

Minimal zero-free regions for results on primes between consecutive perfect k th powers

by

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Abstract. We compute minimal zero-free regions for the Riemann zeta-function which ensure there is always a prime between consecutive perfect k th powers. Our computations cover powers $k \geq 65$ and quantify how far we are away from proving certain milestones toward an infamous open problem (Legendre's conjecture). In addition, we prove there is always a prime between consecutive perfect 86th powers and identify an integer sequence for which there is always a prime between consecutive 70th powers.

1. Introduction. Let ζ denote the Riemann zeta-function. Recall the Riemann Hypothesis (RH) postulates that every non-trivial zero $\varrho = \beta + i\gamma$ of ζ satisfies $\beta = 1/2$ and $\gamma \neq 0$. The Riemann height is a constant $H_R > 0$ such that the RH is known to be true for all $|\gamma| \leq H_R$. Platt and Trudgian [19] have announced $H_R = 3\,000\,175\,332\,800$ is admissible.

A famous open problem from Legendre postulates there is at least one prime between consecutive perfect squares, i.e., in the interval $(n^2, (n+1)^2)$ for every integer $n \geq 1$. A proof of Legendre's conjecture is presently out of reach, even under the assumption of the RH. On the other hand, Ingham [14] proved Theorem 1.1 by applying results on the distribution of the zeros of ζ .

THEOREM 1.1 (Ingham). *If n is sufficiently large, then there is at least one prime in the interval $(n^3, (n+1)^3)$.*

In recent work, Cully-Hugill [6] has proved there is at least one prime in the interval $(n^3, (n+1)^3)$ for every integer $n \geq \exp(\exp(32.892))$. This result makes progress toward a complete explicit version of Theorem 1.1, but the range of n that remains to be made explicit is largely impervious to modern techniques.

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This naturally leads to the problem of computing the least $k \geq 3$ such that we can prove (unconditionally) that there is at least one prime in the interval $(n^k, (n+1)^k)$ for every integer $n \geq 1$. Several authors [6, 7, 8, 11, 17] have proved progressively stronger results; the latest result in this direction claims $k = 90$ is admissible (see [8]). Using similar techniques, we prove Theorem 1.2, which improves these computations.

THEOREM 1.2. *For every integer $n \geq 1$, there is at least one prime in the interval $(n^{86}, (n+1)^{86})$.*

For any fixed k , the method to prove there is at least one prime in the interval $(n^k, (n+1)^k)$ for every integer $n \geq 1$ requires three ingredients:

- (1) Zero-free regions for ζ of the Littlewood form (or stronger).
- (2) Upper bounds for the number $N(\sigma, T)$ of non-trivial zeros $\varrho = \beta + i\gamma$ of ζ in the rectangle $\sigma \leq \beta \leq 1$ and $0 < \gamma \leq T$.
- (3) Explicit descriptions for the error in the truncated Riemann–von Mangoldt formula.

In general, to prove that smaller k are admissible, recent works have applied the latest knowledge on the distribution of the zeros of ζ and exercised tighter control over the error in the truncated Riemann–von Mangoldt formula. While this has been a fruitful endeavour, further improvements to the error in the truncated Riemann–von Mangoldt formula are limited, as the present results are quite sharp save for minor refinements to the constants within.

In this paper, we investigate two alternative strategies to prove complete results (i.e., results which hold on $n \geq 1$) for $k < 86$. First, we fix $k = 70$ and prove a complete result on a subset of the natural numbers (see Theorem 1.3). Second, we compute minimal zero-free regions for ζ which are sufficient to prove complete results for $k < 86$ (see Theorem 1.4).

THEOREM 1.3. *Fix $N \geq 0$ and let $a_n = n(1 + \lfloor N/n \rfloor)$. If $N \geq \exp(\frac{15.951}{70}) - \exp(\frac{2.135}{70})$, then there is at least one prime in the interval (a_n^{70}, a_{n+1}^{70}) for every integer $n \geq 1$.*

Table 1. Table of admissible constants in Theorem 1.4

k	z_k	$\log T_k$
85	17.270	75.853
80	16.198	80.643
75	15.122	90.945
70	14.055	264.334

THEOREM 1.4. *Suppose that $k \in \{85, 80, 75, 70\}$ is fixed and z_k, T_k are the values associated to each choice of k in Table 1. If $\zeta(\sigma + it) \neq 0$ in the*

region

$$1 - \frac{\log \log |t|}{z_k \log |t|} \leq \sigma \leq 1 \quad \text{and} \quad H_R < |t| \leq T_k,$$

then there is at least one prime in the interval $(n^k, (n+1)^k)$ for every integer $n \geq 1$. Computations for every $65 \leq k \leq 85$ not covered by this theorem statement can be found in the main body of the paper (see Table 5).

For example, Theorem 1.4 tells us that if $\zeta(\sigma + it) \neq 0$ in the thin rectangular region

$$0.99168 \leq \sigma \leq 1 \quad \text{and} \quad H_R < |t| \leq e^{264.334}$$

of the critical strip, then there is at least one prime in the interval $(n^{70}, (n+1)^{70})$ for every integer $n \geq 1$. This is interesting, because a proof of the result with $k = 70$ would be a significant theoretical milestone, yet the associated computation appears reasonable. Using the results in this paper, we can repeat this analysis for any $k \in [65, 85]$, and the difficulty of these computations indicate how far away we are from complete proofs of important milestones toward Legendre's conjecture.

To prove Theorems 1.3–1.4, we enhanced the usual methods by applying more than one bound for $N(\sigma, T)$. This strategy yields stronger computations for $67 \leq k \leq 85$, but offered no benefits outside this range. For example, without this refinement to the method, we could only prove Theorem 1.3 with

$$N \geq \exp\left(\frac{580\,009}{70}\right) - \exp\left(\frac{2\,135}{70}\right) = 3.132 \dots \cdot 10^{3598}.$$

In the future, if stronger explicit bounds for $N(\sigma, T)$ with $H_R < T \leq \exp(80)$ of the right shape are proved, then this strategy can be extended to improve the computations in Theorems 1.3–1.4.

REMARK. If $n > N$ in Theorem 1.3, then $a_n = n$. Therefore, Theorem 1.3 also demonstrates there is at least one prime in the interval $(n^{70}, (n+1)^{70})$ for every integer

$$n > \exp\left(\frac{15\,951}{70}\right) - \exp\left(\frac{2\,135}{70}\right) = 9.189 \dots \cdot 10^{98}.$$

Structure. The remainder of this paper is dedicated to proving Theorems 1.2–1.4. An important intermediate result, namely Proposition 3.1, establishes user-friendly parametrised bounds for a difference of prime-counting functions, which are formulated so they can be readily updated or adapted for use in future work. Since obtaining explicit estimates over primes in short intervals is often a technically delicate task, the parametrised bounds we prove may be of independent interest beyond the present paper.

In Section 2, we introduce several properties of the Riemann zeta-function ζ . In Section 3, we prove Proposition 3.1 and apply these bounds to prove the auxiliary results we require. In Section 4, we combine these results to prove Theorems 1.2-1.4. To avoid breaking the flow of the narrative, we have collected all referenced tables in Appendix A.

Computations. To prove our results, we used `Python`. To supplement this work, our code to obtain all the computations in this paper are given here: [press this link](#).

2. The Riemann zeta-function. Results pertaining to the distribution of the non-trivial zeros of ζ , which we denote by $\varrho = \beta + i\gamma$, play a central role in this paper. Therefore, we introduce several results pertaining to the distribution of these zeros in this section. All of the notation that is introduced in this section will be used throughout the rest of this paper.

2.1. Zero-free regions. There are three commonly-seen forms of zero-free region:

(2.1)

$$\sigma \geq 1 - \frac{1}{Z_1 \log t}, \quad \sigma \geq 1 - \frac{\log \log t}{Z_2 \log t}, \quad \text{and} \quad \sigma \geq 1 - \frac{1}{Z_3 (\log t)^{2/3} (\log \log t)^{1/3}}.$$

For $t \geq 3$, Yang has proved in his PhD thesis [21] that $\zeta(\sigma + it) \neq 0$ in the regions defined in (2.1), with

$$(2.2) \quad Z_1 = 4.862, \quad Z_2 = 19.62, \quad \text{and} \quad Z_3 = 51.34.$$

These computations improve earlier results from [18], [20], and [1] respectively. In addition, Ford [12, Thm. 3] has proved that $\zeta(\sigma + it) \neq 0$ in the region $\sigma \geq 1 - Z(t)^{-1}/\log t$ and $t \geq 5.45 \cdot 10^8$ such that

$$J(T) = \min \left\{ \frac{\log T}{4} + 1.8521, \frac{\log T}{6} + \log \log T + \log 3 \right\},$$

$$Z(T) = \frac{J(T) + 0.685 + 0.155 \log \log T}{(0.04962 - \frac{0.0196}{J(T)+1.15}) \log T}.$$

Combined with the zero-free regions in (2.1), we have $\zeta(\sigma + it) \neq 0$ in the region

$$1 - \frac{\log \log |t|}{\widehat{Z}(t) \log |t|} \leq \sigma \leq 1 \quad \text{and} \quad |t| \geq 5.45 \cdot 10^8,$$

in which

$$(2.3) \quad \widehat{Z}(T) = \min \left\{ Z_2, \min \{Z_1, Z(T)\} \log \log T, \frac{Z_3 (\log \log T)^{4/3}}{(\log T)^{1/3}} \right\},$$

which is a non-increasing function.

2.2. Zero-density estimates. For $1/2 \leq \sigma < 1$, let

$$N(\sigma, T) = \#\{\varrho : \sigma < \beta < 1, 0 < \gamma < T\}$$

and

$$N(T) = \#\{\varrho : 0 < \beta < 1, 0 < \gamma < T\}.$$

First, there exist constants a_1, a_2, a_3 such that

$$(2.4) \quad \left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq R(T),$$

where $R(T) = a_1 \log T + a_2 \log \log T + a_3$, for all $T \geq H_R$. In recent work, Bellotti and Wong [3] have computed that $(a_1, a_2, a_3) = (0.10076, 0.24460, 8.08344)$ are admissible; their computations improve earlier work of Hasanlizada, Shen, and Wong [13]. It follows from (2.4) that if $T \geq 1$, then

$$(2.5) \quad N(T) \leq \frac{T \log T}{2\pi}.$$

Next, we import an upper bound for $N(\sigma, T)$ from Bellotti [2], who has computed explicit constants C, B, T_0, T_1 (presented in Table 2) such that if $T_0 \leq T \leq T_1$ and $0.985 < \sigma < 1$, then

$$(2.6) \quad N(\sigma, T) \leq CT^{B(1-\sigma)}.$$

Further, computations from [16] and [15, Appendix B] prove that if $0.6 < \sigma < 1$ and $T \geq H_R$, then

$$(2.7) \quad N(\sigma, T) \leq 17.253(\log T)^{5-2\sigma} T^{\frac{8}{3}(1-\sigma)} + 5.663(\log T)^2.$$

3. Auxiliary bounds. Recall the Chebyshev functions, defined by

$$\theta(x) = \sum_{p \leq x} \log p \quad \text{and} \quad \psi(x) = \sum_{p^r \leq x} \log p,$$

where the first sum is over primes p and the second is over prime powers. To prove the main results of this paper, we require explicit lower bounds for a prescribed difference of θ functions. To establish the bounds we require, we prove the following technical result (Proposition 3.1), which might also be of independent interest.

PROPOSITION 3.1. *Suppose that $k \geq 11$, $h = kx^{1-1/k}$, and $x \geq e^{1000}$. Let $0.6 < \sigma_1 < 1$, $\alpha_1 = 1 + 1.93378 \cdot 10^{-8}$, $\alpha_2 = 1 + 1.936 \cdot 10^{-8}$, and $2T = x^\alpha$ such that $T \geq H_R$ and*

$$\frac{1}{k} < \alpha < \min \left\{ \frac{1}{10}, 1 - \sigma_1 \right\}.$$

Suppose that $\zeta(\sigma + it) \neq 0$ in the region $1 - \nu(2T) \leq \sigma \leq 1$ and $|t| \leq 2T$, where

$$\nu(y) = \frac{\log \log y}{Z \log y}$$

for some constant $Z > 0$. Further, suppose that B_i are positive constants and $f(Y)$ is a non-decreasing function such that if $\sigma_1 < \sigma < 1$, then

$$(3.1) \quad N(\sigma, Y) \leq B_1(\log Y)^{B_2} f(Y)^{1-\sigma} + B_3(\log Y)^{B_4}.$$

If $x \geq ef(x^\alpha)$, then

$$\begin{aligned} & \frac{|\theta(x+h) - \theta(x) - h|}{h} \\ & \leq \frac{\alpha x^{\sigma_1+\alpha-1} \log x}{\pi} + 2B_3(\alpha \log x)^{B_4} ((\alpha \log x)^{-\frac{1}{Z\alpha}} - x^{\sigma_1-1}) \\ & \quad + \frac{2B_1(\alpha \log x)^{B_2} \log x}{\log(xf(x^\alpha)^{-1})} ((\alpha \log x)^{-\frac{\log(xf(x^\alpha)^{-1})}{Z\alpha \log x}} - (xf(x^\alpha)^{-1})^{\sigma_1-1}) \\ & \quad + \frac{2.52}{kx^{\alpha-1/k}} ((1+kx^{-1/k})(\log x + kx^{-1/k})^{8/10} + (\log x)^{8/10}) \\ & \quad + (\alpha_1(1+kx^{-1/k})^{1/2} - 0.999) \frac{x^{1/k-1/2}}{k} \\ & \quad + (\alpha_2(1+kx^{-1/k})^{1/3} - 0.885) \frac{x^{1/k-2/3}}{k}. \end{aligned}$$

If $k \geq 86$, then the constants $1/10$, 2.52 , and $8/10$ can be replaced with $1/85$, 12.782 , and $1/10$ respectively.

REMARK. In Proposition 3.1, we restrict our attention to $k \geq 11$ (resp. $k \geq 86$) because we require $\alpha > 1/k$ and this condition cannot hold on $k \leq 10$ (resp. $k \leq 85$).

Our proof of Proposition 3.1 is deferred to Section 3.1. Next, we use Proposition 3.1 to prove Propositions 3.2, 3.3, and 3.4 which compute broad ranges of x such that

$$(3.2) \quad \theta(x + kx^{1-1/k}) - \theta(x) > 0$$

holds for fixed choices of k . For example, the computations in Proposition 3.2 demonstrate that if $k \geq 70$, then (3.2) holds unconditionally for all $x \geq \exp(15951)$.

PROPOSITION 3.2. Fix any $k \in [65, 90]$ and suppose that x_A and x_B correspond to the values in Table 3.

- If $k \geq 86$ or $k \leq 66$, then (3.2) holds for all $\log x \geq x_A$.
- If $67 \leq k \leq 85$, then (3.2) holds for all $\log x \geq x_B$.

Proof. Fix a $k \in [65, 90]$ and suppose that x_A, c_A, x_B, c_B correspond to the values in Table 3. We only compute values for x_B, c_B for every k where there is a benefit to do so. Recall that we have $\zeta(\sigma + it) \neq 0$ in the region $\sigma \geq 1 - \widehat{Z}(t)^{-1} \log \log t / \log t$ and $t \geq H_R$, where $\widehat{Z}(t)$ has been defined

in (2.3). Further, we take

$$\sigma_1 = 0.675, \quad B_1 = 17.253, \quad B_2 = 3, \quad B_3 = 5.663, \\ B_4 = 2, \quad f(t) = t^{8/3}(\log t)^2, \quad \text{and} \quad \alpha = \frac{c_A}{k}$$

in Proposition 3.1 to see (3.2) holds for all $\log x \geq x_A$. Each of these parameters (except c_A) is chosen in light of (2.7). With this, the result is proved for each $k \geq 86$ and $k \leq 66$. For each $65 \leq k \leq 85$ and $x_A \geq \log x \geq x_B$, we take

$$\sigma_1 = 0.985, \quad B_1 = C, \quad B_2 = B_3 = B_4 = 0, \quad f(t) = t^B, \quad \text{and} \quad \alpha = \frac{c_B}{k}$$

in Proposition 3.1, with C and B corresponding to the values in (2.6) and Table 2. It follows that (3.2) holds in the extended range $\log x \geq x_B$, and hence the result for each $67 \leq k \leq 85$ is complete. ■

PROPOSITION 3.3. *For any $k \geq 21$, the inequality (3.2) is true for all*

$$4 \cdot 10^{18} \leq x \leq e^{k \log(k \cdot 2.51949 \cdot 10^{11} (1 - \frac{1}{2.51949 \cdot 10^{11}})^2)}.$$

Proof. For each pair (Δ, x_0) in Table 4, Cully-Hugill and the author [9, 10] have proved that there exists at least one prime in the interval $(x(1 - \Delta^{-1}), x]$ for all $x \geq x_0$. It follows that if $x \geq x_0$, then there is at least one prime in the interval $(x(1 + kx^{-1/k})(1 - \Delta^{-1}), x(1 + kx^{-1/k})]$. Moreover,

$$x(1 + kx^{-1/k})(1 - \Delta^{-1}) \geq x \iff 1 + kx^{-1/k} \geq (1 - \Delta^{-1})^{-1} \\ \iff kx^{-1/k} \geq (1 - \Delta^{-1})^{-1} - 1,$$

and

$$(1 - \Delta^{-1})^{-1} - 1 = \int_{1 - \Delta^{-1}}^1 t^{-2} dt \leq \frac{\Delta^{-1}}{(1 - \Delta^{-1})^2}.$$

It follows that there is at least one prime in the interval $(x, x + kx^{1-1/k}]$ for all $x \geq x_0$ such that

$$kx^{-1/k} \geq \frac{\Delta^{-1}}{(1 - \Delta^{-1})^2} \iff \log x \leq k \log(k \Delta (1 - \Delta^{-1})^2).$$

Since $\Delta \geq 3.90970 \cdot 10^7$ for any $x \geq 4 \cdot 10^{18}$, it follows that if $k \geq 21$, then there is at least one prime in the interval $(x, x + kx^{1-1/k}]$ for all

$$4 \cdot 10^{18} \leq x \leq e^{615.2373} < e^{21 \log(21 \cdot 3.90970 \cdot 10^7 (1 - \frac{1}{3.90970 \cdot 10^7})^2)}.$$

Similarly, $\Delta \geq 2.51949 \cdot 10^{11}$ for any $x \geq e^{600}$. Therefore, by extension there is at least one prime in the interval $(x, x + kx^{1-1/k}]$ for all

$$4 \cdot 10^{18} \leq x \leq e^{k \log(k \cdot 2.51949 \cdot 10^{11} (1 - \frac{1}{2.51949 \cdot 10^{11}})^2)}.$$

With this, we have proved the result. ■

PROPOSITION 3.4. *Fix any $k \in [65, 90]$. Suppose that x_A, x_B correspond to the values in Table 3 and $\mathcal{Z}, x_{\mathcal{Z}}, c_{\mathcal{Z}}$ correspond to the values in Table 5. We also let $\hat{x}(k) = x_A$ if $k \leq 66$ or $k \leq 66$ and $\hat{x}(k) = x_B$ if $67 \leq k \leq 85$. If $\zeta(\sigma + it) \neq 0$ in the region*

$$1 - \frac{\log \log |t|}{\mathcal{Z} \log |t|} \leq \sigma \leq 1 \quad \text{and} \quad |t| \leq e^{\hat{x}(k)(\frac{c_{\mathcal{Z}}}{k})},$$

then (3.2) holds for all $\exp(x_{\mathcal{Z}}) \leq x \leq \exp(\hat{x}(k))$.

Proof. For any fixed $k \in [65, 90]$, take

$$\begin{aligned} Z = \mathcal{Z}, \quad \sigma_1 = 0.675, \quad B_1 = 17.253, \quad B_2 = 3, \quad B_3 = 5.663, \\ B_4 = 2, \quad f(t) = t^{8/3}(\log t)^2 \quad \text{and} \quad \alpha = \frac{c_{\mathcal{Z}}}{k} \end{aligned}$$

in Proposition 3.1. These parameters are chosen in light of our assumptions and (2.7), and these choices imply (3.2) holds in the range $\log x \geq x_{\mathcal{Z}}$. Further, since we only need to consider $x \leq \exp(\hat{x}(k))$, we only require the desired zero-free region to hold for all t such that $|t| \leq x^{\alpha} \leq \exp(\hat{x}(k)\alpha)$; this can be deduced by analysing the proof of Proposition 3.1. ■

REMARK. To make optimal choices for α in Propositions 3.2 and 3.4, we tested $\alpha = (1 + \varepsilon)/k$ for systematically incremented choices of $\varepsilon > 0$ until the choice $\alpha = (1 + \varepsilon)/k$ yielded the best computation. In Proposition 3.4, we have computed $\mathcal{Z}$ optimally by systematically decreasing choices of $\mathcal{Z}$ until the largest $\mathcal{Z}$ has been found such that the result is proved for all x not covered by Propositions 3.2–3.3. For simplicity, we made choices for each ε and $\mathcal{Z}$ to five decimal places and three decimal places respectively.

3.1. Proof of Proposition 3.1. Make the same assumptions as in the statement of Proposition 3.1. To begin, recall that Cully-Hugill and Johnston [8] have proved that if $x \geq e^{1000}$ and $\max\{51, \log x\} < 2T < (x^{1/10} - 2)/4$, then there exists a point $T \leq T^* \leq 2T$ such that

$$\begin{aligned} (3.3) \quad |\psi(x+h) - \psi(x) - h + \Sigma(T^*)| \\ \leq \frac{1.26}{T} ((x+h)(\log(x+h))^{8/10} + x(\log x)^{8/10}), \end{aligned}$$

where

$$\Sigma(T^*) = \sum_{|\gamma| \leq T^*} \frac{(x+h)^{\varrho} - x^{\varrho}}{\varrho} = \sum_{|\gamma| \leq T^*} \int_x^{x+h} t^{\varrho-1} dt.$$

The constants $1/10$, 1.26 , and $8/10$ can be replaced with $1/85$, 6.391 , and $1/10$ respectively by analysing the computations in [8]. We proceed with $1/10$, 1.26 , and $8/10$ for the remainder of this proof, but one can repeat the same arguments with these updated values to obtain the advertised refinement.

Next, recall from [5, (21), (31)] and [4, Cor. 5.1] that for $\log x \geq 300$, we have

$$(3.4) \quad 0.999x^{1/2} + 0.885x^{1/3} < \psi(x) - \theta(x) < \alpha_1 x^{1/2} + \alpha_2 x^{1/3}.$$

Improvements are available for the constants 0.999 and 0.885 on the final range of x under consideration in this proof, however these updates would make no difference to our final computations. It follows from (3.4) that if $\log x \geq 300$, then

$$\begin{aligned} & |\psi(x+h) - \psi(x) - \theta(x+h) + \theta(x)| \\ & < \left(\alpha_1 \left(1 + \frac{h}{x}\right)^{1/2} - 0.999 \right) \sqrt{x} + \left(\alpha_2 \left(1 + \frac{h}{x}\right)^{1/3} - 0.885 \right) x^{1/3} \\ & \leq (\alpha_1(1 + kx^{-1/k})^{\frac{1}{2}} - 0.999) \sqrt{x} + (\alpha_2(1 + kx^{-1/k})^{1/3} - 0.885) x^{1/3}. \end{aligned}$$

Therefore, this and (3.3) imply

$$\begin{aligned} (3.5) \quad & |\theta(x+h) - \theta(x) - h| \\ & \leq |\Sigma(T^*)| + \frac{1.26}{T} ((x+h)(\log(x+h))^{8/10} + x(\log x)^{8/10}) \\ & \quad + (\alpha_1(1 + kx^{-1/k})^{1/2} - 0.999) \sqrt{x} \\ & \quad + (\alpha_2(1 + kx^{-1/k})^{1/3} - 0.885) x^{1/3}. \end{aligned}$$

We bound $|\Sigma(T^*)|$ by observing

$$|\Sigma(T^*)| \leq \sum_{|\gamma| \leq 2T} \int_x^{x+h} t^{\beta-1} dt \leq h \sum_{|\gamma| \leq 2T} x^{\beta-1}.$$

Further, we also have

$$\begin{aligned} \sum_{|\gamma| \leq 2T} (x^{\beta-1} - x^{-1}) &= \sum_{|\gamma| \leq 2T} \int_0^\beta x^{\sigma-1} \log x dy \\ &= 2 \left(\int_0^{\sigma_1} + \int_{\sigma_1}^{1-\nu(2T)} \right) N(\sigma, 2T) x^{\sigma-1} \log x d\sigma. \end{aligned}$$

Therefore, (2.5) and (3.1) imply

$$\begin{aligned} \frac{|\Sigma(T^*)|}{2h} &\leq N(2T) \left(\frac{1}{x} + \int_0^{\sigma_1} x^{\sigma-1} \log x d\sigma \right) + \int_{\sigma_1}^{1-\nu(2T)} N(\sigma, 2T) x^{\sigma-1} \log x d\sigma \\ &= x^{\sigma_1-1} N(2T) + \int_{\sigma_1}^{1-\nu(2T)} N(\sigma, 2T) x^{\sigma-1} \log x d\sigma \end{aligned}$$

$$\leq \frac{x^{\sigma_1-1}T \log(2T)}{\pi} + B_1(\log(2T))^{B_2} \log x \int_{\sigma_1}^{1-\nu(2T)} (xf(2T)^{-1})^{\sigma-1} d\sigma \\ + B_3(\log(2T))^{B_4}(x^{-\nu(2T)} - x^{\sigma_1-1}).$$

We assume $x \geq ef(2T)$ to ensure the integral evaluates to

$$\int_{\sigma_1}^{1-\nu(2T)} (xf(2T)^{-1})^{\sigma-1} d\sigma = \frac{(xf(2T)^{-1})^{-\nu(2T)} - (xf(2T)^{-1})^{\sigma_1-1}}{\log(xf(2T)^{-1})}.$$

Insert these observations into (3.5) to see

$$\frac{|\theta(x+h) - \theta(x) - h|}{h} \leq \frac{x^{\sigma_1-1}(2T) \log(2T)}{\pi} + 2B_3(\log(2T))^{B_4}(x^{-\nu(2T)} - x^{\sigma_1-1}) \\ + \frac{2B_1(\log(2T))^{B_2} \log x}{\log(xf(2T)^{-1})}((xf(2T)^{-1})^{-\nu(2T)} - (xf(2T)^{-1})^{\sigma_1-1}) \\ + \frac{2 \cdot 1.26x}{h(2T)} \left(\left(1 + \frac{h}{x}\right) (\log(x+h))^{8/10} + (\log x)^{8/10} \right) \\ + (\alpha_1(1 + kx^{-1/k})^{1/2} - 0.999) \frac{\sqrt{x}}{h} \\ + (\alpha_2(1 + kx^{-1/k})^{1/3} - 0.885) \frac{x^{1/3}}{h}.$$

Finally, we take $2T = x^\alpha$ such that $1/k < \alpha < 1/10$ and $\alpha < 1 - \sigma_1$ to ensure that the final bound decreases as $x \rightarrow \infty$. Proposition 3.1 follows naturally, upon combination with the following simple observations:

$$x^{-\nu(x^\alpha)} = (\alpha \log x)^{-\frac{1}{Z\alpha}} \quad \text{and} \quad \log(x+h) \leq \log x + \frac{h}{x}.$$

4. Primes between consecutive powers. Let $a_n \geq 1$ be an increasing sequence of positive integers. For any fixed $k \geq 3$, there is at least one prime in the interval (a_n^k, a_{n+1}^k) for every integer $n \geq n_0$ if and only if $\theta(a_{n+1}^k) - \theta(a_n^k) > 0$. If $n = 1$, then $a_2 \geq 2$ and it is known that there is at least one prime in the interval $(1^k, 2^k)$, so we freely restrict our attention to $n \geq 2$.

Note that

$$a_{n+1}^k - a_n^k \geq (a_{n+1} - a_n)ka_n^{k-1} \geq ka_n^{k-1}.$$

Therefore, to prove the condition $\theta(a_{n+1}^k) - \theta(a_n^k) > 0$ holds for all $a_n \geq 2$, it suffices to prove $\theta(a_n^k + ka_n^{k-1}) - \theta(a_n^k) > 0$. Using the substitution $x = a_n^k$ this is equivalent to the sufficient condition (3.2).

We also have $a_n^k \geq 2^k > 3.68 \cdot 10^{19}$ on $n \geq 2$ for every $k \geq 65$. Therefore, for any fixed $k \geq 65$, to prove there is at least one prime in the interval (a_n^k, a_{n+1}^k) for every integer $n \geq 1$, it suffices to prove (3.2) for all $x \geq 3.68 \cdot 10^{19}$.

Proof of Theorem 1.2. Fix $k = 86$ and take $a_n = n$. Proposition 3.2 tells us the inequality (3.2) is true for all $x \geq \exp(2010)$ and Proposition 3.3 tells us (3.2) is true for all $3.68 \cdot 10^{19} \leq x \leq \exp(2640)$. Therefore, (3.2) is true for all $x \geq 3.68 \cdot 10^{19}$ and the result follows. ■

Proof of Theorem 1.3. Fix $k = 70$, $N \geq 0$, and take $a_n = n(1 + \lfloor N/n \rfloor)$ for each integer $n \geq 1$. Proposition 3.2 implies the inequality (3.2) is true for all $x \geq \exp(15951)$ and Proposition 3.3 tells us (3.2) is true for all $3.68 \cdot 10^{19} \leq x \leq \exp(2135)$. All that remains is to prove the inequality (3.2) is true for all

(4.1)

$$\exp(2135) < x < \exp(15951) \iff \exp\left(\frac{2135}{k}\right) < a_n < \exp\left(\frac{15951}{k}\right).$$

Now, the definition of a_n implies $n \leq a_n \leq n + N$. It follows that (4.1) is true when

$$\exp\left(\frac{2135}{k}\right) < n \quad \text{and} \quad N < \exp\left(\frac{15951}{k}\right) - \exp\left(\frac{2135}{k}\right).$$

Therefore, if $N \geq \exp\left(\frac{15951}{k}\right) - \exp\left(\frac{2135}{k}\right)$, then there are no $n \geq 1$ such that a_n lies in the interval given in (4.1). It follows that if $N \geq \exp\left(\frac{15951}{k}\right) - \exp\left(\frac{2135}{k}\right)$, then the result is proved for every $n \geq 1$. ■

Proof of Theorem 1.4. Fix $k \in \{85, 80, 75, 70\}$ and take $a_n = n$. Proposition 3.2 tells us (3.2) is true for all $x \geq \exp(\hat{x}(k))$, where $\hat{x}(k)$ is defined as in Proposition 3.4. Further, Proposition 3.4 certifies that if $\zeta(\sigma + it) \neq 0$ in the region

$$1 - \frac{\log \log |t|}{z_k \log |t|} \leq \sigma \leq 1 \quad \text{and} \quad |t| \leq e^{\hat{x}(k)(\frac{c_{z_k}}{k})},$$

then (3.2) holds in the extended range $x \geq \exp(x_Z)$, where values for x_Z are given in Table 5. Since

$$x_Z \leq k \log \left(k \cdot 2.51949 \cdot 10^{11} \left(1 - \frac{1}{2.51949 \cdot 10^{11}} \right)^2 \right),$$

it would follow that (3.2) is true for all $x \geq 3.68 \cdot 10^{19}$, which implies the result. Note that the values of z_k and T_k presented in Table 1 correspond to the values of Z and $\hat{x}(k)c_Z/k$ associated to k in Table 5. ■

Appendix A. Tables. We store the tables referred to throughout this paper in this appendix.

Table 2. Admissible computations for B and C in the bound (2.6) in the intervals $[T_0, T_1]$

$\log T_0$	$\log T_1$	C	B	$\log T_0$	$\log T_1$	C	B
80	90	370655.73	5.216	170.2	500	$1.12 \cdot 10^6$	3.337
90	100	425721.47	4.831	500	1000	$6.23 \cdot 10^6$	2.152
100	110	488901.14	4.513	1000	1500	$2.12 \cdot 10^7$	1.820
110	120	545744.21	4.264	1500	2000	$6.47 \cdot 10^7$	1.684
120	130	629490.27	4.032	2000	2500	$1.73 \cdot 10^8$	1.610
130	140	694045.43	3.855	2500	3000	$2.56 \cdot 10^8$	1.577
140	150	771373.78	3.696	3000	481958	$5.76 \cdot 10^8$	1.551
150	160	825913.07	3.572	481958	$6.7 \cdot 10^{12}$	$1.62 \cdot 10^{11}$	1.448
160	170.2	909966.07	3.448				

Table 3. Admissible x_A, c_A, x_B, c_B for $k \in [65, 90]$ in Proposition 3.2

k	x_A	c_A	x_B	c_B	k	x_A	c_A	x_B	c_B
90	1997	1.29478	-	-	77	18427	1.03221	5592	1.10158
89	1999	1.27912	-	-	76	23736	1.03001	5601	1.08553
88	2001	1.26348	-	-	75	29735	1.02151	5880	1.07041
87	2005	1.24663	-	-	74	43186	1.02001	6561	1.06001
86	2010	1.22924	-	-	73	58051	1.01212	7554	1.05201
85	6162	1.08141	5587	1.21712	72	96051	1.01001	9036	1.04341
84	6828	1.07501	5587	1.20280	71	212515	1.01001	11620	1.03505
83	7546	1.07001	5587	1.18848	70	580009	1.01001	15951	1.03001
82	8450	1.06231	5587	1.17416	69	2133517	1.01001	23148	1.02001
81	9685	1.06001	5588	1.15963	68	12318302	1.01001	43001	1.01051
80	10986	1.05001	5588	1.14532	67	37420887	1.01001	238050	1.00301
79	12799	1.04401	5588	1.13100	66	39361821	1.01001	-	-
78	15286	1.04001	5589	1.11648	65	41432145	1.01001	-	-

Table 4. Admissible pairs of x_0 and Δ

$\log x_0$	Δ	$\log x_0$	Δ
$\log(4 \cdot 10^{18})$	$3.90970 \cdot 10^7$	90	$1.84304 \cdot 10^{11}$
43	$4.18168 \cdot 10^7$	105	$1.91886 \cdot 10^{11}$
46	$1.63940 \cdot 10^8$	120	$1.97917 \cdot 10^{11}$
50	$1.06120 \cdot 10^9$	135	$2.02553 \cdot 10^{11}$
55	$1.02884 \cdot 10^{10}$	150	$2.07053 \cdot 10^{11}$
60	$7.69184 \cdot 10^{10}$	300	$2.30126 \cdot 10^{11}$
75	$1.74043 \cdot 10^{11}$	600	$2.51949 \cdot 10^{11}$

Table 5. Admissible $\mathcal{Z}$, $x_{\mathcal{Z}}$, $c_{\mathcal{Z}}$ for $k \in [65, 90]$ in Proposition 3.4

k	$\mathcal{Z}$	$x_{\mathcal{Z}}$	$c_{\mathcal{Z}}$	$\frac{\hat{x}(k)c_{\mathcal{Z}}}{k}$	k	$\mathcal{Z}$	$x_{\mathcal{Z}}$	$c_{\mathcal{Z}}$	$\frac{\hat{x}(k)c_{\mathcal{Z}}}{k}$
90	20.525	2767	1.02401	-	77	15.549	2355	1.16001	84.24384
89	20.281	2734	1.02411	-	76	15.336	2324	1.16001	85.48968
88	20.039	2704	1.02442	-	75	15.122	2292	1.16001	90.94478
87	19.795	2672	1.02352	-	74	14.909	2261	1.16001	102.84899
86	19.460	2640	1.01176	-	73	14.695	2229	1.16001	120.03720
85	17.270	2609	1.15401	75.85240	72	14.482	2198	1.16001	145.58125
84	17.055	2577	1.15402	76.75607	71	14.268	2166	1.16001	189.84952
83	16.841	2545	1.15413	77.68824	70	14.055	2135	1.16001	264.33314
82	16.627	2514	1.15423	78.64248	69	13.841	2103	1.16001	389.15814
81	16.413	2482	1.15434	79.63521	68	13.628	2072	1.16001	733.55279
80	16.198	2450	1.15451	80.64252	67	13.414	2040	1.16001	4121.49822
79	15.985	2419	1.15501	81.69868	66	13.201	2009	1.16001	691819.78755
78	15.770	2387	1.15501	82.76091	65	12.987	1977	1.16001	739410.80802

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