\leq c_1 2^{l(m-sp/v+n\tau(1-p/v))} \|f\|_{\infty}^{1-p/v} \| [f]_{\infty} \|_{\dot{B}_{p,q}^{s,\tau}}^{p/v} \\ &\leq c_2 2^{l(m-sp/v+n\tau(1-p/v))} \|f\|_{\mathcal{B}_{p,q}^{s,\tau}} \quad (\forall l \leq 0). \end{aligned}$$

Estimation of J_l : In (4.14), we successively apply

$$\|(Q_l f)^{(\alpha)}\|_{L_p(P_{k+l,\eta})} \leq 2^{-(k+l)n/p} \|(Q_l f)^{(\alpha)}\|_{\infty},$$

the Bernstein inequality, and the following estimate (cf. (P3), see also (4.5)):

$$\|Q_l f\|_\infty \leq c_1 2^{l(n/p - n\tau - s)} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}} \quad (\forall l \in \mathbb{Z}),$$

to get

$$\begin{aligned} \|(Q_l f)^{(\alpha)}\|_{L_v(P_{k+l,\eta})} &\leq c_1 2^{lm(1-p/v)} \|f\|_\infty^{1-p/v} (2^{-(k+l)n/p} \|(Q_l f)^{(\alpha)}\|_\infty)^{p/v} \\ &\leq c_2 2^{lm - (k+l)n/v} \|f\|_\infty^{1-p/v} \|Q_l f\|_\infty^{p/v} \\ &\leq c_3 2^{lm - (k+l)n/v} 2^{l(n/p - n\tau - s)p/v} \|f\|_\infty^{1-p/v} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}^{p/v}. \end{aligned}$$

Now, we observe that $2^{(k+l)n\tau} \|(Q_l f)^{(\alpha)}\|_{L_v(P_{k+l,\eta})}$ is bounded by

$$c_2^{l(m-sp/v+n\tau(1-p/v))} 2^{kn(\tau-1/v)} \|f\|_{\mathcal{B}_{p,q}^{s,\tau}}.$$

As $k \geq 0$ and $\tau \leq 1/v$ (i.e., $2^{kn(\tau-1/v)} \leq 1$), we obtain

$$(4.16) \quad J_l \leq c_2^{l(m-sp/v+n\tau(1-p/v))} \|f\|_{\mathcal{B}_{p,q}^{s,\tau}}.$$

We now turn to (4.12) (or (4.13) for $w = \infty$). By employing (4.15) and (4.16), and then (4.2) with $t := \min(v, 1)$, we get

$$(4.17) \quad \left\| \sum_{l \leq 0} (Q_l f)^{(\alpha)} \right\|_{\mathcal{M}_v^w} \leq c_1 \left\{ \left(\sum_{l \leq 0} (I_l)^t \right)^{1/t} + \left(\sum_{l \leq 0} (J_l)^t \right)^{1/t} \right\} \\ \leq c_2 \|f\|_{\mathcal{B}_{p,q}^{s,\tau}},$$

where we have used the classical inequality

$$(4.18) \quad (a+b)^d \leq a^d + b^d \leq 2^{1-d} (a+b)^d, \quad a \geq 0, b \geq 0, 0 < d \leq 1;$$

and for $w = \infty$ we get

$$(4.19) \quad \sum_{l \leq 0} \|(Q_l f)^{(\alpha)}\|_\infty \leq \sum_{l \leq 0} I_l + \sum_{l \leq 0} J_l \leq c \|f\|_{\mathcal{B}_{p,q}^{s,1/v}}.$$

Thus, the series $\sum_{l \leq 0} (Q_l f)^{(\alpha)}$ converges in \mathcal{M}_v^w ; if h_α denotes its sum, then

$$(4.20) \quad \|h_\alpha\|_{\mathcal{M}_v^w} \leq c \|f\|_{\mathcal{B}_{p,q}^{s,\tau}}.$$

To justify (4.17) and (4.19), we need the condition

$$(4.21) \quad m - \frac{sp}{v} + n\tau \left(1 - \frac{p}{v} \right) > 0.$$

Then, from (4.7) and (4.21), we choose v such that

$$(4.22) \quad \tau \leq \frac{1}{v} \quad \text{and} \quad m + \frac{n}{p} - s < \frac{n}{v} < \frac{n}{p} \times \frac{m+n\tau}{s+n\tau},$$

which requires

$$m + \frac{n}{p} - s < \frac{n}{p} \times \frac{m+n\tau}{s+n\tau} \quad \text{and} \quad \tau < \frac{1}{p} \times \frac{m+n\tau}{s+n\tau};$$

these inequalities are equivalent to

$$(4.23) \quad (s - m) \left(s + n\tau - \frac{n}{p} \right) > 0 \quad \text{and} \quad s < \frac{m}{\tau p} + \frac{n}{p} - n\tau,$$

respectively. They are satisfied if

$$(4.24) \quad \max \left(m, \frac{n}{p} - n\tau \right) < s < \frac{m}{\tau p} + \frac{n}{p} - n\tau;$$

the first estimate in (4.23) is also satisfied if $s < \min(m, n/p - n\tau)$, but this condition is excluded by (4.9). In checking (4.24), we observe that

$$m < \frac{m}{\tau p} + \frac{n}{p} - n\tau \iff (n\tau + m) \left(\tau - \frac{1}{p} \right) < 0,$$

which is satisfied since $0 \leq \tau < 1/p$.

SUBSTEP 1.3: We claim that $g_\alpha, h_\alpha \in \tilde{C}_0$. Let us take $\varphi \in \mathcal{S}$ and $\lambda > 0$.

1.3.1. *The case of h_α .* We define a sequence $(u_N)_{N \in \mathbb{N}_0}$ by

$$u_N := \sum_{-N \leq l \leq 0} (Q_l f)^{(\alpha)}.$$

We have $0 \notin \text{supp } \hat{u}_N$ since it is a subset of $2^{-N-1} \leq |\xi| \leq 3/2$. By the Bernstein inequality, we also have

$$\begin{aligned} \|u_N\|_\infty &\leq c_1 \sum_{-N \leq l \leq 0} 2^{lm} \|Q_l f\|_\infty \\ &\leq c_1 \|\tilde{\gamma}\|_1 \|f\|_\infty \sum_{l \leq 0} 2^{lm} \leq c_2 \|f\|_\infty. \end{aligned}$$

Hence Lemma 2.10 provides $u_N \in \tilde{C}_0$ for all $N \in \mathbb{N}_0$. On the other hand,

$$\begin{aligned} \|h_\alpha - u_N\|_\infty &\leq c_1 \sum_{l < -N} 2^{lm} \|Q_l f\|_\infty \leq c_1 2^{-Nm} \|\tilde{\gamma}\|_1 \|f\|_\infty \sum_{j < 0} 2^{jm} \\ &\leq c_2 2^{-Nm} \|f\|_\infty \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Now, we write

$$|\langle h_\alpha(\cdot/\lambda), \varphi \rangle| \leq \|h_\alpha - u_N\|_\infty \|\varphi\|_1 + |\langle u_N(\cdot/\lambda), \varphi \rangle|.$$

For every $\varepsilon > 0$ there exists a positive integer $N_0 := N(\varepsilon)$ such that

$$|\langle h_\alpha(\cdot/\lambda), \varphi \rangle| \leq \varepsilon + |\langle u_N(\cdot/\lambda), \varphi \rangle|$$

for all $N \geq N_0$. Thus, we conclude that $h_\alpha \in \tilde{C}_0$.

1.3.2. *The case of g_α with $w = \infty$.* We proceed exactly as in the preceding paragraph by considering the sequence $(v_N)_{N \in \mathbb{N}}$, $v_N := \sum_{l=1}^N (Q_l f)^{(\alpha)}$, which clearly satisfies the assumptions of Lemma 2.10, since v_N is a bounded function (see (4.11)) and \hat{v}_N has support in the set $1 \leq |\xi| \leq 3 \cdot 2^{N-1}$, i.e.

$v_N \in \tilde{C}_0$ for all $N \in \mathbb{N}$. We note that g_α is the limit of (v_N) in L_∞ : indeed, by property (P3) (see also (4.5)) we obtain

$$\begin{aligned} \|g_\alpha - v_N\|_\infty &\leq c_1 \sum_{l>N} 2^{lm} \|Q_l f\|_\infty \leq c_1 \| [f]_\infty \| \dot{B}_{\infty, \infty}^{s+n\tau-n/p} \sum_{l>N} 2^{l(m-s-n\tau+n/p)} \\ &\leq c_2 2^{N(m-s-n/v+n/p)} \| [f]_\infty \| \dot{B}_{p, q}^{s, 1/v} \rightarrow 0 \quad \text{as } N \rightarrow \infty; \end{aligned}$$

recall that $\tau = 1/v$ and $m - s - n/v + n/p < 0$. Now, we write again

$$|\langle g_\alpha(\cdot/\lambda), \varphi \rangle| \leq \|g_\alpha - v_N\|_\infty \|\varphi\|_1 + |\langle v_N(\cdot/\lambda), \varphi \rangle|,$$

and obtain $g_\alpha \in \tilde{C}_0$.

1.3.3. *The case of g_α with $w < \infty$ and $v \geq 1$.* Applying Hölder's inequality with $1/u := 1 - 1/v$, we get

$$(4.25) \quad |\langle g_\alpha(\cdot/\lambda), \varphi \rangle| \leq \sum_{\eta \in \mathbb{Z}^n} \|g_\alpha(\cdot/\lambda)\|_{L_v(P_{0, \eta})} \|\varphi\|_{L_u(P_{0, \eta})}.$$

Moreover,

$$\|g_\alpha(\cdot/\lambda)\|_{L_v(P_{0, \eta})} = 2^{0 \times n\tau} \|g_\alpha(\cdot/\lambda)\|_{L_v(P_{0, \eta})} \leq \|g_\alpha(\cdot/\lambda)\|_{\mathcal{M}_v^w} \quad (\forall \eta \in \mathbb{Z}^n).$$

As $x \in P_{0, \eta}$ implies $1 + |\eta| \leq 2\sqrt{2n}(1 + |x|)$, we have (setting $N := n + 1$)

$$\begin{aligned} \sum_{\eta \in \mathbb{Z}^n} \|\varphi\|_{L_u(P_{0, \eta})} &\leq c_1 \sum_{\eta \in \mathbb{Z}^n} (1 + |\eta|)^{-N} \left(\int_{P_{0, \eta}} (1 + |x|)^{Nu} |\varphi(x)|^u dx \right)^{1/u} \\ &\leq c_1 \left(\int_{\mathbb{R}^n} (1 + |x|)^{Nu} |\varphi(x)|^u dx \right)^{1/u} \sum_{\eta \in \mathbb{Z}^n} (1 + |\eta|)^{-N} \\ &=: C(\varphi) < \infty. \end{aligned}$$

Using a homogeneity argument of \mathcal{M}_v^w , and inserting the preceding two estimates in (4.25), we get

$$|\langle g_\alpha(\cdot/\lambda), \varphi \rangle| \leq c \lambda^{n/w} \|g_\alpha\|_{\mathcal{M}_v^w},$$

where the constant c is independent of λ , but depends on n, m, ω, v and φ . Thus, letting $\lambda \rightarrow 0+$, we deduce that $g_\alpha \in \tilde{C}_0$.

1.3.4. *The case of g_α with $w < \infty$ and $0 < v < 1$.* This case seems more complicated than the previous ones. Let θ and r be such that $0 < \theta < 1 < r \leq \infty$ and $(1 - \theta)/v + \theta/r = 1$. We fix $0 < \lambda < 1$, and apply the Hölder inequality with exponents $v/(1 - \theta)$ and r/θ to get

$$\begin{aligned} (4.26) \quad | \langle (Q_l f)^{(\alpha)}(\cdot/\lambda), \varphi \rangle | &\leq \lambda^{n+m} \|Q_l f\|_\infty^\theta \int_{\mathbb{R}^n} |Q_l f(x)|^{1-\theta} |\varphi^{(\alpha)}(\lambda x)| dx \\ &\leq c \lambda^{n+m} \|f\|_\infty^\theta \sum_{\eta \in \mathbb{Z}^n} \|Q_l f\|_{L_v(P_{0, \eta})}^{1-\theta} \|\varphi^{(\alpha)}(\lambda \cdot)\|_{L_{r/\theta}(P_{0, \eta})}, \end{aligned}$$

where the constant c is independent of λ and φ . We put $s_2 := s - n/p + n/v > 0$, and use the embedding $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{v,\infty}^{s_2,\tau}$. Then (recall that $l \geq 0$)

(4.27)

$$\begin{aligned} \|Q_l f\|_{L_v(P_{0,\eta})} &\leq 2^{-ls_2} (2^{0 \times n\tau}) \sup_{j \geq 0} 2^{js_2} \|Q_j f\|_{L_v(P_{0,\eta})} \leq 2^{-ls_2} \|[f]_\infty\|_{\dot{B}_{v,\infty}^{s_2,\tau}} \\ &\leq c 2^{-ls_2} \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}. \end{aligned}$$

Further, as above, $x \in P_{0,\eta}$ implies

$$\lambda(1 + |\eta|) \leq 2\lambda\sqrt{2n}(1 + |x|) \leq 2\sqrt{2n}(1 + |\lambda x|) \quad (0 < \lambda < 1).$$

Hence, for $N := n + a$, where $a > 0$ will be chosen later,

$$\begin{aligned} (4.28) \quad \|\varphi^{(\alpha)}(\lambda \cdot)\|_{L_{r/\theta}(P_{0,\eta})} &\leq c_1 \lambda^{-N} (1 + |\eta|)^{-N} \left(\int_{P_{0,\eta}} (1 + |\lambda x|)^{Nr/\theta} |\varphi^{(\alpha)}(\lambda x)|^{r/\theta} dx \right)^{\theta/r} \\ &\leq c_2 \lambda^{-N-n\theta/r} (1 + |\eta|)^{-N} \left(\int_{\mathbb{R}^n} (1 + |y|)^{Nr/\theta} |\varphi^{(\alpha)}(y)|^{r/\theta} dy \right)^{\theta/r}. \end{aligned}$$

We insert (4.27) and (4.28) in (4.26), and use $\sum_{\eta \in \mathbb{Z}^n} (1 + |\eta|)^{-N} < \infty$ to get

$$\begin{aligned} \sum_{l > 0} | \langle (Q_l f)^{(\alpha)}(\cdot/\lambda), \varphi \rangle | &\leq c_1 \lambda^{m-a-n\theta/r} \|f\|_\infty^\theta \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}^{1-\theta} \sum_{l > 0} 2^{-ls_2(1-\theta)} \\ &\leq c_2 \lambda^{m-a-n\theta/r} \|f\|_{\mathcal{B}_{p,q}^{s,\tau}}, \end{aligned}$$

where the constants c_1, c_2 are independent of λ . Now, letting $\lambda \rightarrow 0+$, we obtain $g_\alpha \in \tilde{C}_0$ provided we choose θ and a such that $m - a - n\theta/r > 0$. To do this, we choose

$$0 < \theta < \min(1, rm/n) \quad \text{and} \quad 0 < a < m - n\theta/r \quad (\text{recall } m \geq 1);$$

then we may assume that r is chosen such that $\theta = r(1-v)/(r-v) < rm/n$, i.e.

$$r > \max\left(1, v + \frac{n}{m}(1-v)\right).$$

SUBSTEP 1.4: Summarizing the conditions on parameters. We first choose s such that (4.24) is satisfied, then we take v satisfying (4.22). Altogether we get conditions (3.1)–(3.2).

As $f \in L_\infty$, we have $f^{(\alpha)} \in \tilde{C}_0$ since $|\alpha| = m \geq 1$. We know that $f^{(\alpha)} - (g_\alpha + h_\alpha) \in \mathcal{P}_\infty$. Now, since both $f^{(\alpha)}$ and $g_\alpha + h_\alpha$ belong to \tilde{C}_0 , we conclude that they coincide. By combining (4.10) and (4.20), we obtain the desired estimate (4.1).

STEP 2: F -spaces. This is a simple consequence of the embedding $\mathcal{F}_{p,q}^{s,\tau} \hookrightarrow \mathcal{B}_{p,\infty}^{s,\tau}$ and the preceding step. The proof is complete. ■

Proof of Theorem 3.4. Let $f \in \mathcal{E}_{p,q}^{s,\tau}$ and $|\alpha| = m$.

STEP 1: *Preparation.* We begin by estimating $\|\sum_{l \in \mathbb{Z}} (Q_l f)^{(\alpha)}\|_{\mathcal{M}_v^w}$, where $w := n/m$. We will systematically use inequalities of type (4.3) and (4.18). We get

$$\left\| \sum_{l \in \mathbb{Z}} (Q_l f)^{(\alpha)} \right\|_{\mathcal{M}_v^w} \leq c_1(I + J),$$

where

$$I := \sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} 2^{kn\tau} \left\| \sum_{l \geq k} (Q_l f)^{(\alpha)} \right\|_{L_v(P_{k,\eta})},$$

J is defined similarly with $\sum_{l \leq k}$ instead of $\sum_{l \geq k}$ and $c_1 := \max(2^{1/v-1}, 1)$.

SUBSTEP 1.1: *Estimate of J .* We put $t := \min(v, 1)$. By the Minkowski and Bernstein inequalities and $\|Q_l f\|_\infty \leq \|\tilde{\gamma}\|_1 \|f\|_\infty$, we get

$$\begin{aligned} 2^{kn\tau} \left\| \sum_{l \leq k} (Q_l f)^{(\alpha)} \right\|_{L_v(P_{k,\eta})} &\leq 2^{kn\tau} \left(\sum_{l \leq k} \|(Q_l f)^{(\alpha)}\|_{L_v(P_{k,\eta})}^t \right)^{1/t} \\ &\leq c_1 2^{kn(\tau-1/v)} \|f\|_\infty \left(\sum_{l \leq k} 2^{tml} \right)^{1/t} \quad (\text{we put } j := l - k) \\ &= c_1 \|f\|_\infty \left(\sum_{j \leq 0} 2^{jtm} \right)^{1/t} \leq c_2 \|f\|_\infty \quad (\text{here } n\tau - n/v + m = 0). \end{aligned}$$

Hence $J \leq c \|f\|_\infty$.

SUBSTEP 1.2: *Estimate of I in F -spaces.* Trivially, by (P5) we have

$$I \leq \| [f^{(\alpha)}]_\infty \|_{\dot{F}_{v,1}^{0,\tau}} \leq c \| [f]_\infty \|_{\dot{F}_{v,1}^{m,\tau}},$$

and we finish by applying the embedding $\dot{F}_{p,q}^{s,\tau} \hookrightarrow \dot{F}_{v,1}^{m,\tau}$.

SUBSTEP 1.3: *Estimate of I in B -spaces.* If $p \geq q$ we have $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{F}_{p,q}^{s,\tau} \hookrightarrow \dot{F}_{v,1}^{m,\tau}$, then we apply the preceding substep.

If $p < q$ and $q < v$, we introduce p_1 such that $p < p_1 < v$ and $q \leq p_1$, and put $s_1 := s - n/p + n/p_1$. We obtain $\dot{B}_{p,q}^{s,\tau} \hookrightarrow \dot{B}_{p_1,q}^{s_1,\tau} \hookrightarrow \dot{F}_{p_1,q}^{s_1,\tau} \hookrightarrow \dot{F}_{v,1}^{m,\tau}$, and again apply the preceding substep.

If $q \geq v$ and $\tau > 0$, then by the Minkowski inequality we first have (recall $t := \min(v, 1)$)

$$(4.29) \quad 2^{kn\tau} \left\| \sum_{l \geq k} (Q_l f)^{(\alpha)} \right\|_{L_v(P_{k,\eta})} \leq 2^{kn\tau} \left(\sum_{l \geq k} \|(Q_l f)^{(\alpha)}\|_{L_v(P_{k,\eta})}^t \right)^{1/t}.$$

As before, for instance in (4.14), by taking the cube $P_{k,\eta}$ instead of $P_{k+l,\eta}$, we have

$$\|(Q_l f)^{(\alpha)}\|_{L_v(P_{k,\eta})} \leq c 2^{lm(1-p/v)} \|f\|_\infty^{1-p/v} \|(Q_l f)^{(\alpha)}\|_{L_p(P_{k,\eta})}^{p/v}.$$

We then obtain

$$\begin{aligned} & \left(\sum_{l \geq k} \|(Q_l f)^{(\alpha)}\|_{L_v(P_{k,\eta})}^t \right)^{1/t} \\ & \leq c \|f\|_\infty^{1-p/v} \left(\sum_{l \geq k} 2^{lt(m-sp/v)} \left[2^{l(s-m)} \|(Q_l f)^{(\alpha)}\|_{L_p(P_{k,\eta})} \right]^{tp/v} \right)^{1/t}. \end{aligned}$$

Applying the Hölder inequality with exponents $u := qv/tp > 1$ and $1/u' := 1 - 1/u$, we find that the right-hand side above is bounded by

$$\begin{aligned} (4.30) \quad c_1 \|f\|_\infty^{1-p/v} & \left(\sum_{l \geq k} 2^{ltu'(m-sp/v)} \right)^{1/(tu')} \\ & \times \left(\sum_{l \geq k} [2^{l(s-m)} \|(Q_l f)^{(\alpha)}\|_{L_p(P_{k,\eta})}]^q \right)^{p/(qv)} \\ & \leq c_2 2^{k(m-sp/v-n\tau p/v)} \|f\|_\infty^{1-p/v} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}^{p/v}. \end{aligned}$$

It suffices to check that $m - sp/v < 0$; indeed, $m - sp/v = -n\tau(1 - p/v)$; recall here $\tau > 0$. Observe also that $m - sp/v - n\tau p/v = -n\tau$, thus, inserting (4.30) into (4.29) and taking $\sup_{k \in \mathbb{Z}} \sup_{\eta \in \mathbb{Z}^n} \dots$, we obtain

$$I \leq c_1 \|f\|_\infty^{1-p/v} \| [f]_\infty \|_{\dot{B}_{p,q}^{s,\tau}}^{p/v} \leq c_2 \|f\|_{\mathcal{B}_{p,q}^{s,\tau}}.$$

SUBSTEP 1.4. Now, since the series $\sum_{l \in \mathbb{Z}} (Q_l f)^{(\alpha)}$ converges in \mathcal{M}_v^w , we prove that its sum belongs to \tilde{C}_0 . We proceed as in [7, p. 483] or [18, p. 170]. Indeed, let $\lambda > 0$ and $\varphi \in \mathcal{S}$. We introduce an integer r such that $2^{-r-1} < \lambda \leq 2^{-r}$. Then $\text{supp } \mathcal{F}(Q_{l-r} f^{(\alpha)}(\cdot/\lambda))$ (where \mathcal{F} denotes the Fourier transform) is contained in the compact annulus $2^{l-1} \leq |\xi| \leq 3 \cdot 2^l$, and $\mathcal{F}(Q_k [Q_{l-r} f^{(\alpha)}(\cdot/\lambda)]) = 0$ if $k - l \geq 3$ or $k - l \leq -2$. Hence

$$\langle Q_{l-r} f^{(\alpha)}(\cdot/\lambda), \varphi \rangle = \langle Q_{l-r} f^{(\alpha)}(\cdot/\lambda), \tilde{Q}_l \varphi \rangle,$$

where $\tilde{Q}_l := \sum_{k=-2}^3 Q_{l+k}$ (recall that γ is an even function on \mathbb{R}^n). We turn to our assertion: since $\sum_{l \in \mathbb{Z}} (Q_l f)^{(\alpha)} = \sum_{l \in \mathbb{Z}} (Q_{l-r} f)^{(\alpha)}$ (translation invariance), we have

$$\sum_{l \in \mathbb{Z}} \langle (Q_l f)^{(\alpha)}(\cdot/\lambda), \varphi \rangle = \sum_{l \in \mathbb{Z}} \langle Q_{l-r} f^{(\alpha)}(\cdot/\lambda), \tilde{Q}_l \varphi \rangle.$$

We also have, for any $N \in \mathbb{N}_0$, $\|\tilde{Q}_l \varphi\|_1 \leq c 2^{-lN}$ for all $l \in \mathbb{N}_0$ (cf. [18, Proposition 2.5] or [17, Lemma 3.5]), and $\|Q_{l-r} f^{(\alpha)}\|_\infty \leq c 2^{(l-r)m} \|f\|_\infty$. Then

$$\sum_{l \in \mathbb{Z}} |\langle (Q_l f)^{(\alpha)}(\cdot/\lambda), \varphi \rangle| \leq c 2^{-rm} \|f\|_\infty \left(\sum_{l \geq 0} 2^{l(m-N)} + \|\varphi\|_1 \sum_{l \leq 0} 2^{lm} \right).$$

Choosing $N > m$ and letting $r \rightarrow \infty$, we obtain the desired result.

STEP 2: *Proof of (i) and (ii).* We denote by w_α the sum of the series $\sum_{l \in \mathbb{Z}} (Q_l f)^{(\alpha)}$ in \mathcal{M}_v^w , which belongs to \tilde{C}_0 . Proceeding as in the proof of Theorem 3.1, Substep 1.4, we deduce that the functions f and w_α coincide. Now to prove (i), it suffices to observe that Substeps 1.1 and 1.2 yield this statement; for (ii), we replace Substep 1.2 by 1.3. The proof is complete. ■

Proof of Theorem 3.6.

STEP 1: *Preparation.* For $f \in L_{1,\text{loc}}$, we define the associated maximal function

$$Mf(x) := \sup_{r>0} r^{-n} \int_{|x-y|<r} |f(y)| dy.$$

Then the following well-known lemma is useful (see, e.g., [28, Lemma 2.6] for the proof):

LEMMA 4.2.

- (i) If $1 < p \leq \infty$, then $\|Mf\|_p \leq c\|f\|_p$ for all $f \in L_p$.
- (ii) If $b > n$, then

$$\int_{\mathbb{R}^n} (1 + |x - y|)^{-b} |f(y)| dy \leq cMf(x)$$

for all $f \in L_{1,\text{loc}}$ and all $x \in \mathbb{R}^n$.

STEP 2. Let $f \in E \cap \dot{W}^m(\mathcal{M}_p^w)$. For brevity, we set $g := f^{(\alpha)}$ with $|\alpha| = m$.

SUBSTEP 2.1: *B-spaces.* We observe that $|Q_j g(x)| \leq \|\tilde{\gamma}\|_1 \|g\|_\infty$ for all $x \in \mathbb{R}^n$, so

$$2^{kn/p} \|Q_j g\|_{L_p(P_{k,\eta})} \leq c\|g\|_\infty \quad (\forall k, j \in \mathbb{Z}, \forall \eta \in \mathbb{Z}^n).$$

Hence

$$(4.31) \quad \|[g]\|_\infty \|\dot{B}_{p,\infty}^{0,\tau}\| \leq c\|g\|_\infty \quad \text{with } \tau := 1/p.$$

We now introduce two numbers b and β such that $b > n$ and $0 < \beta < \min(1, p)$. We have

$$\begin{aligned} |Q_j g(x)| &\leq \sum_{\nu \in \mathbb{Z}^n} \int_{P_{k,\nu}} 2^{jn} |\mathcal{F}^{-1} \gamma(2^j(x-y))| |g(y)| dy \\ &\leq c \sum_{\nu \in \mathbb{Z}^n} \int_{P_{k,\nu}} 2^{jn} (1 + 2^j|x-y|)^{-N-b} |g(y)| dy \quad (\text{where } N := n+1). \end{aligned}$$

For $j \geq k$, $x \in P_{k,\eta}$ and $y \in P_{k,\nu}$ we have

$$1 + |\eta - \nu| \leq 2\sqrt{2n}(2 + 2^k|x-y|) \leq 2\sqrt{2n}(2 + 2^j|x-y|).$$

Then Lemma 4.2(ii) yields

$$(4.32) \quad |Q_j g(x)| \leq c_1 2^{jn} \|g\|_\infty^{1-\beta} \sum_{\nu \in \mathbb{Z}^n} (1+|\eta-\nu|)^{-N} \int_{P_{k,\nu}} \frac{|g(y)|^\beta}{(1+2^j|x-y|)^b} dy \\ \leq c_2 \|g\|_\infty^{1-\beta} \sum_{\nu \in \mathbb{Z}^n} (1+|\eta-\nu|)^{-N} M(|g|^\beta \chi_{P_{k,\nu}}(2^{-j} \cdot))(2^j x)$$

for all $j \geq k$ and all $x \in P_{k,\eta}$, where $\chi_{P_{k,\nu}}$ is the characteristic function of $P_{k,\nu}$. We have $M(|g|^\beta \chi_{P_{k,\nu}}(2^{-j} \cdot))(2^j x) = M(|g|^\beta \chi_{P_{k,\nu}})(x)$ (an easy proof). Then, by Hölder's inequality with exponents $1/\beta$ and $1/(1-\beta)$, we find that $\|Q_j g\|_{L_p(P_{k,\eta})}$ is bounded by

$$(4.33) \quad c 2^{-kn(1-\beta)/p} \|g\|_\infty^{1-\beta} \left\| \sum_{\nu \in \mathbb{Z}^n} (1+|\eta-\nu|)^{-N} M(|g|^\beta \chi_{P_{k,\nu}}) \right\|_{L_{p/\beta}(P_{k,\eta})} \quad (j \geq k).$$

By using Minkowski's inequality in $L_{p/\beta}(P_{k,\eta}, \ell_1(\mathbb{Z}^n))$, and

$$\|M(|g|^\beta \chi_{P_{k,\nu}})\|_{L_{p/\beta}(P_{k,\eta})} \leq \|M(|g|^\beta \chi_{P_{k,\nu}})\|_{p/\beta},$$

and Lemma 4.2(i), we get

$$\|Q_j g\|_{L_p(P_{k,\eta})} \leq c_1 2^{-kn(1-\beta)/p} \|g\|_\infty^{1-\beta} \sum_{\nu \in \mathbb{Z}^n} (1+|\eta-\nu|)^{-N} \|g\|_{L_p(P_{k,\nu})}^\beta \\ \leq c_1 2^{kn(\beta/w-1/p)} \|g\|_\infty^{1-\beta} \|g\|_{\mathcal{M}_p^w}^\beta \sum_{\nu \in \mathbb{Z}^n} (1+|\eta-\nu|)^{-N} \quad (j \geq k).$$

We set $\tau := 1/p - \beta/w$. As $\sum_{\nu \in \mathbb{Z}^n} (1+|\eta-\nu|)^{-N} =: c < \infty$ for all $\eta \in \mathbb{Z}^n$, we obtain

$$2^{kn\tau} \|Q_j g\|_{L_p(P_{k,\eta})} \leq c \|g\|_\infty^{1-\beta} \|g\|_{\mathcal{M}_p^w}^\beta \quad (\forall k \in \mathbb{Z}, \forall j \geq k, \forall \eta \in \mathbb{Z}^n).$$

Hence

$$\|[g]_\infty\|_{\dot{B}_{p,\infty}^{0,\tau}} \leq c \|g\|_\infty^{1-\beta} \|g\|_{\mathcal{M}_p^w}^\beta.$$

Now, in this estimate together with (4.31), as $g = f^{(\alpha)}$, we apply (P5); then (3.4) follows. Finally, using the elementary inequality $a^{1-\beta} b^\beta \leq a + b$ ($a, b \geq 0$), we get $\|[f]_\infty\|_{\dot{B}_{p,\infty}^{m,\tau}} \leq c(\|f\|_{\dot{W}^m(L_\infty)} + \|f\|_{\dot{W}^m(\mathcal{M}_p^w)})$, which yields the continuity of the embedding.

SUBSTEP 2.2: F -spaces. As above, with the same notation, we arrive at (4.32), i.e.,

$$\sup_{j \geq k} |Q_j g(x)| \leq c_1 \|g\|_\infty^{1-\beta} \sum_{\nu \in \mathbb{Z}^n} (1+|\eta-\nu|)^{-N} M(|g \chi_{P_{k,\nu}}|^\beta)(x) \quad (\forall k \in \mathbb{Z}).$$

Thus, $\|\sup_{j \geq k} |Q_j g|\|_{L_p(P_{k,\eta})}$ is also bounded by (4.33), and the assertion follows exactly as in the preceding step; we omit the details. The proof is complete. ■

5. Conclusion. Further research could address the necessity of the conditions (3.1)–(3.2). For the moment, we only know that the assumptions of Theorem 3.1 are optimal in a certain sense; see Remark 3.3.

It is underlined in the Introduction that one of the aims of this study is the application of $\mathcal{E}_{p,q}^{s,\tau}$ in various domains. For this, the investigation on $\mathcal{E}_{p,q}^{s,\tau}$ can be continued, e.g. to get characterizations by approximation (see e.g. [22]) and by differences. In this direction, we have the following property:

PROPOSITION 5.1. *Let $m = 1, 2, \dots$. If $s - m > n/p - n\tau$, then*

$$\mathcal{E}_{p,q}^{s,\tau} = \{f \in L_\infty : f^{(\alpha)} \in \mathcal{E}_{p,q}^{s-m,\tau}, \forall |\alpha| = m\},$$

and $\|f\|_\infty + \sum_{|\alpha|=m} \|f^{(\alpha)}\|_{\mathcal{E}_{p,q}^{s-m,\tau}}$ is an equivalent quasi-norm in $\mathcal{E}_{p,q}^{s,\tau}$.

Proof. Without loss of generality we take $m = 1$. Assume first $f \in L_\infty$ is such that $\partial_\ell f \in \mathcal{E}_{p,q}^{s-1,\tau}$ ($\ell = 1, \dots, n$). By (P5) we have

$$\|f\|_{\mathcal{E}_{p,q}^{s,\tau}} \leq \|f\|_\infty + c \sum_{\ell=1}^n \|[\partial_\ell f]_\infty\|_{\dot{A}_{p,q}^{s-1,\tau}} \leq \|f\|_\infty + c \sum_{\ell=1}^n \|\partial_\ell f\|_{\mathcal{E}_{p,q}^{s-1,\tau}},$$

and we obtain the assertion in one direction. Next, we assume that $f \in \mathcal{E}_{p,q}^{s,\tau}$. Again, by (P5), it suffices to prove that there exists a constant $c > 0$ independent of f such that

$$\|\partial_\ell f\|_\infty \leq c \|f\|_{\mathcal{E}_{p,q}^{s,\tau}} \quad (\ell = 1, \dots, n).$$

Indeed, applying Proposition 2.5 we obtain $f \in B_{\infty,\infty}^{s+n\tau-n/p}$. By [25, Theorem 2.3.8] we have

$$\|\partial_\ell f\|_{B_{\infty,\infty}^{s+n\tau-1-n/p}} \leq c_1 \|f\|_{B_{\infty,\infty}^{s+n\tau-n/p}} \leq c_2 \|f\|_{\mathcal{E}_{p,q}^{s,\tau}},$$

and we then use the embedding $B_{\infty,\infty}^{s+n\tau-1-n/p} \hookrightarrow C_{ub}$. The desired result follows. ■

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