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On automorphisms of some semidirect product groups and ranks of Iwasawa modules

by

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Abstract. Let p be an odd prime number and k an imaginary quadratic field in which p does not split. Based on some heuristic, Kundu and Washington asked whether the λ- and μ-invariants of the anti-cyclotomic $\mathbb{Z}_p$-extension k_∞^a of k are always trivial. Also, if k_∞^a/k is totally ramified, for $n \geq 1$, they showed that the p-part of the ideal class group of the nth layer of the anti-cyclotomic $\mathbb{Z}_p$-extension of k is not cyclic. Inspired by their paper, we study anti-cyclotomic-like $\mathbb{Z}_p$-extensions, extending both the above question and Kundu–Washington's result. We show that the values of λ of certain anti-cyclotomic-like $\mathbb{Z}_p$-extensions are always even. We also show that the p-parts of the ideal class groups of certain anti-cyclotomic-like $\mathbb{Z}_p$-extensions of CM-fields are always non-cyclic.

1. Introduction. Let p be a fixed prime number. A Galois extension is called a $\mathbb{Z}_p$*-extension* if its Galois group is topologically isomorphic to the additive group of the ring of p-adic integers $\mathbb{Z}_p$. Let $k/\mathbb{Q}$ be a finite extension and K/k a $\mathbb{Z}_p$-extension. For each non-negative integer n, there is a unique intermediate field k_n, called the nth *layer*, of K/k such that $[k_n : k] = p^n$. For a finite extension $F/\mathbb{Q}$, let A_F denote the p-part of the ideal class group of F. For a $\mathbb{Z}_p$-extension K/k, let X_K denote the Galois group of the maximal unramified abelian pro-p extension L_K/K. The module X_K is also defined to be the projective limit $\varprojlim_n A_{k_n}$ with respect to norm maps. By Iwasawa's class number formula, there are non-negative integers λ, μ and an integer ν depending only on K/k such that $\# A_{k_n} = p^{\lambda n + \mu p^n + \nu}$ for all sufficiently large n. The integers λ and μ are structure invariants of X_K as a Galois module. In particular, it is known that $\lambda = \dim_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} X_K$, and that $\lambda = \mu = 0$ if and only if X_K is finite. Here we denote by $\mathbb{Q}_p$ the p-adic number field.

Suppose that p is odd. For each imaginary quadratic field k, there is a unique $\mathbb{Z}_p$-extension k_∞^a/k such that $k_\infty^a/\mathbb{Q}$ is a non-abelian Galois exten-

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sion. The extension k^a_∞/k is called the *anti-cyclotomic* $\mathbb{Z}_p$*-extension* of k. In [KW24], based on some heuristic, Kundu and Washington posed a question: Is $\lambda = \mu = 0$ always true for the anti-cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field k in which p does not split? For each non-negative integer n let k^a_n be the nth layer of k^a_∞/k. Kundu and Washington also showed the following result.

THEOREM A. *Let p be an odd prime number and k an imaginary quadratic field in which p does not split. Suppose that k^a_∞/k is totally ramified at the unique prime above p. If $A_k \neq 0$ then $A_{k^a_n}$ is not cyclic for $n \geq 1$.*

In the present article, by giving tentatively the following definition, we study more general settings than anti-cyclotomic $\mathbb{Z}_p$-extensions of imaginary quadratic fields.

DEFINITION. Let p be an odd prime number. Let $F/\mathbb{Q}$ be a finite extension and k/F a quadratic extension. A $\mathbb{Z}_p$-extension k^a_∞/k is *anti-cyclotomic-like* with respect to k/F if k^a_∞/F is a non-abelian Galois extension.

Let k and F be totally real fields such that k/F is a quadratic extension. If Leopoldt's conjecture holds true for p and k, then k has only the cyclotomic $\mathbb{Z}_p$-extension, and hence there are no anti-cyclotomic-like $\mathbb{Z}_p$-extensions with respect to k/F. Also, let k be a CM-field and k^+ the maximal totally real subfield of k. If $[k:\mathbb{Q}] > 2$ then there are infinitely many anti-cyclotomic-like $\mathbb{Z}_p$-extensions with respect to k/k^+.

Let k^a_∞/k be an anti-cyclotomic-like $\mathbb{Z}_p$-extension with respect to a quadratic extension k/F. Let J be the generator of $\mathrm{Gal}(k/F)$. Since k^a_∞/F is a non-abelian Galois extension, J acts on $\mathrm{Gal}(k^a_\infty/k)$ as the inverse. For each non-negative integer n, denote by k^a_n the nth layer of k^a_∞/k. The first result of this article concerns λ-invariants.

THEOREM 1. *Let p be an odd prime number, k/F a quadratic extension over a finite extension $F/\mathbb{Q}$ and k^a_∞/k an anti-cyclotomic-like $\mathbb{Z}_p$-extension with respect to k/F. Suppose that the prime p does not split in $k^a_\infty/\mathbb{Q}$. Then* $\lambda \equiv 0 \bmod 2$.

Theorem 1 asserts that odd positive integers, half of all positive integers, actually do not appear as λ if p does not split in $k^a_\infty/\mathbb{Q}$. We remark that the values of λ for cyclotomic $\mathbb{Z}_p$-extensions are often odd. We also remark that, when $p = 3$ or 5, there is an imaginary quadratic field k in which p splits such that λ of k^a_∞/k is p; see examples below in Theorem 2 and [F13, Theorem 4.2]. To prove Theorem 1, the argument of the proof of [CK81, Theorem 3], based on the structure theorem for $\mathbb{Z}_p[\![T]\!]$-modules, can be applied. In this article, we will prove Theorem 1 using a group-theoretical method which can be seen as a variant of [KW24, Lemma 6.6]; see Lemma 2 in Section 2. For CM-fields k, Caputo and Nuccio have clarified the meaning of the integer

$\lambda/2$ for anti-cyclotomic-like $\mathbb{Z}_p$-extensions with respect to k/k^+; see [CN23, Corollary 4.16].

The second result is a generalization of Theorem A for CM-fields.

THEOREM 2. *Let p be an odd prime number, k a CM-field and k^+ the maximal totally real subfield of k. Let k_∞^a/k be an anti-cyclotomic-like $\mathbb{Z}_p$-extension with respect to k/k^+ such that p does not split in $k_\infty^a/\mathbb{Q}$. Suppose that k contains no primitive pth roots of unity, $A_k \neq 0$ and $A_{k^+} = 0$. Then $A_{k_n^a}$ is not cyclic for $n \geq 1$.*

If $p = 3$ and $k = \mathbb{Q}(\sqrt{-3})$ then $A_k = 0$. If k is an imaginary quadratic field then $k^+ = \mathbb{Q}$, and hence $A_{k^+} = A_{\mathbb{Q}} = 0$. Thus we can say that Theorem 2 is a generalization of Theorem A for CM-fields. We will see that the proof of Theorem A can be applied almost directly to the proof of Theorem 2, we further consider the structure of the ideal class groups and the unit groups as $\mathrm{Gal}(k/k^+)$-modules.

In the rest of this section we give some notations and recall certain fundamental facts. For a finite cyclic group G_1 and a G_1-module M, let $\hat{H}^i(G_1, M)$ be the ith Tate cohomology group for $i = -1, 0$. For a group G and a G-module N, put

$$N^G = \{x \in N \mid gx = x \text{ for all } g \in G\}.$$

Let p be a prime number. For an algebraic extension $K/\mathbb{Q}$, let L_K/K be the maximal unramified abelian pro-p extension and put $X_K = \mathrm{Gal}(L_K/K)$. If K/F is a Galois extension, it follows that L_K/F is also a Galois extension.

Here, we briefly explain this fact. Let $\alpha \in L_K$ and $\alpha_1 = \alpha, \dots, \alpha_n$ be all conjugates of α over F. Since K/F is a Galois extension and $K(\alpha)/K$ is a finite unramified abelian p-extension, it follows that $L_\alpha = K(\alpha_1, \dots, \alpha_n)$ is also a finite unramified abelian p-extension of K, and L_α/F is a Galois extension. By the maximality of L_K, we have $L_\alpha \subseteq L_K$. Thus, since $L_K = \bigcup_{\alpha \in L_K} L_\alpha$, it follows that L_K/F is a Galois extension.

We then define an action of $\mathrm{Gal}(K/F)$ on X_K as follows. Let $g \in \mathrm{Gal}(K/F)$ and $x \in X_K$, and denote by $\tilde{g} \in \mathrm{Gal}(L_K/F)$ an extension of g. Then $\mathrm{Gal}(K/F)$ acts on X_K as the inner automorphism $g(x) = \tilde{g}x\tilde{g}^{-1}$.

When $K/\mathbb{Q}$ is a finite extension, by class field theory, the Artin map induces an isomorphism $A_K \simeq X_K$ of $\mathrm{Gal}(K/F)$-modules. Let $\langle J \rangle$ be a cyclic group of order 2. For an odd prime number p and a $\mathbb{Z}_p[\langle J \rangle]$-module M, put $M^- = \frac{1-J}{2}M$. Also, for a $\mathbb{Z}[\langle J \rangle]$-module M, put $M_p^- = (\mathbb{Z}_p \otimes_{\mathbb{Z}} M)^-$. It is known that the functor $M \mapsto M_p^-$ is exact. For a positive integer m, let μ_m be the group of all mth roots of unity.

2. Automorphisms of semidirect products of some pro-p abelian groups. In this section, let p be an odd prime number. Let $G_1 = \langle \tau \rangle$ be a cyclic group of order p generated by τ, and let A_1 be a cyclic group of order

p^{u+1} for some positive integer u. Suppose that G_1 acts on A_1 non-trivially. Let $G = A_1 \rtimes G_1$. Kundu and Washington proved the following.

LEMMA 1 ([KW24, Lemma 6.6]). *There is no automorphism ϕ of G such that $\phi(\tau) = y\tau^{-1}$ with $y \in A_1$.*

Kundu and Washington have used a presentation of G by 2×2 matrices. We give another brief proof.

Proof of Lemma 1. Suppose that such an automorphism ϕ of G exists. Let x be a generator of A_1. We may assume that $\tau x \tau^{-1} = x^{1+p^u}$. We remark that $(1+p^u)^r \equiv 1 + rp^u \bmod p^{u+1}$ for all $r \in \mathbb{Z}$. Put $y = x^a$ and $\phi(x) = x^b\tau^c$ for some $a, b, c \in \mathbb{Z}$. Since $\tau^p = 1$ and

$$\begin{aligned} 1 = \phi(\tau)^p = (x^a\tau^{-1})^p &= x^a\tau^{-1}x^a\tau\tau^{-2}x^a\tau^2 \cdots \tau^{-(p-1)}x^a\tau^{p-1}\tau^{-p} \\ &= x^a x^{(1+p^u)^{-1}a} x^{(1+p^u)^{-2}a} \cdots x^{(1+p^u)^{-(p-1)}a} = x^{a\sum_{t=0}^{p-1}(1-tp^u)} = x^{ap}, \end{aligned}$$

we have $ap \equiv 0 \bmod p^{u+1}$. Also, since $x^{p^u} \neq 1$ and

$$1 \neq \phi(x)^{p^u} = (x^b\tau^c)^{p^u} = x^{b\sum_{t=0}^{p^u-1}(1+ctp^u)} = x^{bp^u},$$

we have $b \not\equiv 0 \bmod p$. Then

$$\begin{aligned} \phi(\tau x \tau^{-1}) = \phi(\tau)\phi(x)\phi(\tau)^{-1} &= x^a\tau^{-1}x^b\tau^c\tau x^{-a} = x^a\tau^{-1}x^b\tau\tau^c x^{-a}\tau^{-c}\tau^c \\ &= x^a x^{(1+p^u)^{-1}b} x^{-(1+p^u)^c a}\tau^c = x^{(1-p^u)b+a-a(1+cp^u)}\tau^c \\ &= x^{(1-p^u)b-acp^u}\tau^c = x^{(1-p^u)b}\tau^c. \end{aligned}$$

On the other hand, we have

$$\phi(\tau x \tau^{-1}) = \phi(x^{1+p^u}) = (x^b\tau^c)^{1+p^u} = x^{b\sum_{t=0}^{p^u}(1+tcp^u)}\tau^{(1+p^u)c} = x^{(1+p^u)b}\tau^c.$$

Thus it follows that $(1-p^u)b \equiv (1+p^u)b \bmod p^{u+1}$. However, since p is an odd prime number and $b \not\equiv 0 \bmod p$, this congruence does not hold. ∎

Here, we give a variant of the above lemma for semidirect products of certain pro-p abelian groups. Let Γ be a pro-p group which is isomorphic to $\mathbb{Z}_p$ with a topological generator γ. Let X be a pro-p abelian group isomorphic to $\mathbb{Z}_p^r$ for some $r > 0$. Let $\{x_1, \ldots, x_r\}$ be a basis of X over $\mathbb{Z}_p$. Suppose that Γ acts on X and that $X^\Gamma = 0$. We shall write the action $\sigma \in \Gamma$ on $x \in X$ as $\sigma(x) \in X$. Put $\mathcal{G} = X \rtimes \Gamma$.

LEMMA 2. *Here we regard $\mu_{p-1} \subseteq \mathbb{Z}_p^\times$. Let $\zeta \in \mu_{p-1}$ and d be the order of ζ. If there is a topological automorphism ψ of $\mathcal{G}$ such that $\psi(X) = X$ and that $\psi(\gamma) = x\gamma^\zeta$ with $x \in X$, then $r \equiv 0 \bmod d$.*

Proof. If $\zeta = 1$ then $d = 1$, and hence there is nothing to do. Let $\zeta \neq 1$. Suppose that there is such an automorphism ψ. Let $(\alpha_{ij}), (\delta_{ij})$ be invertible $r \times r$ matrices with entries in $\mathbb{Z}_p$ such that $\gamma(x_j) = \gamma x_j \gamma^{-1} = \prod_{i=1}^r x_i^{\alpha_{ij}}$ and $\psi(x_j) = \prod_{i=1}^r x_i^{\delta_{ij}}$. Since Γ is a pro-p group, we can define

$(\alpha_{ij})^\zeta$. Let $\eta_1, \dots, \eta_t$ be all the distinct eigenvalues of (α_{ij}) with multiplicities $m_1, \dots, m_t$. Thus $m_1 + \cdots + m_t = r$. By our assumption that Γ is a pro-p group and $X^\Gamma = 0$, we have $\eta_i \equiv 1 \bmod \pi$ and $\eta_i \neq 1$ for all $1 \leq i \leq t$, where π denotes a prime element of $L = \mathbb{Q}_p(\eta_i \mid 1 \leq i \leq t)$. Indeed, let η be one of $\eta_1, \dots, \eta_t$ and let $v \in L \otimes_{\mathbb{Z}_p} X$. Suppose that v is an eigenvector of eigenvalue η with respect to the transformation γ, that is, $\gamma(v) = \eta v$. Since $\gamma^{-1}(v) = \eta^{-1} v$, η is a unit of L. Write $\eta = wu$ with a root of unity w of order prime to p and a principal unit u of L. Since $\gamma^{p^n} \to 1$ as $n \to \infty$, we have $(wu)^{p^n} \to 1$ as $n \to \infty$. It follows that $u^{p^n} \to 1$ as $n \to \infty$. Thus we also have $w^{p^n} \to 1$ as $n \to \infty$. Let f be the relative degree of $L/\mathbb{Q}_p$. By $p^f \equiv 1 \bmod p^f - 1$, we have $(p^f)^m = p^{mf} \equiv 1 \bmod p^f - 1$ for all $m \in \mathbb{Z}$. Also, it is known that $w^{p^f - 1} = 1$. In particular, $w^{p^{mf}} = w$ for all $m \in \mathbb{Z}$. Thus, if $w \neq 1$ then $1 \neq w = \lim_{m \to \infty} w^{p^{mf}} = 1$, a contradiction. Therefore $\eta = u$ is a principal unit of L, namely, $\eta \equiv 1 \bmod \pi$.

Moreover, all the distinct eigenvalues of $(\alpha_{ij})^\zeta$ are $\eta_1^\zeta, \dots, \eta_t^\zeta$ with multiplicities $m_1, \dots, m_t$. Indeed, we can easily see that $\eta_1^\zeta, \dots, \eta_t^\zeta$ are all the eigenvalues of $(\alpha_{ij})^\zeta$. If $\eta_i^\zeta = \eta_j^\zeta$, then $\eta_i = (\eta_i^\zeta)^{\zeta^{-1}} = (\eta_j^\zeta)^{\zeta^{-1}} = \eta_j$. Thus elements $\eta_1^\zeta, \dots, \eta_t^\zeta$ are all the distinct eigenvalues of $(\alpha_{ij})^\zeta$ with multiplicities $m_1, \dots, m_t$. Let $(\beta_{ij}) = (\alpha_{ij})^\zeta$. Let j be an integer with $1 \leq j \leq r$. Since $\gamma^\zeta(x_j) = \gamma^\zeta x_j \gamma^{-\zeta} = \prod_{i=1}^r x_i^{\beta_{ij}}$, it follows that

$$\begin{aligned}\psi(\gamma x_j \gamma^{-1}) &= \psi(\gamma)\psi(x_j)\psi(\gamma)^{-1} = (x\gamma^\zeta)\Big(\prod_{k=1}^r x_k^{\delta_{kj}}\Big)(\gamma^{-\zeta}x^{-1}) \\ &= x\Big(\prod_{k=1}^r \gamma^\zeta x_k^{\delta_{kj}} \gamma^{-\zeta}\Big)x^{-1} = x\Big(\prod_{k=1}^r \prod_{i=1}^r x_i^{\beta_{ik}\delta_{kj}}\Big)x^{-1} = \prod_{i=1}^r x_i^{\sum_{i=1}^r \beta_{ik}\delta_{kj}}.\end{aligned}$$

On the other hand, it also follows that

$$\begin{aligned}\psi(\gamma x_j \gamma^{-1}) &= \psi\Big(\prod_{k=1}^r x_k^{\alpha_{kj}}\Big) = \prod_{k=1}^r \psi(x_k)^{\alpha_{kj}} \\ &= \prod_{k=1}^r \prod_{i=1}^r x_i^{\delta_{ik}\alpha_{kj}} = \prod_{i=1}^r x_i^{\sum_{k=1}^r \delta_{ik}\alpha_{kj}}.\end{aligned}$$

Thus $(\alpha_{ij})^\zeta(\delta_{ij}) = (\delta_{ij})(\alpha_{ij})$, and hence

$$\prod_{i=1}^t (T - \eta_i^\zeta)^{m_i} = \prod_{i=1}^t (T - \eta_i)^{m_i}$$

(here we denote by T a variable), because the multiplicity of η_i^ζ as an eigenvalue of the matrix $(\alpha_{ij})^\zeta$ is m_i as stated above. The correspondence $\eta_i \mapsto \eta_i^\zeta$ defines an action of $\langle \zeta \rangle$ on $\{\eta_1, \dots, \eta_t\}$ without fixed points. Indeed, suppose that $\eta_i = \eta_i^{\zeta^c}$ for some i with $1 \leq i \leq t$ and an integer c. We remark here that $\eta_i \neq 1$. If the order of η_i is finite then it is a power of p, hence $\zeta^c \equiv 1 \bmod p$. Since $\zeta \in \mu_{p-1}$, we have $\zeta^c = 1$. If the order of η_i is infinite then $\zeta^c = 1$. Thus $c \equiv 0 \bmod d$ in both cases. Hence, if necessary, by rearranging, we can see that all the distinct eigenvalues of (α_{ij}) can be represented as

$$\eta_1, \ldots, \eta_s, \eta_1^{\zeta}, \ldots, \eta_s^{\zeta}, \ldots, \eta_1^{\zeta^{d-1}}, \ldots, \eta_s^{\zeta^{d-1}}$$

with multiplicities

$$m_1, \ldots, m_s, m_1, \ldots, m_s, \ldots, m_1, \ldots, m_s,$$

for some s. Therefore, $r = d(m_1 + \cdots + m_s)$. ■

3. Lemmas.

In this section, we give some lemmas.

Lemma 3. *Let p be a prime number and $F/\mathbb{Q}$ a finite extension. Let K/F be a Galois extension. Assume that there is at least one prime $\mathfrak{l}$ of k such that K/F is totally ramified at $\mathfrak{l}$.*

(1) *Let M/K be an unramified abelian extension such that M/F is a Galois extension. Then $\mathrm{Gal}(M/F) \simeq \mathrm{Gal}(M/K) \rtimes \mathrm{Gal}(K/F)$.*

(2) *Suppose that $\mathrm{Gal}(K/F)$ is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}/p^r\mathbb{Z}$ for some positive integer r. Let τ be a topological generator of $\mathrm{Gal}(K/F)$. If K/F is ramified at only the prime $\mathfrak{l}$, then $X_K/(\tau - 1)X_K \simeq X_F$.*

Proof. (1) Let $\mathcal{T}$ be the inertia subgroup of a prime of M above $\mathfrak{l}$ in $\mathrm{Gal}(M/F)$. Since K/F is totally ramified at $\mathfrak{l}$ and since $\mathcal{T} \cap \mathrm{Gal}(M/K) = 1$, it follows that

$$\mathcal{T} \simeq \mathrm{Gal}(M/K)\mathcal{T}/\mathrm{Gal}(M/K) \simeq \mathrm{Gal}(K/F).$$

Hence $\mathrm{Gal}(M/F) = \mathrm{Gal}(M/K)\mathcal{T}$. For $x, x' \in \mathrm{Gal}(M/K)$ and $t, t' \in \mathcal{T}$, assume that $xt = x't'$. Since ${x'}^{-1}x = t't^{-1} \in \mathrm{Gal}(M/K) \cap \mathcal{T} = 1$, we have $x = x'$ and $t = t'$. This shows that

$$\mathrm{Gal}(M/F) = \mathrm{Gal}(M/K)\mathcal{T} \simeq \mathrm{Gal}(M/K) \rtimes \mathrm{Gal}(K/F).$$

(2) Let L be the maximal intermediate field of L_K/F such that L/F is abelian. Since the extensions L_F/F and K/F are abelian, it follows that $L_F K \subseteq L$. Let T be the inertia subgroup at $\mathfrak{l}$ in $\mathrm{Gal}(L/F)$. Then the fixed field of T in L is L_F. Thus $\mathrm{Gal}(L/L_F K) = T \cap \mathrm{Gal}(L/K) = 1$ since L/K is unramified, and hence $L = L_F K$. One can easily see that $\mathrm{Gal}(L_K/L) = (\tau - 1)X_K$. Therefore,

$$X_F \simeq \mathrm{Gal}(L_F K/K) = \mathrm{Gal}(L/K) \simeq X_K/(\tau - 1)X_K. \ ■$$

Let $F/\mathbb{Q}$ be a finite extension, k/F a quadratic field and J the generator of $\mathrm{Gal}(k/F)$. Let k_∞^a/k be an anti-cyclotomic-like $\mathbb{Z}_p$-extension with respect to k/F.

LEMMA 4. *If the prime number p does not split in $k_\infty^a/\mathbb{Q}$, then k_∞^a/k is totally ramified at the unique prime of k above p.*

Proof. Suppose that k_1^a/k is an unramified extension. By class field theory, the Artin map induces an isomorphism $A_k \simeq \mathrm{Gal}(L_k/k)$ as $\langle J\rangle$-modules. Since k_1^a is a subfield of L_k and J acts on $\mathrm{Gal}(k_1^a/k)$ as -1, there is a surjective map $A_k^- \to \mathrm{Gal}(k_1^a/k)$. By our assumption that the prime $\mathfrak{p}$ of k above p does not split in $k_\infty^a/\mathbb{Q}$, $\mathfrak{p}$ is inert in k_1^a/k. Also, $J(\mathfrak{p}) = \mathfrak{p}$. This implies that the Artin symbol of $\mathfrak{p}$ in k_1^a/k is trivial, and thus $\mathfrak{p}$ splits in k_1^a/k. This is a contradiction. Hence k_∞^a/k is totally ramified at the unique prime above p. ∎

4. Proof of Theorem 1. Until the end of this article, let p be an odd prime number. In this section we show a somewhat general result which yields the assertion of Theorem 1. Let $F/\mathbb{Q}$ be a finite extension k/F a finite Galois extension, and put $\Delta = \mathrm{Gal}(k/F)$. Let K/k be a $\mathbb{Z}_p$-extension such that K/F is a Galois extension and put $\Gamma = \mathrm{Gal}(K/k)$. Let γ be a topological generator of Γ. Then there is $\chi \in \mathrm{Hom}(\Delta, \mu_{p-1})$ such that $\delta_1\gamma\delta_1^{-1} = \gamma^{\chi(\delta)}$ for each $\delta \in \Delta$ and an extension $\delta_1 \in \mathrm{Gal}(K/F)$ of δ. Indeed, since $\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ and Δ is finite, there is $\chi \in \mathrm{Hom}(\Delta, \mu_{p-1})$ such that Δ acts on Γ as χ.

THEOREM 3. *Let the notations be as above. Let d be the order of a finite cyclic group $\chi(\Delta)$. Let s be the multiplicity of T in the characteristic polynomial, lying in $\mathbb{Z}_p[T]$, of the linear map $\gamma - 1$ on $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} X_K$. Then $\lambda \equiv s \bmod d$.*

Proof. We know that X_K is a $\mathrm{Gal}(K/F)$-module. Thus the $\mathbb{Z}_p$-torsion submodule $\mathrm{Tor}_{\mathbb{Z}_p}X_K$ of X_K is also a $\mathrm{Gal}(K/F)$-submodule. This shows that the fixed field of $\mathrm{Tor}_{\mathbb{Z}_p}X_K$ is a Galois extension over F. Since $\lambda = \dim_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} X_K$, we may assume that X_K is a free $\mathbb{Z}_p$-module of rank λ.

For each element $\delta \in \Delta$, we denote by $\tilde\delta \in \mathrm{Gal}(L_K/F)$ an extension of δ. Also, let $\tilde\gamma \in \mathrm{Gal}(L_K/k)$ be an extension of γ. Since Δ acts on Γ as χ, there is $z_\delta \in X_K$ such that $\tilde\delta\tilde\gamma\tilde\delta^{-1} = \tilde\gamma^{\chi(\delta)}z_\delta$. Also, each element $g \in \mathrm{Gal}(L_K/F)$ can be written as $g = y\tilde\gamma^a\tilde\delta$ for some $y \in X_K$, $a \in \mathbb{Z}_p$ and $\delta \in \Delta$. Let $x \in X_K^\Gamma$. Then

$$\begin{aligned}\tilde\gamma(g^{-1}xg)\tilde\gamma^{-1} &= \tilde\gamma(\tilde\delta^{-1}\tilde\gamma^{-a}y^{-1}xy\tilde\gamma^a\tilde\delta)\tilde\gamma^{-1} = \tilde\delta^{-1}\tilde\delta\tilde\gamma\tilde\delta^{-1}\tilde\gamma^{-a}x\tilde\gamma^a\tilde\delta\tilde\gamma^{-1}\tilde\delta^{-1}\tilde\delta \\ &= \tilde\delta^{-1}\tilde\gamma^{\chi(\delta)}z_\delta x z_\delta^{-1}\tilde\gamma^{-\chi(\delta)}\tilde\delta = \tilde\delta^{-1}\tilde\gamma^{\chi(\delta)}x\tilde\gamma^{-\chi(\delta)}\tilde\delta \\ &= \tilde\delta^{-1}x\tilde\delta = \tilde\delta^{-1}\tilde\gamma^{-a}y^{-1}xy\tilde\gamma^a\tilde\delta = g^{-1}xg,\end{aligned}$$

and hence X_K^Γ is a closed normal subgroup of $\mathrm{Gal}(L_K/F)$. Suppose that $s > 0$. By the theory of Jordan normal forms, the multiplicity of T in the characteristic polynomial of the linear map $\gamma - 1$ on $X_K/X_K^\Gamma \simeq (\gamma - 1)X_K \subseteq X_K$ is less than s. Thus, by repeating the same argument, we can find an intermediate field M of L_K/K such that M/F is a Galois extension with the property that $\mathrm{Gal}(M/K)$ is a free $\mathbb{Z}_p$-module, and that the characteristic polynomial of $\gamma-1$ on $\mathrm{Gal}(L_K/M)$ is T^s. Put $X=\mathrm{Gal}(M/K)$. Then $X^{\varGamma}=0$ and $X\simeq\mathbb{Z}_p^{\lambda-s}$. By Lemma 3, we have $\mathrm{Gal}(M/k)\simeq X\rtimes\varGamma$. Let $\delta\in\varDelta$ be an element such that the order of $\zeta=\chi(\delta)$ is d. Let $\tilde{\delta}\in\mathrm{Gal}(M/F)$ be an extension of δ and ψ be an automorphism of $\mathrm{Gal}(M/k)$ defined by $\psi(g)=\tilde{\delta}g\tilde{\delta}^{-1}$. Since $\varDelta$ acts on $\varGamma$ as χ, there is $x\in X$ such that $\psi(\tilde{\gamma})=x\tilde{\gamma}^{\chi(\delta)}=x\tilde{\gamma}^{\zeta}$. By Lemma 2, we have $\lambda-s\equiv 0 \bmod d$. ∎

We show Theorem 1. Let k be a quadratic extension of a finite extension $F/\mathbb{Q}$. Let k_∞^a/k be an anti-cyclotomic-like $\mathbb{Z}_p$-extension with respect to k/F. Since k_∞^a/k is totally ramified at the unique prime above p by Lemma 4, it follows that $A_k\simeq X_{k_\infty^a}/(\gamma-1)X_{k_\infty^a}$. From the exact sequence

$$0\to X_{k_\infty^a}^{\varGamma}\to X_{k_\infty^a}\xrightarrow{\gamma-1}X_{k_\infty^a}\to X_{k_\infty^a}/(\gamma-1)X_{k_\infty^a}\to 0,$$

we have $s=0$. The fields k, F and k_∞^a satisfy the conditions of Theorem 3 with $d=[k:F]=2$, and therefore we have $\lambda\equiv 0 \bmod 2$. ∎

5. Proof of Theorem 2. Let k be a CM-field and k^+ the totally real subfield of k. Let J be the generator of $\mathrm{Gal}(k/k^+)$. Let k_∞^a/k be an anti-cyclotomic-like $\mathbb{Z}_p$-extension with respect to k/k^+. We remark that k_∞^a/k is totally ramified at the unique prime above p. For each non-negative integer n, put $A_n=A_{k_n^a}$. Since norm maps $A_m\to A_n$ for each pair m and n of non-negative integers with $m\geq n$ are surjective, it suffices to show that A_1 is not cyclic. If A_0 is not cyclic then A_1 is not either. Suppose that A_0 is a non-trivial cyclic group. Suppose further that $A_{k^+}=0$. Then $A_0=A_0^-$. Put $G_1=\mathrm{Gal}(k_1^a/k)\simeq\mathbb{Z}/p\mathbb{Z}$. Following the method of the proof of [KW24, Theorem 6.1], we show here that $\#A_0<\#A_1$. Let τ be a generator of G_1. By Lemma 3,

$$\#A_0=\#A_1/(\tau-1)A_1=\#A_1^{G_1}.$$

Suppose that $\#A_0=\#A_1$. Then $A_1^{G_1}=A_1$, and hence $\mathrm{Gal}(k/k^+)=\langle J\rangle$ acts on A_1 canonically. The norm map $A_1^-\to A_0^-=A_0$ is surjective. Further, since

$$\#A_0=\#A_0^-\leq\#A_1^-\leq\#A_1=\#A_0,$$

we have $A_1^-=A_1$.

For $i=0$ or 1, let I_i, C_i and E_i be the ideal group, the ideal class group and the unit group of k_i^a. Let P_1 be the principal ideal group of k_1^a. From the exact sequence

$$0\to P_1\to I_1\to C_1\to 0,$$

we have an exact sequence

$$I_1^{G_1}\to C_1^{G_1}\to\hat{H}^{-1}(G_1,P_1)\to 0$$

of abelian groups. One also sees that

$$(I_1^{G_1})_p^- \to A_1^- \to \hat{H}^{-1}(G_1, P_1)_p^- \to 0$$

is exact since $\mathbb{Z}_p \otimes_{\mathbb{Z}} C_1^{G_1} = A_1^{G_1} = A_1$. From the exact sequence

$$0 \to E_1 \to (k_1^a)^\times \to P_1 \to 0,$$

we have an exact sequence

$$0 \to \hat{H}^{-1}(G_1, P_1) \to \hat{H}^0(G_1, E_1).$$

Let $\mu(k)$ be the group of all roots of unity of k, and let E_0^+ be the unit group of k^+. By Hasse's unit index, it is known that $[E_0 : \mu(k)E_0^+] = 1$ or 2 (see for example [W97, Theorem 4.12]). From the assumptions that p is odd and k contains no primitive pth roots of unity, it follows that

$$(E_0)_p^- = (\mu(k)E_0^+/E_0^+)_p^- = 0.$$

Since $\hat{H}^0(G_1, E_1)_p^-$ is a quotient of $(E_0)_p^-$, it follows that $\hat{H}^0(G_1, E_1)_p^- = 0$. Hence $\hat{H}^{-1}(G_1, P_1)_p^- = 0$, and then it turns out that $(I_1^{G_1})_p^- \to A_1^-$ is surjective. Let $\mathfrak{p}$ be the prime of k above p, and $\mathfrak{p}_1$ be the prime of k_1^a above $\mathfrak{p}$. Then $I_1^{G_1} = I_0\langle \mathfrak{p}_1 \rangle$. There is the following exact sequence:

$$0 \to I_0 \to I_1^{G_1} \to \langle \mathfrak{p}_1 \rangle I_0/I_0 \to 0.$$

Since $J(\mathfrak{p}) = \mathfrak{p}$, it follows that $J(\mathfrak{p}_1) = \mathfrak{p}_1$, and hence $(\langle \mathfrak{p}_1 \rangle I_0/I_0)_p^- = 0$. Thus $(I_1^{G_1})_p^- = (I_0)_p^-$. This shows that the lifting map $A_0 = A_0^- \to A_1^- = A_1$ is surjective. However, the composition of the norm map $A_1 \to A_0$ and the lifting map $A_0 \to A_1$ is the pth power map because $A_1 = A_1^{G_1}$, and thus the lifting map $A_0 \to A_1$ is never surjective. This is a contradiction. Therefore, we have $\#A_0 < \#A_1$. Hence G_1 acts on A_1 non-trivially.

By Lemma 3, we have $\mathrm{Gal}(L_{k_1^a}/k) = A_1 \rtimes G_1$. Let $\tilde{J} \in \mathrm{Gal}(L_{k_1^a}/k^+)$ be an extension of J. Let ϕ be an automorphism of $\mathrm{Gal}(L_{k_1^a}/k)$ defined by $\phi(g) = \tilde{J} g \tilde{J}^{-1}$ for $g \in \mathrm{Gal}(L_{k_1^a}/k)$. Since J acts as -1 on $\mathrm{Gal}(k_1^a/k)$, it follows that $\phi(\tau) \equiv \tau^{-1} \bmod A_1$, and hence there is $y \in A_1$ such that $\phi(\tau) = y\tau^{-1}$. By Lemma 1, such an automorphism ϕ does not exist if A_1 is cyclic. Therefore, A_1 is not cyclic. ∎

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