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LOCAL TIMES OF DETERMINISTIC PATHS WITH FINITE VARIATION

BY

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Abstract. We define the numbers of level crossings by a càdlàg (RCLL) real function $x\colon [0,+\infty) \to \mathbb{R}$ and, in analogy to the work of Bertoin and Yor (2014), we prove that for x with locally finite total variation, these numbers are densities of relevant occupation measures associated with x. Next, depending on the regularity of x and $f:\mathbb{R}\to\mathbb{R}$, we derive change of variable formulas, which may be seen as analogs of the Itô or Tanaka–Meyer formulas. Some of these formulas were already given by Bertoin and Yor (2014), but we also present some generalizations.

1. Introduction. The total variation of a function $x\colon [0,+\infty)\to\mathbb{R}$ on the time interval $[s,t]$, $0\le s<t<+\infty$, is defined by

$$\mathrm{TV}(x,[s,t]) := \sup_{\pi\in\Pi(s,t)} \sum_{[u,v]\in\pi} |x_v - x_u|, \tag{1.1}$$

where the supremum is taken over all finite partitions π of the interval $[s,t]$, that is, finite sets of nonoverlapping (with disjoint interiors) subintervals $[u,v]$ of $[s,t]$ such that $\bigcup_{[u,v]\in\pi}[u,v]=[s,t]$. The family of all such partitions is denoted by $\Pi(s,t)$.

x is said to have *locally finite total variation* if $\mathrm{TV}(x,[s,t])$ is finite for any s,t satisfying $0\le s<t<+\infty$. In [BY14], the authors introduce local times $\ell^z(t)$ for càdlàg functions with locally finite variation and state change of variable formulas which may be viewed as analogs of similar, fundamental formulas known in stochastic calculus (Itô formula or Itô–Tanaka–Meyer formula). The local times also allow one to generalize the Banach Indicatrix Theorem [Ban25, Vit26] to the case of càdlàg (RCLL) functions. Other generalizations of the Banach indicatrix may be found in [Loz48, Loz58, Łoc17].

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In this short note, we define the numbers of level (up-, down-) crossings by a càdlàg real function $x\colon [0,+\infty) \to \mathbb{R}$, $\mathrm{u}^z(x,[0,t])$, $\mathrm{d}^z(x,[0,t])$, and prove that they may also be viewed as densities of relevant occupation measures associated with x. Our approach seems to be more uniform and we obtain, among other results, a simpler (and more general) change of variable formula, namely

$$\begin{aligned} f(x_t) - f(x_0) &= \int_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z \\ &= \int_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z + \sum_{0<s\le t,\, \Delta x_s \ne 0} \{f(x_s) - f(x_{s-})\}, \end{aligned}$$

for any locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ with derivative g. In order to define the summation over the jumps of a càdlàg function, we apply the concept of summation over general sets. Let I be a set, let $b\colon I \to \mathbb{R}$ be a real valued function and let $\mathcal{I}$ be the family of all finite subsets of I. Since $\mathcal{I}$ is directed when ordered by inclusion, the summation over I can be defined by

$$\sum_{i\in I} b_i := \lim_{\Gamma\in\mathcal{I}} \sum_{i\in\Gamma} b_i \tag{1.2}$$

as the limit of a net, i.e., $\lim_{\Gamma\in\mathcal{I}} \sum_{i\in\Gamma} b_i =: l \in (-\infty,\infty)$ if, for any $\varepsilon > 0$, there is $\Gamma \in \mathcal{I}$ such that for all $\tilde{\Gamma} \in \mathcal{I}$ satisfying $\tilde{\Gamma} \supseteq \Gamma$, one has $\sum_{i\in\tilde{\Gamma}} b_i \in (l-\varepsilon, l+\varepsilon)$.

This paper is organized as follows. In Section 2, we introduce the necessary notations, definitions, and the main result of our paper. Section 3 provides an alternative proof to the main result. Finally, in Section 4, we present change of variable formulas.

2. Definitions, notation and the main result. We set $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $C([0,+\infty);\mathbb{R})$ the space of all continuous functions $x\colon [0,+\infty) \to \mathbb{R}$, and by $D([0,+\infty);\mathbb{R})$ the space of all càdlàg (RCLL) functions $x\colon [0,+\infty) \to \mathbb{R}$, that is, $x \in D([0,+\infty);\mathbb{R})$ if it is right-continuous at each $t \in [0,+\infty)$ and has a left limit at each $t \in (0,+\infty)$. Moreover, let $V = V([0,+\infty);\mathbb{R})$ be the subset of $D = D([0,+\infty);\mathbb{R})$ of functions with locally finite total variation.

Let $y \in \mathbb{R}$ and $c > 0$. In what follows, we introduce the *numbers of (up-, down-) crossings* of the interval $[y-c/2, y+c/2]$ by $x \in D([0,+\infty);\mathbb{R})$ during the time interval $[s,t]$. We define $\sigma_0^c := s$ and for $n \in \mathbb{N}_0$, we set

$$\tau_n^c := \inf\{u \in [\sigma_n^c, t] : x_u \ge y + c/2\},$$
$$\sigma_{n+1}^c := \inf\{u \in [\tau_n^c, t] : x_u < y - c/2\},$$

where we apply the conventions $\inf\emptyset = +\infty$, $[+\infty, t] = \emptyset$.

In Figure 1, we present the graph of some $x \in D([0,+\infty);\mathbb{R})$ and illustrate the times τ_n, σ_n, $n = 0, 1, 2$, for $y = 0$, $c = 0.05$ and the time interval $[s,t] = [0,10]$ by marking the points (τ_0, x_{τ_0}), (τ_1, x_{τ_1}) and (τ_2, x_{τ_2}) with squares (□) and the points (σ_1, x_{σ_1}) and (σ_2, x_{σ_2}) with bullets (•).

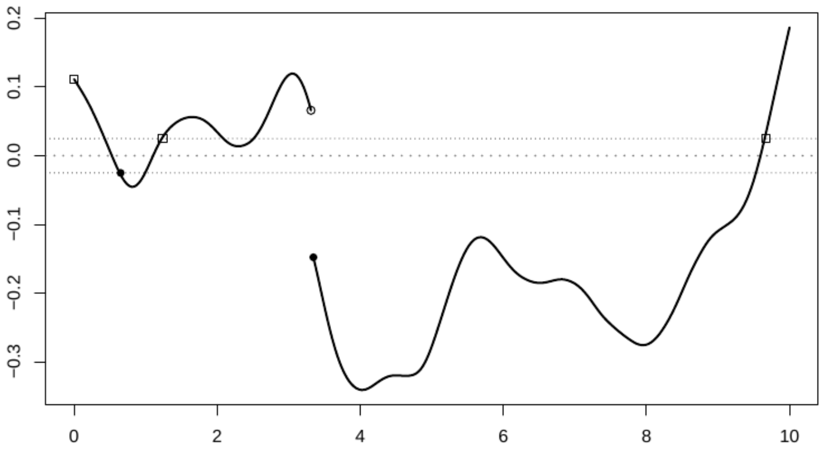

Fig. 1. Points corresponding to times τ_0, τ_1, τ_2 (□) and σ_1, σ_2 (•)

Definition 2.1. The *number of down-crossings* of $[y-c/2, y+c/2]$ by x during the time interval $[s,t]$ is defined by

$$\mathrm{d}^{y,c}(x,[s,t]) := \max\{n : \sigma_n^c \le t\}.$$

As an illustration, we can see that for the function x of Figure 1, there are *two* down-crossings of $[-0.025, 0.025]$ by x during the time interval $[0,10]$.

Definition 2.2. The *number of up-crossings* of $[y-c/2, y+c/2]$ by x during the time interval $[s,t]$ is defined by

$$\mathrm{u}^{y,c}(x,[s,t]) := \mathrm{d}^{-y,c}(-x,[s,t]).$$

Definition 2.3. The *number of crossings* of $[y-c/2, y+c/2]$ by x during the time interval $[s,t]$ is defined by

$$\mathrm{n}^{y,c}(x,[s,t]) := \mathrm{u}^{y,c}(x,[s,t]) + \mathrm{d}^{y,c}(x,[s,t]).$$

Remark 2.4. If $y + c/2$ is not a local maximum of x, then $\mathrm{d}^{y,c}(x,[s,t])$ may also be defined by

$$\mathrm{d}^{y,c}(x,[s,t]) := \sup_{n\in\mathbb{N}} \sup_{s\le s_1<t_1<\ldots<s_n<t_n\le t;\, s_1,\ldots,s_n,t_1,\ldots,t_n\in\mathbb{Q}} \sum_{i=1}^{n} \mathrm{D}^{y,c}(x_{s_i}, x_{t_i}),$$

where

$$\mathrm{D}^{y,c}(a,b) := \begin{cases} 1 & \text{if } a \ge y + c/2 \text{ and } b < y - c/2, \\ 0 & \text{otherwise.} \end{cases}$$

The numbers $\mathrm{u}^{y,c}(x,[s,t])$ of up-crossings may be defined analogously.

From Remark 2.4, the measurability of $y \mapsto \mathrm{d}^{y,c}(x,[s,t])$, $y \mapsto \mathrm{u}^{y,c}(x,[s,t])$ and $y \mapsto \mathrm{n}^{y,c}(x,[s,t])$ follows immediately.

Using the numbers of interval (down-, up-) crossings we define the numbers of *level* (*down-, up-*) *crossings.*

DEFINITION 2.5. Let $y \in \mathbb{R}$. The number of times that the function x *downcrosses* the level y during the time interval $[s,t]$ is defined by

$$\mathrm{d}^{y}(x,[s,t]) = \lim_{c\to 0+} \mathrm{d}^{y,c}(x,[s,t]) \in \mathbb{N}_0 \cup \{+\infty\}.$$

Analogously, we define the number of *upcrosses* by

$$\mathrm{u}^{y}(x,[s,t]) = \lim_{c\to 0+} \mathrm{u}^{y,c}(x,[s,t]) \in \mathbb{N}_0 \cup \{+\infty\}.$$

Finally, we define the number of times the function x *crosses* the level y during $[s,t]$ by

$$\mathrm{n}^{y}(x,[s,t]) = \lim_{c\to 0+} \mathrm{n}^{y,c}(x,[s,t]) \in \mathbb{N}_0 \cup \{+\infty\}.$$

Next, let us also define the numbers of *level* (*down-, up-*) *crossings via jumps.* For $s \in (0,+\infty)$, Δx_s denotes the jump of x at time s, $\Delta x_s = x_s - x_{s-}$.

DEFINITION 2.6. Let $y \in \mathbb{R}$. The number of times the function x *downcrosses* the level y during the time interval $[s,t]$ *via jumps* is defined by

$$\Delta\mathrm{d}^{y}(x,[s,t]) = \mathrm{Card}\,\{u \in (s,t] : \Delta x_u < 0 \text{ and } y \in (x_u, x_{u-})\}.$$

Analogously we define the number of *upcrosses via jumps* by

$$\Delta\mathrm{u}^{y}(x,[s,t]) = \mathrm{Card}\,\{u \in (s,t] : \Delta x_u > 0 \text{ and } y \in (x_{u-}, x_u)\}.$$

Finally, we define the number of times the function x *crosses* the level y *via jumps* during $[s,t]$ by

$$\Delta\mathrm{n}^{y}(x,[s,t]) = \Delta\mathrm{d}^{y}(x,[s,t]) + \Delta\mathrm{u}^{y}(x,[s,t]).$$

As an illustration, for the function x of Figure 1, there is *one* downcrossings of the level 0 by x via jump during the time interval $[0,10]$.

REMARK 2.7. The numbers $\Delta\mathrm{d}^{y}(x,[s,t])$ may also be defined by

$$\Delta\mathrm{d}^{y}(x,[s,t]) := \sup_{c\in\mathbb{Q},\, c>0}\ \inf_{m\in\mathbb{N}}\ \sup_{n\in\mathbb{N}}\ \sup_{s\le s_1<t_1<\cdots<s_n<t_n\le t;\, s_i,t_i\in\mathbb{Q},\, t_i-s_i\le 1/m, i=1,\ldots,n}\ \sum_{i=1}^{n} \mathrm{D}^{y,c}(x_{s_i}, x_{t_i}),$$

where the numbers $\mathrm{D}^{y,c}(a,b)$ were defined in Remark 2.4.

The numbers $\Delta\mathrm{u}^{y}(x,[s,t])$ of upcrossings may be defined analogously.

From Remark 2.7, the measurability of $y \mapsto \Delta\mathrm{d}^{y}(x,[s,t])$, $y \mapsto \Delta\mathrm{u}^{y}(x,[s,t])$ and $y \mapsto \Delta\mathrm{n}^{y}(x,[s,t])$ follows immediately.

One of the main goals of this note is to prove a relationship between the numbers of level crossings just defined and the occupation measures associated with x and with the continuous part of x which we will denote by x^{cont} and which is defined by the relation

$$x(t) = x^{\mathrm{cont}}(t) + \sum_{0<s\le t} \Delta x_s, \qquad t \in [0, +\infty).$$

Let us now define two companion quantities of the total variation: *positive* and *negative total variation*, also called *upward* and *downward total variation.* They are defined respectively by

$$\mathrm{UTV}(x, [s, t]) := \sup_{\pi\in\Pi(s,t)} \sum_{[u,v]\in\pi} (x_v - x_u)_+$$

and

$$\mathrm{DTV}(x, [s, t]) := \sup_{\pi\in\Pi(s,t)} \sum_{[u,v]\in\pi} (x_v - x_u)_-,$$

where for a real y, $(y)_+ := \max(0, y)$, $(y)_- := \max(0, -y)$.

Let $|\mu|$ be the unique measure on $[0, +\infty)$ which assigns to the set $\{0\}$ the value 0 and to the interval $(a, b] \subset [0, +\infty)$ with $0 \le a < b < +\infty$ the value

$$|\mu|(a, b] := \mathrm{TV}(x, [0, b]) - \mathrm{TV}(x, [0, a]) = \mathrm{TV}(x, [a, b]).$$

In what follows, we will denote the Lebesgue integrals $\int_A h \,\mathrm{d}|\mu|$ by

$$\int_A h_s \,|\mathrm{d}x_s|.$$

Two other associated measures are μ_+ and μ_- which are defined as the unique measures on $[0, +\infty)$ which assign to the set $\{0\}$ the value 0 and to the interval $(a, b] \subset [0, +\infty)$ with $0 \le a < b < +\infty$ the values

$$\mu_+(a, b] := \mathrm{UTV}(x, [0, b]) - \mathrm{UTV}(x, [0, a]) = \mathrm{UTV}(x, [a, b]),$$
$$\mu_-(a, b] := \mathrm{DTV}(x, [0, b]) - \mathrm{DTV}(x, [0, a]) = \mathrm{DTV}(x, [a, b]).$$

We will denote the Lebesgue integrals $\int_A h \,\mathrm{d}\mu_+$ and $\int_A h \,\mathrm{d}\mu_-$ by

$$\int_A h_s \,(\mathrm{d}x_s)_+ \quad \text{and} \quad \int_A h_s \,(\mathrm{d}x_s)_-$$

respectively.

Similarly to the integrals $\int_A h_s \,|\mathrm{d}x_s|$, $\int_A h_s \,(\mathrm{d}x_s)_+$ and $\int_A h_s \,(\mathrm{d}x_s)_-$, one defines the integrals $\int_A h_s \,|\mathrm{d}x_s^{\mathrm{cont}}|$, $\int_A h_s \,(\mathrm{d}x_s)_+$ and $\int_A h_s \,(\mathrm{d}x_s)_-$.

Now we are ready to state the following.

THEOREM 2.8. *For any function $x \in V$, if $t > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is Borel-measurable and locally bounded, then*

$$\int_{\mathbb{R}} \mathrm{u}^z(x,[0,t])g(z)\,\mathrm{d}z \tag{2.1}$$
$$= \int_{(0,t]} g(x_{s-})\,(\mathrm{d}x_s)_+ + \sum_{0<s\le t,\,\Delta x_s>0} \int_{x_{s-}}^{x_s} \{g(z)-g(x_{s-})\}\,\mathrm{d}z$$
$$= \int_{(0,t]} g(x_{s-})\,(\mathrm{d}x_s^{\mathrm{cont}})_+ + \sum_{0<s\le t,\,\Delta x_s>0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z$$
$$= \int_{(0,t]} g(x_{s})\,(\mathrm{d}x_s^{\mathrm{cont}})_+ + \sum_{0<s\le t,\,\Delta x_s>0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z$$

and

$$\int_{\mathbb{R}} \mathrm{d}^z(x,[0,t])g(z)\,\mathrm{d}z \tag{2.2}$$
$$= \int_{(0,t]} g(x_{s-})\,(\mathrm{d}x_s)_- + \sum_{0<s\le t,\,\Delta x_s<0} \int_{x_{s}}^{x_{s-}} \{g(z)-g(x_{s-})\}\,\mathrm{d}z$$
$$= \int_{(0,t]} g(x_{s-})\,(\mathrm{d}x_s^{\mathrm{cont}})_- + \sum_{0<s\le t,\,\Delta x_s<0} \int_{x_{s}}^{x_{s-}} g(z)\,\mathrm{d}z$$
$$= \int_{(0,t]} g(x_{s})\,(\mathrm{d}x_s^{\mathrm{cont}})_- + \sum_{0<s\le t,\,\Delta x_s<0} \int_{x_{s}}^{x_{s-}} g(z)\,\mathrm{d}z.$$

REMARK 2.9. Companion formulas read

$$\int_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t])-\mathrm{d}^z(x,[0,t])\}g(z)\,\mathrm{d}z \tag{2.3}$$
$$= \int_{(0,t]} g(x_{s-})\,\mathrm{d}x_s + \sum_{0<s\le t,\,\Delta x_s\neq 0} \int_{x_{s-}}^{x_s} \{g(z)-g(x_{s-})\}\,\mathrm{d}z$$
$$= \int_{(0,t]} g(x_{s-})\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{0<s\le t,\,\Delta x_s\neq 0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z$$
$$= \int_{(0,t]} g(x_{s})\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{0<s\le t,\,\Delta x_s\neq 0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z$$

and

$$\int_{\mathbb{R}} \mathrm{n}^z(x,[0,t])g(z)\,\mathrm{d}z \tag{2.4}$$
$$= \int_{(0,t]} g(x_{s-})\,|\mathrm{d}x_s| + \sum_{0<s\le t,\,\Delta x_s\neq 0} \int_{x_{s-}\wedge x_s}^{x_s\vee x_s} \{g(z)-g(x_{s-})\}\,\mathrm{d}z$$

$$= \int\limits_{(0,t]} g(x_{s-})\,|\mathrm{d}x_s^{\mathrm{cont}}| + \sum_{0<s\le t,\,\Delta x_s\neq 0} \int\limits_{x_{s-}\wedge x_s}^{x_s\vee x_s} g(z)\,\mathrm{d}z$$

$$= \int\limits_{(0,t]} g(x_s)\,|\mathrm{d}x_s^{\mathrm{cont}}| + \sum_{0<s\le t,\,\Delta x_s\neq 0} \int\limits_{x_{s-}\wedge x_s}^{x_s\vee x_s} g(z)\,\mathrm{d}z.$$

Theorem 2.8 is an almost immediate consequence of [BY14, Theorem 1]. In [BY14], for a real z, the authors introduce positive times at which the function x increases continuously through the level z and times at which x decreases continuously through that level. They denote the sets of the former times by $\mathcal{I}(z)$ and of the latter by $\mathcal{D}(z)$. They are defined in the following way:

- $t \in \mathcal{I}(z)$ iff (i) x is continuous at time t and $x(t) = z$ and (ii) the sign of $x(t) - x(s)$ is the same as the sign of $t - s$ for all s in some neighborhood of t.
- $t \in \mathcal{D}(z)$ iff (i) x is continuous at time t and $x(t) = z$ and (ii) the sign of $x(t) - x(s)$ is the same as the sign of $s - t$ for all s in some neighborhood of t.

For $t > 0$ and $z \in \mathbb{R}$, Bertoin and Yor also introduce the number $n^z(t)$ (known as Banach's indicatrix) defined by

$$n^z(t) := \mathrm{Card}\,\{s \in [0,t] : x(s) = z\}.$$

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. For $z \in \mathbb{R}$ and $t > 0$, set

$$r^z(t) := n^z(t) - \mathrm{Card}(\mathcal{I}(z) \cap (0,t]) - \mathrm{Card}(\mathcal{D}(z) \cap (0,t]) \ge 0.$$

It is not difficult to see that for $z \in \mathbb{R}$ and $c, t > 0$ we have

$$\mathrm{Card}(\mathcal{I}(z) \cap (0,t]) + \Delta \mathrm{u}^z(x,[0,t]) \le \mathrm{u}^z(x,[s,t])$$

and

$$\mathrm{u}^{z,c}(x,[s,t]) \le \mathrm{Card}(\mathcal{I}(z) \cap (0,t]) + \Delta \mathrm{u}^z(x,[0,t]) + r^z(t).$$

Letting $c \to 0+$, we obtain

$$\begin{aligned}\mathrm{Card}(\mathcal{I}(z) \cap (0,t]) + \Delta \mathrm{u}^z(x,[0,t]) &\le \mathrm{u}^z(x,[s,t]) \\ &\le \mathrm{Card}(\mathcal{I}(z) \cap (0,t]) + \Delta \mathrm{u}^z(x,[0,t]) + r^z(t),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathrm{Card}(\mathcal{I}(z) \cap (0,t]) &\le \mathrm{u}^y(x,[s,t]) - \Delta \mathrm{u}^z(x,[0,t]) \\ &\le \mathrm{Card}(\mathcal{I}(z) \cap (0,t]) + r^z(t).\end{aligned} \tag{2.5}$$

By [BY14, Theorem 1], we have

$$\int_{(0,t]} g(x_s)\,(\mathrm{d}x_s^{\mathrm{cont}})_+ = \frac{1}{2}\int_{(0,t]} g(x_s)\,(\mathrm{d}x_s^{\mathrm{cont}} + |\mathrm{d}x_s^{\mathrm{cont}}|) \tag{2.6}$$
$$= \frac{1}{2}\int_{\mathbb{R}} g(z)\{\ell^z(t)+\lambda^z(t)\}\,\mathrm{d}z = \int_{\mathbb{R}} g(z)\,\mathrm{Card}(\mathcal{I}(z)\cap(0,t])\,\mathrm{d}z,$$

where

$$\ell^z(t) = \mathrm{Card}(\mathcal{I}(z)\cap(0,t]) - \mathrm{Card}(\mathcal{D}(z)\cap(0,t]),$$
$$\lambda^z(t) = \mathrm{Card}(\mathcal{I}(z)\cap(0,t]) + \mathrm{Card}(\mathcal{D}(z)\cap(0,t]).$$

Next, we observe that

$$\int_{\mathbb{R}} g(z)\{\mathrm{u}^z(x,[0,t]) - \Delta\mathrm{u}^z(x,[0,t])\}\,\mathrm{d}z \tag{2.7}$$
$$= \int_{\mathbb{R}} g(z)\mathrm{u}^z(x,[0,t])\,\mathrm{d}z - \sum_{0<s\le t,\,\Delta x_s>0}\int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z.$$

(All terms on the right side of (2.7) are well defined since x has locally finite variation and g is locally bounded.) The last ingredient we need is the equality

$$\int_{\mathbb{R}} g(z)r^z(t)\,\mathrm{d}z = 0, \tag{2.8}$$

which also follows from [BY14, Theorem 1], namely from the equality of the measures $n^z(t)\,\mathrm{d}z$ and $\lambda^z(t)\,\mathrm{d}z$. From (2.8) and (2.6), we obtain

$$\int_{\mathbb{R}} g(z)\{\mathrm{Card}(\mathcal{I}(z)\cap(0,t]) + r^z(t)\}\,\mathrm{d}z = \int_{(0,t]} g(x_s)\,(\mathrm{d}x_s^{\mathrm{cont}})_+.$$

This together with (2.5)–(2.7) also gives

$$\int_{\mathbb{R}} g(z)\mathrm{u}^z(x,[0,t])\,\mathrm{d}z - \sum_{0<s\le t,\,\Delta x_s>0}\int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z = \int_{(0,t]} g(x_s)\,(\mathrm{d}x_s^{\mathrm{cont}})_+.$$

From the continuity of x^{cont}, we further obtain

$$\int_{(0,t]} g(x_s)\,(\mathrm{d}x_s^{\mathrm{cont}})_+ = \int_{(0,t]} g(x_{s-})\,(\mathrm{d}x_s^{\mathrm{cont}})_+$$
$$= \int_{(0,t]} g(x_{s-})\,(\mathrm{d}x_s)_+ - \sum_{0<s\le t,\,\Delta x_s>0}\int_{x_{s-}}^{x_s} g(x_{s-})\,\mathrm{d}z.$$

The last two displays give (2.1).

The proof of (2.2) is similar. ∎

3. Alternative proof of Theorem 2.8. In the proof of Theorem 2.8, we used the main result of [BY14], that is, [BY14, Theorem 1]. To prove their theorem, Bertoin and Yor use change of variable formulas which are analogs of the Itô and Tanaka–Meyer formulas from stochastic calculus, namely, if $f \in C^1(\mathbb{R}, \mathbb{R})$ then for $t > 0$,

$$f(x_t) - f(x_0) = \int_0^t f'(x_s)\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{0<s\le t,\, \Delta x_s>0} (f(x_s) - f(x_{s-})), \tag{3.1}$$

$$f(x_t) - f(x_0) = \int_{\mathbb{R}} f'(z)\ell^z(t)\,\mathrm{d}z + \sum_{0<s\le t,\, \Delta x_s>0} (f(x_s) - f(x_{s-})), \tag{3.2}$$

$$\begin{aligned} \mathbf{1}_{[z,+\infty)}(x_t) &= \mathbf{1}_{[z,+\infty)}(x_0) + \ell^z(t) \\ &\quad + \sum_{0<s\le t,\, \Delta x_s>0} (\mathbf{1}_{[z,+\infty)}(x_s) - \mathbf{1}_{[z,+\infty)}(x_{s-})). \end{aligned} \tag{3.3}$$

Formula (3.3) holds under the condition that z is a *simple level*, which means that for any $t > 0$,

$$\operatorname{Card}\{s \in (0,t] : x_{s-} < z < x_s \text{ or } x_s < z < x_{s-} \text{ or } x_s = z\} < +\infty$$

and there is no jump of x that starts or ends at the level z.

In what follows, we present an alternative proof, from which, depending on the regularity of g or x, we extend formula (3.2) to the case of less regular (not necessarily continuous) g.

Let V^0 denote the subset of D (and V) consisting of functions which are piecewise monotone, that is, for any $t > 0$ there exists a *finite* sequence $\{I_i\}_{i=1}^N$, $N \in \mathbb{N}$, of intervals which are mutually disjoint, $\bigcup_{i=1}^N I_i = [0,t]$ and the function x is monotone on each I_i.

First, we will prove (2.1) and (2.2) for $x \in V^0$.

LEMMA 3.1. *Let $x \in V^0$, $t > 0$ and $g : \mathbb{R} \to \mathbb{R}$ be Borel-measurable and locally integrable. Then the equalities* (2.1) *and* (2.2) *hold.*

Proof. Let $\{I_i\}_{i=1}^N$, $N \in \mathbb{N}$, be a sequence of intervals which are mutually disjoint, $\bigcup_{i=1}^N I_i = [0,t]$ and the function x is monotone on each I_i. Since x is càdlàg, we may and will assume that these intervals, except the last one containing t, are of the form $I_i = [t_i, t_{i+1})$, where $t_i < t_{i+1}$. Moreover, we will assume that they are the largest intervals possible on which x is monotone. Thus, if for some $i > 1$, x is non-decreasing on $[t_{i-1}, t_i)$ and non-decreasing on $[t_i, t_{i+1})$ then there has to be a downward jump at time t_i. Similarly, if for some $i > 1$, x is non-increasing on $[t_{i-1}, t_i)$ and non-increasing on $[t_i, t_{i+1})$ then there has to be an upward jump at time t_i. Let S denote the set of such times s.

Let us define $R = \bigcup_{i=1}^N \{x_{t_i}, x_{t_{i+1}-}, x_{t_{i+1}}\}$. Given a level $z \in \mathbb{R} \setminus R$, for any interval I_i or any $s \in S$, we can have at most one associated upcrossing.

This happens when $x_{t_i} < x_{t_{i+1}}$ or $x_{s-} < x_s$ and

$$z \in (x_{t_i}, x_{t_{i+1}}) \quad \text{or} \quad z \in (x_{s-}, x_s).$$

Clearly, no interval where x is decreasing induces any upcrossing. Let $J \subset \{1, \dots, N\}$ be the set of $i \in \{1, \dots, N\}$ that x is increasing on $[t_i, t_{i+1})$, and T be the set of $s \in S$ such that x has an upward jump at s. Then

$$\mathrm{u}^z(x, [0,t]) = \sum_{i \in J} \mathbf{1}_{(x_{t_i}, x_{t_{i+1}})}(z) + \sum_{s \in T} \mathbf{1}_{(x_{s-}, x_s)}(z).$$

Consequently,

$$\int_{\mathbb{R}} g(z)\mathrm{u}^z(x,t)\,\mathrm{d}z = \int_{\mathbb{R}\setminus R} g(z)\mathrm{u}^z(x,t)\,\mathrm{d}z = \sum_{i\in J} \int_{x_{t_i}}^{x_{t_{i+1}}} g(z)\,\mathrm{d}z + \sum_{s \in T} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z.$$

To deal with these integrals, we apply the classical idea of opening temporal windows at jump times. We may require that the sum of the lengths of these windows is finite and consider the interpolated continuous $\tilde{x}$. We have

$$\int_{x_{t_i}}^{x_{t_{i+1}}} g(z)\,\mathrm{d}z = \int_{\tilde{x}_u}^{\tilde{x}_{u'}} g(z)\,\mathrm{d}z \quad \text{and} \quad \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z = \int_{\tilde{x}_v}^{\tilde{x}_{v'}} g(z)\,\mathrm{d}z$$

for properly defined u, u', v, v'. Then we apply the change of variable for the Riemann–Stieltjes integral,

$$\int_{\tilde{x}_u}^{\tilde{x}_{u'}} g(z)\,\mathrm{d}z = \int_u^{u'} g(\tilde{x}_s)\,\mathrm{d}\tilde{x}_s = \int_u^{u'} g(\tilde{x}_s)\,(\mathrm{d}\tilde{x}_s)_+,$$

$$\int_{\tilde{x}_v}^{\tilde{x}_{v'}} g(z)\,\mathrm{d}z = \int_v^{v'} g(\tilde{x}_s)\,\mathrm{d}\tilde{x}_s = \int_v^{v'} g(\tilde{x}_s)\,(\mathrm{d}\tilde{x}_s)_+.$$

Clearly, for a properly defined $\tilde{t}$ we have

$$\int_{\mathbb{R}} g(z)\mathrm{u}^z(x,t)\,\mathrm{d}z = \int_0^{\tilde{t}} g(\tilde{x}_s)\,(\mathrm{d}\tilde{x}_s)_+. \tag{3.4}$$

Let us consider the decomposition of the measure $\mathrm{UTV}(x,\, \mathrm{d}s) = (\mathrm{d}x_s)_+$ into the continuous part μ^{cont} and the atomic part. Moreover, let W be the set of temporal windows. We obtain

$$\int_0^{\tilde{t}} \mathbf{1}_{s\notin W} g(\tilde{x}_s)\,(\mathrm{d}\tilde{x}_s)_+ = \int_0^t g(x_{s-})\,\mathrm{d}\mu^{\mathrm{cont}}(s),$$

$$\int_0^{\tilde{t}} \mathbf{1}_{s\in W} g(\tilde{x}_s)\,(\mathrm{d}\tilde{x}_s)_+ = \sum_{0<s\le t,\, \Delta x_s > 0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z.$$

Using these and (3.4) we obtain (2.1) by simple calculations.

In a similar way one proves (2.2). ∎

Now we will present another proof of Theorem 2.8.

Alternative proof of Theorem 2.8. First, we will assume that g is continuous.

By [ŁG14, Proposition 2.9 and Lemma 3.4], for any $c > 0$ there exists a function $x^c \in V^0$ with the following properties: for any $s \in [0, +\infty)$,

$$|x(s) - x^c(s)| \le c/2, \qquad |\Delta x^c_s| \le |\Delta x_s|, \tag{3.5}$$

$$\mathrm{TV}(x^c, [0, s]) \le \mathrm{TV}(x, [0, t]) + c \tag{3.6}$$

and

$$\mathrm{u}^z(x^c, [0, s]) = \mathrm{u}^{z,c}(x, [0, s]).$$

Using Lemma 3.1 for x^c, we have

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{u}^z(x^c, [0, t]) g(z)\,\mathrm{d}z &= \int_{\mathbb{R}} \mathrm{u}^{z,c}(x, [0, t]) g(z)\,\mathrm{d}z \\ &= \int_{(0,t]} g(x^c_s)\,(\mathrm{d}x^{c,\mathrm{cont}}_s)_+ + \sum_{0<s\le t,\, \Delta x^c_s > 0} \int_{x^c_{s-}}^{x^c_s} g(z)\,\mathrm{d}z,\end{aligned} \tag{3.7}$$

where $x^{c,\mathrm{cont}}$ denotes the continuous part of x^c.

By the Lebesgue dominated convergence theorem and by the definition of $\mathrm{u}^z(x, [0, t])$, we get

$$\lim_{c\to 0+} \int_{\mathbb{R}} \mathrm{u}^{z,c}(x, [0, t]) g(z)\,\mathrm{d}z = \int_{\mathbb{R}} \mathrm{u}^z(x, [0, t]) g(z)\,\mathrm{d}z.$$

Also, to deal with the last expression in (3.7), for arbitrary $\varepsilon > 0$ we can approximate x by a step function $\tilde{x} = \sum a_i \mathbf{1}_{[t_i, t_{i+1})}$ such that

$$\sup_{s\in[0,t]} |g(x_s) - g(\tilde{x}_s)| \le \varepsilon/2.$$

From (3.6) we know that

$$\lim_{c\to 0+} \left| \int_{(0,t]} g(x^c_s)\,(\mathrm{d}x^{c,\mathrm{cont}}_s)_+ - \int_{(0,t]} g(\tilde{x}_s)\,(\mathrm{d}x^{c,\mathrm{cont}}_s)_+ \right| \le \frac{\varepsilon}{2}\,\mathrm{TV}(x, [0, t]).$$

Also, by (3.6) and by the fact that $\tilde{x}$ is a step function, we have

$$\lim_{c\to 0+} \left| \int_{(0,t]} g(\tilde{x}_s)\,(\mathrm{d}x^{c,\mathrm{cont}}_s)_+ - \int_{(0,t]} g(\tilde{x}_s)\,(\mathrm{d}x^{\mathrm{cont}}_s)_+ \right| = 0.$$

Finally,

$$\lim_{c\to 0+} \left| \int_{(0,t]} g(x_s)\,(\mathrm{d}x^{\mathrm{cont}}_s)_+ - \int_{(0,t]} g(\tilde{x}_s)\,(\mathrm{d}x^{\mathrm{cont}}_s)_+ \right| \le \frac{\varepsilon}{2}\,\mathrm{TV}(x, [0, t]).$$

Since ε may be arbitrarily close to 0, the last three relations imply

$$\lim_{c\to 0+} \int_{(0,t]} g(x^c_s)\,(\mathrm{d}x^{c,\mathrm{cont}}_s)_+ = \int_{(0,t]} g(x_s)\,(\mathrm{d}x^{\mathrm{cont}}_s)_+.$$

Similarly, dividing for example the jumps of x into a set of small jumps (smaller than some threshold) whose sum of magnitudes is close to 0, and a finite number of bigger jumps (larger than or equal to some threshold), we get

$$\lim_{c\to 0+} \sum_{0<s\le t,\, \Delta x_s^c>0} \int_{x_{s-}^c}^{x_s^c} g(z)\,\mathrm{d}z = \sum_{0<s\le t,\, \Delta x_s>0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z.$$

Thus, we deduce that the relation (2.1) also holds for any continuous $g:\mathbb{R}\to\mathbb{R}$ and any $x\in V$.

Now we are going to drop the assumption about continuity of g. Let us consider the measures assigning to a Borel set A the value

$$\mu^x(A) := \int_{x^{-1}(A)\cap(0,t]} (\,\mathrm{d}x_s^{\mathrm{cont}})_+.$$

Notice that for any Borel-measurable and bounded $g:\mathbb{R}\to\mathbb{R}$,

$$\int_{(0,t]} g(x_s)\,(\mathrm{d}x_s^{\mathrm{cont}})_+ = \int_{\mathbb{R}} g(y)\,\mathrm{d}\mu^x(y).$$

Let $\mathcal{B}$ be the Borel σ-field of subsets of real numbers and consider the measure ν^x such that

$$\mathrm{d}\nu^x(z) = \mathrm{u}^z(x,[0,t])\,\mathrm{d}z + \mathrm{d}\mu^x(z) + \sum_{0<s\le t,\, \Delta x_s>0} \mathbf{1}_{[x_{s-},x_s]}(z)\,\mathrm{d}z.$$

Let $\delta>0$. By Luzin's theorem [Fol99, Theorem 7.10], for any $\varepsilon>0$ there exists a continuous $\tilde{g}:\mathbb{R}\to\mathbb{R}$ and $E\in\mathcal{B}$ such that $\nu^x(E^c)<\varepsilon$ and $\tilde{g}=g$ on E. Since g is bounded, taking ε sufficiently small we get

$$\Big|\int_{\mathbb{R}} \mathrm{u}^z(x,[0,t])g(z)\,\mathrm{d}z - \int_{\mathbb{R}} \mathrm{u}^z(x,[0,t])\tilde{g}(z)\,\mathrm{d}z\Big| < \delta,$$

$$\Big|\int_{\mathbb{R}} g(y)\,\mathrm{d}\mu^x(y) - \int_{\mathbb{R}} \tilde{g}(y)\,\mathrm{d}\mu^x(y)\Big| < \delta,$$

$$\Big|\sum_{0<s\le t,\, \Delta x_s>0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z - \sum_{0<s\le t,\, \Delta x_s>0} \int_{x_{s-}}^{x_s} \tilde{g}(z)\,\mathrm{d}z\Big| < \delta.$$

Relation (2.1) holds for $\tilde{g}$, since it is continuous; thus, as δ is arbitrary, it must hold for g as well. ∎

4. Change of variable formulas. In this section, we extend formula (3.2) in two ways. The first extension assumes that $x\in V^0$ and $g:\mathbb{R}\to\mathbb{R}$ is Borel-measurable and locally integrable (with respect to the Lebesgue measure). Under these assumptions we have:

PROPOSITION 4.1. *If $x \in V^0$, $g : \mathbb{R} \to \mathbb{R}$ is Borel-measurable and locally integrable (with respect to the Lebesgue measure), and one defines $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(y) := \begin{cases} f_0 + \int_{[0,y]} g(z)\,\mathrm{d}z & \text{if } y \geq 0, \\ f(y) := f_0 - \int_{[y,0]} g(z)\,\mathrm{d}z & \text{if } y < 0, \end{cases} \tag{4.1}$$

with some $f_0 \in \mathbb{R}$, or equivalently if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous and

$$f(y) = \begin{cases} f(0) + \int_{[0,y]} g(z)\,\mathrm{d}z = f(0) + \int_{[0,y]} f'(z)\,\mathrm{d}z & \text{if } y \geq 0, \\ f(0) - \int_{[y,0]} g(z)\,\mathrm{d}z = f(0) - \int_{[0,y]} f'(z)\,\mathrm{d}z & \text{if } y < 0, \end{cases} \tag{4.2}$$

then

$$\begin{aligned} f(x_t) - f(x_0) &= \int_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z \\ &= \int_{(0,t]} g(x_{s-})\,\mathrm{d}x_s + \sum_{0<s\leq t,\, \Delta x_s \neq 0} \{f(x_s) - f(x_{s-}) - g(x_{s-})\Delta x_s\} \\ &= \int_{(0,t]} g(x_s)\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{0<s\leq t,\, \Delta x_s \neq 0} \{f(x_s) - f(x_{s-})\} \\ &= \int_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z + \sum_{0<s\leq t,\, \Delta x_s \neq 0} \{f(x_s) - f(x_{s-})\}. \end{aligned} \tag{4.3}$$

Proof. Let us apply the same notation as in the proof of Lemma 3.1. For $i \in \{1, \ldots, N\}$ the function x is monotone on $[t_i, t_{i+1})$. This gives

$$\int_{\mathbb{R}} \{\mathrm{u}^z(x,[t_i,t_{i+1}]) - \mathrm{d}^z(x,[t_i,t_{i+1}])\} g(z)\,\mathrm{d}z = f(x_{t_{i+1}}) - f(x_{t_i}).$$

Also, for $z \notin R$, we have

$$\sum_{i=1}^{N} \mathrm{u}^z(x,[t_i,t_{i+1}]) = \mathrm{u}^z(x,[0,t]), \qquad \sum_{i=1}^{N} \mathrm{d}^z(x,[t_i,t_{i+1}]) = \mathrm{d}^z(x,[0,t]).$$

Moreover, by (2.3),

$$\begin{aligned} &\int_{\mathbb{R}} \{\mathrm{u}^z(x,[t_i,t_{i+1}]) - \mathrm{d}^z(x,[t_i,t_{i+1}])\} g(z)\,\mathrm{d}z \\ &\qquad = \int_{(t_i,t_{i+1}]} g(x_{s-})\,\mathrm{d}x_s + \sum_{t_i<s\leq t_{i+1},\, \Delta x_s \neq 0} \int_{x_{s-}}^{x_s} \{g(z) - g(x_{s-})\}\,\mathrm{d}z \\ &\qquad = \int_{(t_i,t_{i+1}]} g(x_s)\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{t_i<s\leq t_{i+1},\, \Delta x_s \neq 0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z. \end{aligned}$$

Summing the last display over $i = 1, \ldots, N$ and applying the previous two displays, we get

$$\begin{aligned}
f(x_t) - f(x_0) &= \int_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z \\
&= \int_{(0,t]} g(x_{s-})\,\mathrm{d}x_s + \sum_{0<s\le t,\, \Delta x_s \ne 0} \{f(x_s) - f(x_{s-}) - g(x_{s-})\Delta x_s\} \\
&= \int_{(0,t]} g(x_s)\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{0<s\le t,\, \Delta x_s \ne 0} \{f(x_s) - f(x_{s-})\}.
\end{aligned}$$

From (2.5) and the analogous relations for $\mathrm{Card}(\mathcal{D}(z)\cap(0,t])$ and $\mathrm{d}^y(x,[s,t])$, we also infer

$$\begin{aligned}
&\int_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z \\
&\qquad = \int_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z + \sum_{0<s\le t,\, \Delta x_s \ne 0} \int_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z \\
&\qquad = \int_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z + \sum_{0<s\le t,\, \Delta x_s \ne 0} \{f(x_s) - f(x_{s-})\}.
\end{aligned}$$

The last two displays give (4.3). ∎

In the next proposition, we relax the assumption that $x \in V^0$ but then we need to assume that g is locally bounded. At the end of this section, we give an example showing that this assumption is necessary. (Interestingly, it is possible to go in a different direction – assume that x and f are càdlàg and non-decreasing – and obtain a relevant change of variable formula [BY13, Theorem 2.1]).

PROPOSITION 4.2. *If $x \in V$, $g : \mathbb{R} \to \mathbb{R}$ is Borel-measurable and locally bounded, and one defines $f : \mathbb{R} \to \mathbb{R}$ by* (4.1) *or equivalently if $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and satisfies* (4.2) *then*

$$\begin{aligned}
f(x_t) - f(x_0) &= \int_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z + \sum_{0<s\le t,\, \Delta x_s \ne 0} \{f(x_s) - f(x_{s-})\}.
\end{aligned} \tag{4.4}$$

Proof. Let $c > 0$ and let us apply the same notation as in the second proof of Theorem 2.8. Using Proposition 4.1 for x^c, and (3.7), we get

$$\begin{aligned}
f(x_t^c) - f(x_0^c) &= \int_{(0,t]} g(x_s^c)\,\mathrm{d}x_s^{c,\mathrm{cont}} + \sum_{0<s\le t,\, \Delta x_s^c \ne 0} \{f(x_s^c) - f(x_{s-}^c)\} \\
&= \int_{(0,t]} g(x_s^c)\,\mathrm{d}x_s^{c,\mathrm{cont}} + \sum_{0<s\le t,\, \Delta x_s^c \ne 0} \int_{x_{s-}^c}^{x_s^c} g(z)\,\mathrm{d}z.
\end{aligned}$$

Assuming that g is continuous and letting $c \to 0$, as in the second proof of

Theorem 2.8 we get

$$f(x_t) - f(x_0) = \int\limits_{(0,t]} g(x_s)\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{0<s\le t,\, \Delta x_s \neq 0} \int\limits_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z.$$

Now, applying Theorem 2.8 we get

$$\begin{aligned} f(x_t) - f(x_0) &= \int\limits_{(0,t]} g(x_s)\,\mathrm{d}x_s^{\mathrm{cont}} + \sum_{0<s\le t,\, \Delta x_s \neq 0} \int\limits_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z \\ &= \int\limits_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z. \end{aligned}$$

From (2.5) and the analogous relations for $\mathrm{Card}(\mathcal{D}(z) \cap (0,t])$ and $\mathrm{d}^y(x,[s,t])$, we also infer

$$\begin{aligned} &\int\limits_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z \\ &\qquad = \int\limits_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z + \sum_{0<s\le t,\, \Delta x_s \neq 0} \int\limits_{x_{s-}}^{x_s} g(z)\,\mathrm{d}z \\ &\qquad = \int\limits_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z + \sum_{0<s\le t,\, \Delta x_s \neq 0} \{f(x_s) - f(x_{s-})\}. \end{aligned}$$

Using the last two displays, we get (4.4) for continuous g.

To prove (4.4) for any bounded but not necessarily continuous g, we apply a C^∞ mollifier function $h : \mathbb{R} \to [0,+\infty)$ such that $h(y) = 0$ if $|y| > 1$ and $\int_{\mathbb{R}} h(y)\,\mathrm{d}y = 1$. Define

$$g_n(z) = n \int\limits_{\mathbb{R}} h(ny) g(z-y)\,\mathrm{d}y = n \int\limits_{\mathbb{R}} h(n(z-y)) g(y)\,\mathrm{d}y, \qquad n \in \mathbb{N}.$$

Then g_n is continuous and bounded on any interval $[-K,K]$, $K > 0$ (by the supremum of $|g|$ on $[-K-1, K+1]$). Moreover,

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}} |g_n(z) - g(z)|\,\mathrm{d}z = 0$$

(see [Fol99, Theorem 8.14(a)]). The Lebesgue dominated convergence theorem together with the local boundedness of $|g_n|$, $|g|$ and $|g_n - g|$ gives

$$\int\limits_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g_n(z)\,\mathrm{d}z \to \int\limits_{\mathbb{R}} \{\mathrm{u}^z(x,[0,t]) - \mathrm{d}^z(x,[0,t])\} g(z)\,\mathrm{d}z,$$

$$\int\limits_{\mathbb{R}} \ell^z(t) g_n(z)\,\mathrm{d}z \to \int\limits_{\mathbb{R}} \ell^z(t) g(z)\,\mathrm{d}z,$$

$$\sum_{0<s\le t,\, \Delta x_s \neq 0} \Bigg| \int\limits_{x_{s-}}^{x_s} |g_n(z) - g(z)|\,\mathrm{d}z \Bigg| \to 0.$$

The last convergence is easily obtained when we divide the jumps of x into two groups: a finite number of large jumps, and jumps whose sum of magnitudes is less than a given $\varepsilon > 0$. Now, for any $y \in \mathbb{R}$, defining $f_n(y) = f_0 + \operatorname{sgn}(y) \int_{[0\wedge y, 0\vee y]} g_n(z)\,\mathrm{d}z$ we get

$$f_n(y) \to f_0 + \operatorname{sgn}(y) \int_{[0\wedge y, 0\vee y]} g(z)\,\mathrm{d}z = f(y)$$

as $n \to \infty$. Using these limits and applying formula (4.4) for f_n (and g_n) we obtain (4.4) for f and g. ∎

We end this section with an easy example showing that the assumption that f is locally Lipschitz in Proposition 4.2 is necessary. Let

$$x(t) = \begin{cases} \dfrac{1 + (-1)^{\lfloor \frac{1}{1-t} \rfloor^2}}{2\lfloor \frac{1}{1-t} \rfloor^2} & \text{if } t \in [0,1), \\ 0 & \text{if } t \geq 1; \end{cases} \qquad f(y) = \begin{cases} \sqrt{y} & \text{if } y \in [0, +\infty), \\ 0 & \text{if } y < 0. \end{cases}$$

Then f is absolutely continuous, while x has finite variation, it has no continuous part and has two jumps, from $1/(2n)^2$ to 0 and from 0 to $1/(2n)^2$, for each $n \in \mathbb{N}$, on the interval $t \in (0,1)$. It follows that the sum

$$\sum_{0<s\leq 1,\, \Delta x_s \neq 0} \{f(x_s) - f(x_{s-})\}$$

is not well defined.

5. Conclusion. We studied local times and occupation measures for càdlàg real functions with locally finite total variation. Building upon the framework developed by Bertoin and Yor [BY14], we introduced notions of level crossings including up-crossings, down-crossings, and jump crossings, and established their connection to occupation densities. Furthermore, we derived change-of-variable formulas analogous to the Itô and Tanaka–Meyer formulas in stochastic calculus. The main results were proved using both a direct proof, grounded in prior work, and an alternative approach based on approximation by piecewise monotone functions.

A natural (but technically challenging) direction for further research would be the generalization of the formulas obtained to general, locally finite variation functions, not necessarily càdlàg ones. The first possible step to do this may be the observation that any locally finite variation function is regulated, that is, it has left and right limits.

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