

An improvement of Konstantoulas' density constant

by

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Abstract. Let $A \subseteq \mathbb{N}$, and define its ordered representation function

$$r(n) = \#\{(a, b) \in A \times A : a + b = n\}.$$

The Erdős–Turán conjecture asserts that, if $r(n) > 0$ for all sufficiently large n , then $r(n)$ is unbounded. Konstantoulas proved a density-theoretic version: if the upper density of $E = \mathbb{N} \setminus (A + A)$ is less than $1/10$, then $\limsup_{n \rightarrow \infty} r(n) > 5$. In this paper, we improve Konstantoulas' constant to $7/32$. We also prove that if E has upper density less than 1 , then $\limsup_{n \rightarrow \infty} r(n) > 3$, and give a conditional criterion forcing $\limsup_{n \rightarrow \infty} r(n) > 7$.

1. Introduction. Throughout the paper, $\mathbb{N}$ denotes the set of positive integers. A set $A \subseteq \mathbb{N}$ is called an *asymptotic basis* of order 2 if every sufficiently large integer can be written as a sum of two elements of A . We write

$$r(n) = \#\{(a, b) \in A \times A : a + b = n\}$$

for the ordered representation function. The Erdős–Turán conjecture, first proposed in 1941 [6], asserts the following.

CONJECTURE 1 (Erdős–Turán). *If A is an asymptotic basis of order 2, then $r(n)$ is unbounded. Equivalently,*

$$\limsup_{n \rightarrow \infty} r(n) = \infty.$$

Even very small lower bounds for $\limsup_{n \rightarrow \infty} r(n)$ are difficult to obtain. Grekos, Haddad, Helou and Pihko [7] in 2003 proved that, for bases of order 2 in the stronger sense that every positive integer is represented, $r(n) \geq 6$ for infinitely many n . Borwein, Choi and Chu [1] in 2006 showed that $\sup_n r(n) \geq 8$, again under this stronger basis assumption. These two

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results use this stronger basis assumption in an essential way and are computational in nature. Konstantoulas [9] considered a different, density-theoretic setting: if the exceptional set $E = \mathbb{N} \setminus (A + A)$ has upper density less than $1/10$, then $\limsup_{n \rightarrow \infty} r(n) > 5$.

Several related variants have also been studied. In the setting of $\mathbb{N}$, Chen [3] constructed an asymptotic basis of order 2 whose two-fold representation function has the minimal value on a density-one set. Over $\mathbb{Z}$, Nathanson [10] constructed arbitrarily sparse unique representation bases, showing that the direct integer analogue of the Erdős–Turán conjecture is false. Finite cyclic analogues over $\mathbb{Z}/m\mathbb{Z}$ also exhibit bounded representation phenomena: Ruzsa [11] introduced the underlying finite construction, and Chen [2] proved the uniform bound $R_m \leq 288$ for Ruzsa’s numbers. Subsequent work by Sándor and Yang [12] refined the lower bound; Ding and Zhao [5], followed more recently by Ding, Sun, and Zhao [4], improved the upper bound.

The present paper remains in the setting $A \subseteq \mathbb{N}$. For $E \subseteq \mathbb{N}$, write

$$D(E) = \limsup_{n \rightarrow \infty} \frac{|E \cap \{1, \dots, n\}|}{n}$$

for its upper density. Our first result improves Konstantoulas’ constant.

THEOREM 1. *Let $A \subseteq \mathbb{N}$, and let $E = \mathbb{N} \setminus (A + A)$. If*

$$D(E) < \frac{7}{32},$$

then

$$\limsup_{n \rightarrow \infty} r(n) > 5.$$

The improvement from $1/10$ to $7/32$ comes from one simple structural observation. If $r(n)$ is eventually bounded, then the levels on which $r(n)$ is odd have zero Abel density. This leaves only the even levels in the main term of the final L^1/L^2 estimate. Optimizing the remaining one-parameter inequality gives the constant $7/32$.

Using a classical result of Erdős, we obtain the following theorem.

THEOREM 2. *Let $A \subseteq \mathbb{N}$, and let $E = \mathbb{N} \setminus (A + A)$. If*

$$D(E) < 1,$$

then

$$\limsup_{n \rightarrow \infty} r(n) > 3.$$

The strict threshold in Theorem 2 is optimal. It cannot be replaced by $D(E) \leq 1$: for example, if $A = \{2^m : m \geq 1\}$, then $r(n) \leq 2$ for all n , while $A + A$ has zero density and hence $D(E) = 1$. The proof of Theorem 2 uses the following classical theorem of Erdős, recorded in Halberstam and

Roth [8, Ch. II, §3, Thm. 8, p. 89]. In its usual Sidon form, it says that for some absolute constant c ,

$$\liminf_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n^{1/2}} (\log n)^{1/2} \leq c$$

for every infinite Sidon set A . We shall use the standard bounded difference multiplicity consequence of the same proof.

The next result records the additional information on the level $r(n) = 4$ that is needed when the same method is pushed toward the stronger conclusion $\limsup_{n \rightarrow \infty} r(n) > 7$.

THEOREM 3. *Let $A \subseteq \mathbb{N}$, let $E = \mathbb{N} \setminus (A + A)$, and put*

$$N_4 = N_4(A) = \{n \in \mathbb{N} : r(n) = 4\}, \quad S_4(z) = \sum_{n \in N_4} z^n.$$

Define the lower Abel density

$$\underline{\delta}_4 = \liminf_{t \rightarrow 1^-} (1 - t)S_4(t).$$

If

$$\underline{\delta}_4 > \frac{1}{16} + 3D(E),$$

then

$$\limsup_{n \rightarrow \infty} r(n) > 7.$$

The paper is organized as follows. In Section 2 we record the common generating-function setup and prove the vanishing of the odd levels. In Section 3 we prove Theorems 1 and 2, explain the one-parameter optimality of the constant $7/32$, and record the sharpness of the threshold 1 in Theorem 2. Finally, in Section 4 we discuss higher representation thresholds and prove Theorem 3.

2. The generating-function setup. Let $A \subseteq \mathbb{N}$. Suppose, for contradiction, that

$$r(n) \leq k$$

for all sufficiently large n . For $0 \leq j \leq k$, let

$$N_j = N_j(A) = \{n \in \mathbb{N} : r(n) = j\}, \quad S_j(z) = \sum_{n \in N_j} z^n.$$

Then $N_0 = E$. Let

$$g(z) = \sum_{a \in A} z^a$$

be the generating function of A . After absorbing finitely many initial terms into polynomials, we have

$$(2.1) \quad S_0(z) + \sum_{i=1}^k S_i(z) = \frac{1}{1-z} - P_{1,k}(z),$$

$$(2.2) \quad g(z)^2 = P_{2,k}(z) + \sum_{i=1}^k i S_i(z),$$

where $P_{1,k}$ and $P_{2,k}$ are polynomials.

We shall use the following elementary Abel-density bound.

LEMMA 1. *Let $E \subseteq \mathbb{N}$, let $S_E(t) = \sum_{n \in E} t^n$, and let $D(E)$ be the upper density of E . Then*

$$\limsup_{t \rightarrow 1^-} (1-t)S_E(t) \leq D(E).$$

Proof. See [9, Lemma 2 and Corollary 3]. Let $E(x) = |E \cap \{1, \dots, x\}|$. Fix $\eta > 0$. For all sufficiently large n , we have $E(n) \leq (D(E) + \eta)n$. Since

$$S_E(t) = (1-t) \sum_{n \geq 1} E(n)t^n,$$

it follows, after separating finitely many initial terms, that

$$\limsup_{t \rightarrow 1^-} (1-t)S_E(t) \leq D(E) + \eta.$$

Letting $\eta \rightarrow 0$ proves the lemma. ■

Fix $\varepsilon > 0$. By Lemma 1, there is a sequence $r_m \rightarrow 1^-$ such that

$$S_0(r_m^2) \leq \frac{D(E) + \varepsilon}{1 - r_m^2}.$$

Passing to a subsequence if necessary, we may assume that all limits

$$\ell_j = \lim_{m \rightarrow \infty} (1 - r_m^2) S_j(r_m^2), \quad 0 \leq j \leq k,$$

exist. Then (2.1) gives

$$(2.3) \quad \sum_{j=0}^k \ell_j = 1, \quad \ell_0 \leq D(E) + \varepsilon.$$

The following proposition is the reason the constant in Konstantoulas' theorem can be improved here: once the odd representation levels have zero limiting mass, the final optimization is carried out only over the even levels.

PROPOSITION 1. *For each odd j , $1 \leq j \leq k$, one has*

$$\ell_j = 0.$$

Proof. For each n , the non-diagonal representations $n = a + b$, $a \neq b$, occur in pairs (a, b) and (b, a) . Hence the parity of $r(n)$ is determined by the diagonal representation $n = 2a$, if such a representation exists. Apart from the finitely many n for which $r(n) > k$, this gives

$$(2.4) \quad \sum_{\substack{1 \leq i \leq k \\ i \text{ odd}}} S_i(z) = g(z^2) - P_{3,k}(z),$$

where $P_{3,k}$ is a polynomial accounting for the exceptional initial terms.

For $0 < r < 1$, (2.4) gives

$$\sum_{\substack{1 \leq i \leq k \\ i \text{ odd}}} S_i(r^2) \leq g(r^4) + O(1).$$

On the other hand, the assumption $r(n) \leq k$ eventually implies

$$g(r)^2 = \sum_{n \geq 0} r(n) r^n = O\left(\frac{1}{1-r}\right) \quad (r \rightarrow 1^-).$$

Hence $g(r^4) = O((1-r^4)^{-1/2})$. Therefore

$$(1-r^2) \sum_{\substack{1 \leq i \leq k \\ i \text{ odd}}} S_i(r^2) \rightarrow 0 \quad (r \rightarrow 1^-),$$

and so every odd ℓ_j is zero. ■

We shall also use the following one-parameter form of Konstantoulas' L^1/L^2 estimate.

LEMMA 2. Assume that $r(n) \leq 2h + 1$ for all sufficiently large n . Let $\lambda \in \mathbb{R}$, and suppose that $\sum_{s=1}^h 2s \ell_{2s} > 0$. With the notation above, we have

$$(2.5) \quad \sqrt{\frac{\sum_{s=0}^h (2s - \lambda)^2 \ell_{2s}}{\sum_{s=1}^h 2s \ell_{2s}}} \geq 1.$$

Proof. Put $K = 2h + 1$. Combining (2.1) and (2.2), we get

$$(2.6) \quad \begin{aligned} g(z)^2 &= P_{2,K}(z) + \lambda(S_0(z) + \cdots + S_K(z)) + \sum_{j=0}^K (j - \lambda) S_j(z) \\ &= \frac{\lambda}{1-z} + P_{2,K}(z) - \lambda P_{1,K}(z) + \sum_{j=0}^K (j - \lambda) S_j(z). \end{aligned}$$

Let $z = re^{i\theta}$. Since $|g(z)^2| = |g(z)|^2$, we have

$$\int_{-\pi}^{\pi} |g(re^{i\theta})|^2 d\theta = \int_{-\pi}^{\pi} |g(re^{i\theta})|^2 d\theta = 2\pi g(r^2).$$

The polynomial terms in (2.6) contribute $O(1)$ to this integral, and

$$\int_{-\pi}^{\pi} \left| \frac{1}{1 - re^{i\theta}} \right| d\theta = O\left(\log \frac{1}{1 - r}\right).$$

By Proposition 1 and Cauchy's inequality,

$$\int_{-\pi}^{\pi} |S_j(re^{i\theta})| d\theta \leq 2\pi S_j(r^2)^{1/2} = o((1 - r^2)^{-1/2})$$

for every odd j , along the chosen sequence $r = r_m$.

Thus the main contribution comes from the even levels. Cauchy's inequality and the disjointness of the coefficient supports of $S_0, S_2, \dots, S_{2h}$ give

$$\int_{-\pi}^{\pi} \left| \sum_{s=0}^h (2s - \lambda) S_{2s}(re^{i\theta}) \right| d\theta \leq 2\pi \left(\sum_{s=0}^h (2s - \lambda)^2 S_{2s}(r^2) \right)^{1/2}.$$

Consequently,

$$g(r^2) \leq \left(\sum_{s=0}^h (2s - \lambda)^2 S_{2s}(r^2) \right)^{1/2} + o((1 - r^2)^{-1/2})$$

along the chosen sequence $r = r_m$.

Finally, (2.2) and Proposition 1 imply

$$(1 - r^2)g(r^2)^2 \rightarrow \sum_{s=1}^h 2s \ell_{2s}.$$

Multiplying the preceding inequality by $(1 - r^2)^{1/2}$ and letting $r = r_m \rightarrow 1^-$, we obtain

$$\left(\sum_{s=1}^h 2s \ell_{2s} \right)^{1/2} \leq \left(\sum_{s=0}^h (2s - \lambda)^2 \ell_{2s} \right)^{1/2}.$$

Since the denominator is positive by hypothesis, this is (2.5). ■

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Assume, to the contrary, that $r(n) \leq 5$ for all sufficiently large n . Choose $\varepsilon > 0$ so small that

$$D(E) + \varepsilon < \frac{7}{32}.$$

Form the sequence $r_m \rightarrow 1^-$ and the limits ℓ_j as in Section 2. By Proposition 1,

$$\ell_1 = \ell_3 = \ell_5 = 0.$$

Hence, by (2.3),

$$\ell_0 + \ell_2 + \ell_4 = 1, \quad \ell_0 \leq D(E) + \varepsilon < \frac{7}{32}.$$

In particular,

$$\ell_2 + \ell_4 = 1 - \ell_0 > 0,$$

so the denominator $2\ell_2 + 4\ell_4$ in Lemma 2 is positive. Taking $h = 2$ and $\lambda = 5/2$ in Lemma 2, we obtain

$$(3.1) \quad 1 \leq \sqrt{\frac{\frac{25}{4}\ell_0 + \frac{1}{4}\ell_2 + \frac{9}{4}\ell_4}{2\ell_2 + 4\ell_4}}.$$

On the other hand,

$$\frac{25}{4}\ell_0 + \frac{1}{4}\ell_2 + \frac{9}{4}\ell_4 < 2\ell_2 + 4\ell_4.$$

Indeed, after multiplying by 4, this inequality is equivalent to

$$25\ell_0 + \ell_2 + 9\ell_4 < 8\ell_2 + 16\ell_4,$$

or

$$25\ell_0 < 7(\ell_2 + \ell_4) = 7(1 - \ell_0),$$

which is precisely $\ell_0 < 7/32$. Thus the right-hand side of (3.1) is strictly smaller than 1, a contradiction. ■

Proof of Theorem 2. Assume, to the contrary, that $r(n) \leq 3$ for all sufficiently large n . Choose n_0 such that $r(n) \leq 3$ for all $n \geq n_0$, and choose M with $2M \geq n_0$. Put

$$B = A \cap \{M, M+1, \dots\}, \quad F = A \setminus B.$$

If B is finite, then $A + A$ has zero density, and hence $D(E) = 1$. Thus we may suppose that B is infinite.

We first check that positive differences in B have multiplicity at most two. Fix $d \geq 1$, and suppose that both

$$u, u+d, v, v+d \in B, \quad u < v.$$

If $v \neq u+d$, then

$$u + (v+d) = v + (u+d)$$

gives two distinct non-diagonal representations of an integer at least $2M$. This contributes at least four ordered representations, contrary to the choice of M . Therefore two distinct representations of the same difference d must have lower endpoints differing by exactly d , and there can be at most two of them, since three lower endpoints cannot be pairwise separated by d .

We now apply Erdős' difference-counting argument for infinite Sidon sequences; see Halberstam and Roth [8, Ch. II, §3, Thm. 8, p. 89]. The Sidon hypothesis in that proof is used only to bound the multiplicity of each positive difference. Since every positive difference of B has multiplicity at most two, the same argument gives an absolute constant C such that

$$\liminf_{x \rightarrow \infty} \frac{B(x)(\log x)^{1/2}}{x^{1/2}} \leq C,$$

where $B(x) = |B \cap \{1, \dots, x\}|$. In particular, there is a sequence $x_m \rightarrow \infty$ such that

$$B(x_m) = o(x_m^{1/2}).$$

Along this sequence,

$$\begin{aligned} |(A + A) \cap \{1, \dots, x_m\}| &\leq |(B + B) \cap \{1, \dots, x_m\}| \\ &\quad + |(F + B) \cap \{1, \dots, x_m\}| + O(1) \\ &\leq B(x_m)^2 + |F|B(x_m) + O(1) = o(x_m). \end{aligned}$$

Consequently, $D(E) = 1$, contradicting the hypothesis $D(E) < 1$. ■

However, using the same L^1/L^2 estimate as in the proof of Theorem 1, we obtain only the threshold $1/2$ in Theorem 2.

The threshold 1 in Theorem 2 is sharp. Indeed, let

$$A = \{2^m : m \geq 1\}.$$

Every integer has at most one unordered representation as a sum of two elements of A , and hence $r(n) \leq 2$ for all n . On the other hand,

$$|(A + A) \cap \{1, \dots, x\}| = O((\log x)^2),$$

so $A + A$ has density zero and $D(\mathbb{N} \setminus (A + A)) = 1$. Thus the strict condition $D(E) < 1$ cannot be weakened to $D(E) \leq 1$.

We now explain why the constant $7/32$ in Theorem 1 cannot be improved merely by changing the coefficient λ in Lemma 2 in the one-parameter decomposition above. After the odd classes have been shown to be negligible, the relevant data for the proof of Theorem 1 are

$$\ell_0, \ell_2, \ell_4 \geq 0, \quad \ell_0 + \ell_2 + \ell_4 = 1.$$

The method needs a choice of λ for which

$$(3.2) \quad \lambda^2 \ell_0 + (2 - \lambda)^2 \ell_2 + (4 - \lambda)^2 \ell_4 < 2\ell_2 + 4\ell_4.$$

For fixed ℓ_0, ℓ_2, ℓ_4 , the left-hand side is minimized when

$$\lambda = 2\ell_2 + 4\ell_4.$$

The threshold $7/32$ is sharp for this optimization. Indeed, take

$$\ell_0 = \frac{7}{32}, \quad \ell_2 = \frac{5}{16}, \quad \ell_4 = \frac{15}{32}.$$

Then

$$\ell_0 + \ell_2 + \ell_4 = 1, \quad 2\ell_2 + 4\ell_4 = \frac{5}{2}.$$

The minimizing parameter is $\lambda = 5/2$, and

$$\left(\frac{5}{2}\right)^2 \frac{7}{32} + \left(2 - \frac{5}{2}\right)^2 \frac{5}{16} + \left(4 - \frac{5}{2}\right)^2 \frac{15}{32} = \frac{5}{2}.$$

Therefore equality holds in (3.1). The argument cannot produce a contradiction at $\ell_0 = 7/32$, and hence cannot prove any threshold larger than $7/32$ without additional information.

4. Proof of Theorem 3. We recall the notation from Theorem 3:

$$N_4 = N_4(A) = \{n \in \mathbb{N} : r(n) = 4\}, \quad S_4(z) = \sum_{n \in N_4} z^n,$$

and define the lower Abel density

$$\underline{\delta}_4 = \liminf_{t \rightarrow 1^-} (1-t)S_4(t).$$

The calculation below shows that this method needs extra mass on the level $r(n) = 4$ in order to force the stronger conclusion $r(n) > 7$.

Proof of Theorem 3. Assume, to the contrary, that $r(n) \leq 7$ for all sufficiently large n . Choose $\varepsilon > 0$ so small that

$$\underline{\delta}_4 > \frac{1}{16} + 3(D(E) + \varepsilon).$$

As before, after passing to a suitable sequence $r_m \rightarrow 1^-$, all limits

$$\ell_j = \lim_{m \rightarrow \infty} (1 - r_m^2) S_j(r_m^2)$$

exist for $j = 0, 2, 4, 6$, and the odd classes are lower order. Hence

$$\ell_0 + \ell_2 + \ell_4 + \ell_6 = 1, \quad \ell_0 \leq D(E) + \varepsilon, \quad \ell_4 \geq \underline{\delta}_4$$

by the definition of the lower Abel density. In particular, $\ell_4 > 0$, so the denominator condition in Lemma 2 is satisfied. Taking $h = 3$ in Lemma 2, the optimized parameter argument gives the necessary inequality

$$(4.1) \quad \min_{\lambda} (\lambda^2 \ell_0 + (2 - \lambda)^2 \ell_2 + (4 - \lambda)^2 \ell_4 + (6 - \lambda)^2 \ell_6) \geq 2\ell_2 + 4\ell_4 + 6\ell_6.$$

Write

$$\mu = 2\ell_2 + 4\ell_4 + 6\ell_6.$$

The minimum on the left of (4.1) is

$$4\ell_2 + 16\ell_4 + 36\ell_6 - \mu^2.$$

Using $\ell_0 + \ell_2 + \ell_4 + \ell_6 = 1$, one checks that

$$4\ell_2 + 16\ell_4 + 36\ell_6 = 8\mu - 12 + 12\ell_0 - 4\ell_4.$$

Thus the inequality opposite to (4.1), namely the desired contradiction, is

$$\mu^2 - 7\mu + 12 - 12\ell_0 + 4\ell_4 > 0.$$

The quadratic $\mu^2 - 7\mu + 12$ has global minimum $-1/4$, attained at $\mu = 7/2$. Therefore a sufficient condition for contradiction is

$$-\frac{1}{4} - 12\ell_0 + 4\ell_4 > 0,$$

that is,

$$\ell_4 > \frac{1}{16} + 3\ell_0.$$

Since $\ell_0 \leq D(E) + \varepsilon$ and $\ell_4 \geq \underline{\delta}_4$, our choice of ε gives

$$\ell_4 > \frac{1}{16} + 3\ell_0,$$

which yields the contradiction. ■

In particular, for an asymptotic basis, where $D(E) = 0$, any counterexample to the conclusion $r(n) > 7$ must satisfy

$$\underline{\delta}_4 \leq \frac{1}{16}.$$

Thus, if an asymptotic basis satisfied $r(n) \leq 7$ eventually, then the set of integers with exactly four ordered representations would have lower Abel density at most $1/16$.

The same calculation also explains why this one-parameter L^1/L^2 method cannot prove an unconditional theorem forcing $r(n) > 7$, even when $\ell_0 = 0$. For example, take

$$\ell_2 = \frac{5}{8}, \quad \ell_4 = 0, \quad \ell_6 = \frac{3}{8}.$$

Then

$$2\ell_2 + 4\ell_4 + 6\ell_6 = \frac{7}{2}.$$

The left hand side of (4.1) is minimized at

$$\lambda = 2\ell_2 + 4\ell_4 + 6\ell_6 = \frac{7}{2},$$

and the minimum is

$$\left(2 - \frac{7}{2}\right)^2 \frac{5}{8} + \left(6 - \frac{7}{2}\right)^2 \frac{3}{8} = \frac{15}{4}.$$

Since $15/4 > 7/2$, no contradiction follows.

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