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Potential theory for Green functions of Schrödinger-type operators

by

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Abstract. A potential theory for Green kernels of Schrödinger-type operators is considered. We give a condition for the Green kernel to satisfy Ugaheri's maximum principle in terms of the bottom of the spectrum of the time-changed process. The condition also leads to the continuity principle. For the proofs, we use probabilistic arguments, in particular, the strong Markov property and strong Feller property.

1. Introduction. Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Denote by $E_\triangle := E \cup \{\triangle\}$ the one-point compactification of E. Let $X = (\mathbb{P}_x, X_t, \zeta)$ be the m-symmetric Hunt process with lifetime $\zeta = \inf\{t > 0 \mid X_t = \triangle\}$ generated by a regular, transient Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E; m)$. We assume that X is irreducible, transient and strong Feller. Let μ be a positive measure in the Kato class. We denote by A_t^μ the positive continuous additive functional in the Revuz correspondence to μ, and define the Feynman–Kac semigroup $\{p_t^\mu\}_{t\geq 0}$ by

$$p_t^\mu f(x) = \mathbb{E}_x[e^{A_t^\mu} f(X_t)].$$

We define the resolvent $\{R_\alpha^\mu\}_{\alpha\geq 0}$ by

$$R_\alpha^\mu f(x) = \int_0^\infty e^{-\alpha t} p_t^\mu f(x)\, dt = \mathbb{E}_x\Big[\int_0^\zeta e^{-\alpha t + A_t^\mu} f(X_t)\, dt\Big].$$

By the strong Feller property of X, R_α^μ has an integral kernel $r_\alpha^\mu(x,y)$:

$$R_\alpha^\mu f(x) = \int_E r_\alpha^\mu(x,y) f(y)\, dm(y).$$

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For a positive Borel measure ν, define the r^μ-potential by

$$R^\mu\nu(x) = \int_E r^\mu(x,y)\,d\nu(y), \qquad r^\mu(x,y) = r_0^\mu(x,y).$$

The objective of this paper is to prove some basic principles (*Ugaheri's maximum principle*, *Cartan's maximum principle*, *the continuity principle*, *the energy principle*, *the balayage principle*) of potential theory for the kernel function $r^\mu(x,y)$.

In [13], we study the criticality theory for Schrödinger forms and give analytic criteria in terms of the principal eigenvalue of time-changed processes. More precisely, let $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu)\ (= \mathcal{D}(\mathcal{E})))$ be the Schrödinger form defined by

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_E u^2\,d\mu, \qquad u \in \mathcal{D}(\mathcal{E}^\mu),$$

where μ is a positive Green-tight Kato measure (Definition 2.1(2)). Define

$$\lambda(\mu) := \inf\left\{\mathcal{E}(u,u)\;\middle|\; u \in \mathcal{D}(\mathcal{E}),\ \int_E u^2\,d\mu = 1\right\}. \tag{1.1}$$

We prove in [13] that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is *subcritical* (resp. *critical*, *supercritical*) if and only if $\lambda(\mu) > 1$ (resp. $\lambda(\mu) = 1$, $\lambda(\mu) < 1$). Here $\lambda(\mu)$ can be identified with the principal eigenvalue of the time-changed process by A_t^μ.

This paper is a continuation of [13]. We prove that if $\lambda(\mu) > 1$, then $r^\mu(x,y)$ satisfies some principles in potential theory, for example, *Ugaheri's maximum principle* [14]: Let ν be a positive Radon measure with compact topological support S_ν. Then there exists a positive constant C such that

$$\sup_{x\in E} R^\mu\nu(x) \le C \sup_{x\in S_\nu} R^\mu\nu(x).$$

For the proof of Ugaheri's maximum principle, a necessary and sufficient condition for the *gaugeability* of the Feynman–Kac functional, $\exp(A_\zeta^\mu)$, is crucial. Indeed, let $g^\mu(x)$ be the *gauge function* defined by

$$g^\mu(x) = \mathbb{E}_x[e^{A_\zeta^\mu}]. \tag{1.2}$$

We then show in Theorem 4.7 that

$$\sup_{x\in E} R^\mu\nu(x) \le \sup_{x\in E} g^\mu(x) \cdot \sup_{x\in S_\nu} R^\mu\nu(x). \tag{1.3}$$

Hence applying the fact proved in [2] that $\sup_{x\in E} g^\mu(x) < \infty$ is equivalent to $\lambda(\mu) > 1$, we can conclude that if $\lambda(\mu) > 1$, then Ugaheri's maximum principle holds. If $\mu \equiv 0$, then equation (1.3) is *Frostman's maximum principle*:

$$\sup_{x\in E} R\nu(x) \le \sup_{x\in S_\nu} R\nu(x), \tag{1.4}$$

where R is the 0-resolvent of X. The condition that $\lambda(\mu) > 1$ expresses the smallness of perturbation by μ; indeed, if $\mu_1 \le \mu_2$, that is, $\mu_1(A) \le \mu_2(A)$ for all Borel sets A, then $\lambda(\mu_1) \ge \lambda(\mu_2)$. Hence, $\lambda(\mu)$ measures the size of μ, and equation (1.3) says that Ugaheri's maximum principle is stable under small perturbation of the potential μ. To the best of my knowledge, Ugaheri's maximum principle is proved under conditions on the shape of the kernel (cf. [9] ([1])). One of our point of emphasis is that this principle follows from the strong Markov property of X.

It is known that Ugaheri's maximum principle leads to the *continuity principle*: If $R^\mu \nu$ is continuous on S_ν, then it is continuous on the whole space E. We give another proof of this implication using the *strong Feller property* of p_t^μ, i.e., $p_t^\mu(\mathscr{B}_b(E)) \subset C_b(E)$. Here $\mathscr{B}_b(E)$ and $C_b(E)$ are the sets of bounded Borel functions and bounded continuous functions. Moreover, using the h-transform by the gauge function g^μ, we consider the energy principle and balayage principle.

2. Schrödinger forms. Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular irreducible Dirichlet form on $L^2(E;m)$. We write $u \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ if for any relatively compact open set $D \subset E$ there exists a function $v \in \mathcal{D}(\mathcal{E})$ such that $u = v$ m-a.e. on D. We denote by $\mathcal{D}_e(\mathcal{E})$ the family of m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}_{n=1}^\infty$ of functions in $\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $X = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \{\mathbb{P}_x\}_{x\in E}, \{X_t\}_{t\ge 0}, \zeta)$ be the symmetric Hunt process generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where $\{\mathscr{F}_t\}_{t\ge 0}$ is the augmented filtration and ζ is the lifetime of X. Denote by $\{p_t\}_{t\ge 0}$ and $\{R_\alpha\}_{\alpha\ge 0}$ the semigroup and the resolvent of X:

$$p_t f(x) = \mathbb{E}_x(f(X_t)), \qquad R_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x)\, dt.$$

We assume that X satisfies the next two conditions:

Irreducibility (I). If a Borel set A is p_t-*invariant*, that is, $p_t(1_A f)(x) = 1_A p_t f(x)$ m-a.e. for any $f \in L^2(E;m) \cap \mathscr{B}_b(E)$ and $t > 0$, then either $m(A) = 0$ or $m(E \setminus A) = 0$. Here $\mathscr{B}_b(E)$ is the space of bounded Borel functions on E.

Strong Feller property (SF). For each t, $p_t(\mathscr{B}_b(E)) \subset C_b(E)$, where $C_b(E)$ is the space of bounded continuous functions on E.

([1]) In [9], Ugaheri's maximum principle is called the generalized maximum principle.

We remark that (SF) implies

Absolute continuity condition (AC). The transition probability of X is absolutely continuous with respect to m: $p(t,x,dy)=p(t,x,y)m(dy)$ for each $t>0$ and $x\in E$.

Under (AC), there exists a non-negative, jointly measurable α-resolvent kernel $r_\alpha(x,y)$, $\alpha\geq 0$:

$$R_\alpha f(x)=\int_E r_\alpha(x,y)f(y)\,m(dy),\qquad x\in E,\ f\in\mathscr{B}_b(E).$$

Moreover, $r_\alpha(x,y)$ is α-excessive in x and in y (see [5, Lemma 4.2.4]). We then see that $r_\alpha(x,y)$ is uniquely determined. In this paper, we assume, in addition,

Transcience (T). The process X is *transient*, i.e. $r_0(x,y)<\infty$, $x\neq y$.

We simply write $r(x,y)$ for $r_0(x,y)$. For a measure μ, we define the *α-potential* of μ by

$$R_\alpha\mu(x)=\int_E r_\alpha(x,y)\,\mu(dy).$$

We define the *capacity* Cap associated with the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ as follows: for an open set $O\subset E$,

$$\mathrm{Cap}(O)=\inf\{\mathcal{E}_1(u,u)\mid u\in\mathcal{D}(\mathcal{E}),\ u\geq 1\ m\text{-a.e. on }O\},$$

and for a Borel set $A\subset E$,

$$\mathrm{Cap}(A)=\inf\{\mathrm{Cap}(O)\mid O\text{ is open, }O\supset A\},$$

where $\mathcal{E}_\alpha(u,u)=\mathcal{E}(u,u)+\alpha(u,u)_m$. When we wish to show the dependence on the Dirichlet form, we write $\mathcal{E}$-Cap.

A statement depending on $x\in E$ is said to hold *q.e.* (*quasi-everywhere*) on E if there exists a set $N\subset E$ of zero capacity such that the statement is true for every $x\in E\setminus N$. A real-valued function u defined q.e. on E is said to be *quasi-continuous* if for any $\epsilon>0$ there exists an open set $G\subset E$ such that $\mathrm{Cap}(G)<\epsilon$ and $u|_{E\setminus G}$ is finite and continuous. Here, $u|_{E\setminus G}$ denotes the restriction of u to $E\setminus G$. Each function u in $\mathcal{D}_e(\mathcal{E})$ admits a quasi-continuous version $\tilde{u}$, that is, $u=\tilde{u}$ m-a.e. In what follows, we always assume that every function $u\in\mathcal{D}_e(\mathcal{E})$ is represented by its quasi-continuous version.

Denote by $\mathcal{M}$ the set of positive Borel measures on E. We say $\nu\in\mathcal{M}$ is *of α-order finite energy integral* if

$$\mathcal{E}_\alpha(\nu):=\int_E\int_E r_\alpha(x,y)\,\nu(dx)\,\nu(dy)<\infty.$$

Denote by $\mathcal{S}_0^{(\alpha)}$ the set of positive Borel measures of α-order finite energy integral. Let $\mathcal{S}_{00}^{(\alpha)}$ $(\subset\mathcal{S}_0^{(\alpha)})$ be the set of positive Borel measures μ such that

$\mu(E) < \infty$ and $R_\alpha \mu$ is bounded. Note that the spaces $\mathcal{S}_0^{(\alpha)}$ and $\mathcal{S}_{00}^{(\alpha)}$ are independent of the choice $\alpha > 0$. We write $\mathcal{S}_0$ and $\mathcal{S}_{00}$ for $\mathcal{S}_0^{(1)}$ and $\mathcal{S}_{00}^{(1)}$, respectively.

An increasing sequence $\{K_n\}_{n=1}^\infty$ of compact sets is called a *generalized compact nest* if for any compact set K,

$$\lim_{n\to\infty} \mathrm{Cap}(K \setminus F_n) = 0. \tag{2.1}$$

We call a measure $\mu \in \mathcal{M}$ on E *smooth* if there exists a generalized compact nest $\{K_n\}_{n=1}^\infty$ such that $\mu\big(E \setminus \bigcup_{n=1}^\infty K_n\big) = 0$ and $1_{K_n} \cdot \mu \in S_{00}$ for each n (see [5, Theorem 2.2.4]). We denote by $\mathcal{S}$ the set of smooth measures. We call a measure $\mu \in \mathcal{M}$ on E *smooth in the strict sense* if there exists a sequence $\{E_n\}_{n=1}^\infty$ of Borel sets such that $1_{E_n} \cdot \mu \in \mathcal{S}_{00}$ for each n and

$$\mathbb{P}_x\Big(\lim_{n\to\infty} \sigma_n \geq \zeta\Big) = 1, \qquad \forall x \in E,$$

where $\sigma_n = \inf\{t > 0 \mid X_t \in E \setminus E_n\}$. In particular, if a Radon measure μ satisfies $\sup_{x\in E} R_\alpha\mu(x) < \infty$, then μ is smooth in the strict sense. We denote by $\mathcal{S}_1$ the set of measures smooth in the strict sense.

A stochastic process $\{A_t\}_{t\geq 0}$ is said to be an *additive functional* (AF for short) if the following conditions hold:

(i) $A_t(\cdot)$ is $\mathscr{F}_t$-measurable for all $t \geq 0$.
(ii) There exists a set $\Lambda \in \mathscr{F}_\infty = \sigma(\bigcup_{t\geq 0} \mathscr{F}_t)$ and an exceptional set $N \subset E$ such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in E\setminus N$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is a function satisfying $A_0 = 0$, $A_t(\omega) < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$. Here, θ_t is the shift operator on Ω defined by $X_s(\theta_t(\omega)) = X_{s+t}(\omega)$.

If an AF $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF). The set of all PCAF's is denoted by $\mathbf{A}_c^+$. In [5], $\{A_t\}_{t\geq 0}$ in $\mathbf{A}_c^+$ is called a *PCAF in the strict sense* if $N = \emptyset$. The set of all PCAF's in the strict sense is denoted by $\mathbf{A}_{c,1}^+$. The families $\mathcal{S}$ and $\mathcal{S}_1$ are in one-to-one correspondence to $\mathbf{A}_c^+$ and $\mathbf{A}_{c,1}^+$ respectively (*Revuz correspondence*): For each $\mu \in \mathcal{S}$ (resp. $\mu \in \mathcal{S}_1$), there exists a unique $\{A_t\}_{t\geq 0} \in \mathbf{A}_c^+$ (resp. $\{A_t\}_{t\geq 0} \in \mathbf{A}_{c,1}^+$) such that for any $f \in \mathscr{B}^+(E)$ and any γ-*excessive* function h ($\gamma \geq 0$), that is, $\lim_{t\downarrow 0} e^{-\gamma t} p_t h \uparrow h$, we have

$$\lim_{t\to 0} \frac{1}{t} \mathbb{E}_{h\cdot m}\Big[\int_0^t f(X_s)\, dA_s\Big] = \int_E f(x)h(x)\, \mu(dx) \tag{2.2}$$

(see [5, Theorem 5.1.7]). Here, $\mathbb{E}_{h\cdot m}[\,\cdot\,] = \int_E \mathbb{E}_x[\,\cdot\,]h(x)\, m(dx)$. We denote by A_t^μ the PCAF corresponding to $\mu \in S$. For a signed smooth measure $\mu = \mu^+ - \mu^-$, we define $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$.

Definition 2.1.

(1) A measure $\mu \in \mathcal{M}$ is said to be in the *Kato class* of X (briefly, in $\mathcal{K}$) if

$$\lim_{\alpha\to\infty} \|R_\alpha \mu\|_\infty = 0.$$

A measure $\mu \in \mathcal{M}$ is said to be in the *local Kato class* (briefly, in $\mathcal{K}_{\mathrm{loc}}$) if for any compact set K, $1_K \cdot \mu$ is in $\mathcal{K}$.

(2) Suppose that X is transient. A measure $\mu \in \mathcal{K}$ is said to be in the class $\mathcal{K}_\infty$ if for any $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$ such that

$$\sup_{x\in E} \int_{K^c} r(x,y)\,\mu(dy) < \epsilon.$$

A measure in $\mathcal{K}_\infty$ is called *Green-tight*.

We note that $\mathcal{K}_{\mathrm{loc}} \subset \mathcal{S}_1$. Indeed, a measure in $\mathcal{K}_{\mathrm{loc}}$ is a Radon measure because of the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the *Stollmann–Voigt inequality* [12]: for $\mu \in \mathcal{K}$,

$$\int_E u^2\,d\mu \le \|R_\alpha\mu\|_\infty \cdot \mathcal{E}_\alpha(u,u) \quad \text{for any } u \in \mathcal{D}(\mathcal{E}), \tag{2.3}$$

where $\mathcal{E}_\alpha(u,u) = \mathcal{E}(u,u) + \alpha(u,u)$ and (u,u) is the $L^2(E;m)$-inner product. We also see from [1, Theorem 3.9] that $\mu \in \mathcal{K}$ if and only if

$$\lim_{t\downarrow 0} \sup_{x\in E} \mathbb{E}_x[A_t^\mu] = \lim_{t\downarrow 0} \sup_{x\in E} \int_0^t \int_E p(s,x,y)\,\mu(dy)\,ds = 0. \tag{2.4}$$

We denote the Green-tight class by $\mathcal{K}_\infty(X)$ if we want to emphasize the dependence on the Markov process X. Chen [2] defines the Green-tight class in a slightly different way; however, the two definitions are equivalent under (SF) (see [7, Lemma 4.1]).

3. h-transform. For $\mu \in \mathcal{K}$ we define the Schrödinger form by

$$\begin{cases} \mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_E u^2\,d\mu, \\ \mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\mathcal{E}). \end{cases} \tag{3.1}$$

We see from (2.3) that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is a closed symmetric form on $L^2(E;m)$. Denoting by $\mathcal{L}$ and $\mathcal{L}^\mu = \mathcal{L} + \mu$ the self-adjoint operators generated by the closed symmetric forms $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ respectively, we see that the associated semigroup $p_t^\mu := \exp(t\mathcal{L}^\mu)$ is expressed as

$$p_t^\mu f(x) = \mathbb{E}_x[e^{A_t^\mu} f(X_t)]$$

(cf. [1]). We remark that the strong Feller property of p_t^μ follows from that of p_t (see [3]).

Let us introduce a function space:

$$\mathcal{H}_+^\mu = \{h \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap C(E) \mid h > 0,\ p_t^\mu h \le h\}. \tag{3.2}$$

Assume that $\mathcal{H}_+^\mu$ is not empty, and let $h \in \mathcal{H}_+^\mu$. We define the bilinear form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ on $L^2(E; h^2m)$ by

$$\begin{cases} \mathcal{E}^{\mu,h}(u,u) = \mathcal{E}^\mu(hu, hu), \\ \mathcal{D}(\mathcal{E}^{\mu,h}) = \{u \in L^2(E; h^2 m) \mid hu \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{cases} \tag{3.3}$$

The next lemma is proved in [13, Lemma 2.6].

Lemma 3.1. *$(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is a regular Dirichlet form on $L^2(E; h^2m)$.*

Let us denote by $X^{\mu,h}$ the Hunt process generated by the regular Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$. By the definition of $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$, the semigroup $\{p_t^{\mu,h}\}$ of $X^{\mu,h}$ is h^2m-symmetric and expressed by

$$p_t^{\mu,h} f(x) = \frac{1}{h(x)} \mathbb{E}_x[e^{A_t^\mu} h(X_t) f(X_t)], \qquad f \in \mathscr{B}_b(E).$$

Since $\exp(A_t^\mu) h(X_t) > 0$, the irreducibility of $X^{\mu,h}$ follows from that of X. Moreover, $X^{\mu,h}$ satisfies

Lower semicontinuity (LSC). For $\alpha > 0$, every α-excessive function with respect to $p_t^{\mu,h}$ is lower-semicontinuous.

Indeed, let R_α^μ and $R_\alpha^{\mu,h}$ be α-resolvents: for $f \in \mathscr{B}_b(X)$,

$$R_\alpha^\mu f(x) = \int_0^\infty e^{-\alpha t} p_t^\mu f(x)\, dt, \qquad R_\alpha^{\mu,h} f(x) = \int_0^\infty e^{-\alpha t} p_t^{\mu,h} f(x)\, dt. \tag{3.4}$$

Then

$$\frac{1}{h(x)} R_\alpha^\mu (f(h \wedge n))(x) \uparrow R_\alpha^{\mu,h} f(x) \quad \text{as } n \to \infty.$$

The function $R_\alpha^\mu(f(h \wedge n))$ is continuous on E by the strong Feller property of p_t^μ (see [3, Theorem 1.1]), and thus $R_\alpha^{\mu,h} f$ is lower-semicontinuous. Since for every α-excessive function g, $\beta R_{\beta+\alpha}^{\mu,h} g \uparrow g$ as $\beta \uparrow \infty$, (LSC) holds. Hence, we see from [6, Corollary 2.3] that if $X^{\mu,h}$ is transient then

$$\sup_{x \in E} R^{\mu,h} 1_K(x) < \infty \quad \text{for any compact set } K.$$

Let $\mathcal{D}_e(\mathcal{E}^{\mu,h})$ be the extended Dirichlet space of the regular Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$. We define the *extended Schrödinger space* $\mathcal{D}_e(\mathcal{E}^\mu)$ by

$$\begin{cases} \mathcal{E}^\mu(u,v) = \mathcal{E}^{\mu,h}\left(\dfrac{u}{h}, \dfrac{v}{h}\right), \\ \mathcal{D}_e(\mathcal{E}^\mu) = \left\{u \;\middle|\; \dfrac{u}{h} \in \mathcal{D}_e(\mathcal{E}^{\mu,h})\right\}. \end{cases} \tag{3.5}$$

We give another definition of the extended Schrödinger space similar to that of the extended Dirichlet form, that is, the family of m-measurable functions u such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}^\mu$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E}^\mu)$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. Denote by $\widetilde{\mathcal{D}_e}(\mathcal{E}^\mu)$ this family, and for $u \in \widetilde{\mathcal{D}_e}(\mathcal{E}^\mu)$ and a sequence $\{u_n\}_{n=1}^\infty$ define

$$\widetilde{\mathcal{E}}^\mu(u,u) = \lim_{n\to\infty} \mathcal{E}^\mu(u_n,u_n). \tag{3.6}$$

We then see that $(\widetilde{\mathcal{E}}^\mu, \widetilde{\mathcal{D}_e}(\mathcal{E}^\mu))$ is well-defined and we have

Lemma 3.2 ([13, Lemma 2.8]). *$\widetilde{\mathcal{D}_e}(\mathcal{E}^\mu) = \mathcal{D}_e(\mathcal{E}^\mu)$ and $\widetilde{\mathcal{E}}^\mu = \mathcal{E}^\mu$.*

Lemma 3.2 shows that the space $\mathcal{D}_e(\mathcal{E}^\mu)$ is independent of $h \in \mathcal{H}_+^\mu$. We call the sequence $\{u_n\}_{n=1}^\infty$ in the definition of $\widetilde{\mathcal{D}_e}(\mathcal{E}^\mu)$ an *approximating sequence* for $u \in \widetilde{\mathcal{D}_e}(\mathcal{E}^\mu)$.

We see from [2, Theorem 5.1] that for $\mu \in \mathcal{K}_\infty$,

$$\lambda(\mu) > 1 \iff 1 \le g^\mu(x) \le \sup_{x\in E} g^\mu(x) < \infty,$$

where g^μ is the gauge function in (1.2). Let us write simply g for g^μ. We know from [13, Definition 4.4, Theorem 5.6] that g belongs to $\mathcal{H}_+^\mu$ and $(\mathcal{E}^{\mu,g}, \mathcal{D}(\mathcal{E}^{\mu,g}))$ is transient. As a result, the extended Dirichlet space $(\mathcal{E}^{\mu,g}, \mathcal{D}_e(\mathcal{E}^{\mu,g}))$ turns out to be a Hilbert space.

Lemma 3.3. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. If $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ satisfies $\mathcal{E}^\mu(u,u) = 0$, then $u = 0$.*

Proof. By the definition of $\mathcal{D}_e(\mathcal{E}^\mu)$ we have $u/g \in \mathcal{D}_e(\mathcal{E}^{\mu,g})$ and

$$\mathcal{E}^{\mu,g}(u/g, u/g) = \mathcal{E}^\mu(u,u) = 0.$$

The transience of $(\mathcal{E}^{\mu,g}, \mathcal{D}(\mathcal{E}^{\mu,g}))$ yields $u/g = 0$ and so $u = 0$. ∎

Let $X^{\mu,g} = (\mathbb{P}_x^{\mu,g}, X_t, \zeta)$ be the Hunt process generated by $(\mathcal{E}^{\mu,g}, \mathcal{D}(\mathcal{E}^{\mu,g}))$ and denote by $\{p_t^{\mu,g}\}$ its semigroup. We define the 0-*order resolvent density $r^{\mu,g}(x,y)$ of $R^{\mu,g}$ with respect to $g^2 m$*:

$$\mathbb{E}_x^{\mu,g}\Big[\int_0^\zeta f(X_t)\,dt\Big] = \int_E r^{\mu,g}(x,y) f(y) g^2(y)\,dm(y), \qquad f \in \mathscr{B}_b(E).$$

We can take $r^{\mu,g}(x,y)$ symmetric and $p_t^{\mu,g}$-excessive in x for each y, and in y for each x (see [8, Theorem 1]). Then the 0-resolvent $r^{\mu,g}(x,y)$ is uniquely determined.

The 0-*order energy integral* $\mathcal{E}^{\mu,g}(\nu)$ of ν is defined by

$$\mathcal{E}^{\mu,g}(\nu) := \iint_{E\times E} r^{\mu,g}(x,y)\,\nu(dx)\,\nu(dy), \qquad \nu \in \mathcal{M},$$

and denote by $\mathcal{S}_0^{(0)}(\mathcal{E}^{\mu,g})$ the set of measures of finite 0-order energy integral:

$$\mathcal{S}_0^{(0)}(\mathcal{E}^{\mu,g}) = \{\mu \in \mathcal{M} \mid |\mathcal{E}^{\mu,g}(\nu) < \infty\}. \tag{3.7}$$

Let $r^\mu(x,y)$ be the *Green kernel*, the integral kernel of the 0-resolvent R^μ in (3.4) with respect to m. Since

$$R^\mu f(x) = \mathbb{E}_x\Big[\int_0^\zeta e^{A_t^\mu} f(X_t)\,dt\Big] = g(x)\bigg(\frac{1}{g(x)}\mathbb{E}_x\Big[\int_0^\zeta e^{A_t^\mu} g(X_t)\frac{f(X_t)}{g(X_t)}\,dt\Big]\bigg),$$

and the right-hand side equals

$$\begin{aligned} g(x)\mathbb{E}_x^{\mu,g}\Big[\int_0^\zeta \frac{f(X_t)}{g(X_t)}\,dt\Big] &= g(x)\int_E r^{\mu,g}(x,y)\frac{f(y)}{g(y)}g^2(y)\,dm(y) \\ &= \int_E g(x)r^{\mu,g}(x,y)g(y)f(y)\,dm(y), \end{aligned}$$

the kernel of R^μ is uniquely defined by

$$r^\mu(x,y) = g(x)r^{\mu,g}(x,y)g(y). \tag{3.8}$$

Note that $r^\mu(x,y)$ is equivalent to $r^{\mu,g}(x,y)$:

$$r^{\mu,g}(x,y) \le r^\mu(x,y) \le C^2\cdot r^{\mu,g}(x,y), \qquad C = \sup_{x\in E} g(x). \tag{3.9}$$

However, we do not know whether it is equivalent to $r(x,y)$; if μ belongs to $\mathcal{S}_\infty \subset \mathcal{K}_\infty$ (for the definition of $\mathcal{S}_\infty$, see [2]) and is conditionally gaugeable, then $r^\mu(x,y)$ is equivalent to $r(x,y)$.

4. Some principles in potential theory. In this section, we assume that X is transient.

Lemma 4.1. *For a measure ν in $\mathcal{S}_0^{(0)}$,*

$$\int_E |u|\,d\nu \le \Big(\int_E R\nu\,d\nu\Big)^{1/2}\mathcal{E}(u,u)^{1/2}, \qquad u \in \mathcal{D}_e(\mathcal{E}). \tag{4.1}$$

Proof. Since

$$\int_E\int_E r_\alpha(x,y)\,\nu(dx)\,\nu(dy) \le \int_E\int_E r(x,y)\,\nu(dx)\,\nu(dy) < \infty,$$

the measure ν is of α-order finite energy integral by [5, Exercise 4.2.2]. Noting that

$$\mathcal{E}(R_\alpha\nu, R_\alpha\nu) \le \mathcal{E}_\alpha(R_\alpha\nu, R_\alpha\nu) = \int_E R_\alpha\nu\,d\nu \le \int_E R\nu\,d\nu < \infty,$$

and $\lim_{\alpha\uparrow\infty} R_\alpha\nu = R\nu$, we see that $R\nu$ belongs to $\mathcal{D}_e(\mathcal{E})$.

Since for $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$,

$$\int_E |u|\,d\nu = \mathcal{E}_\alpha(R_\alpha\nu, |u|) \le \mathcal{E}_\alpha(R_\alpha\nu, R_\alpha\nu)^{1/2}\mathcal{E}_\alpha(u,u)^{1/2} \le \Big(\int_E R\nu\,d\nu\Big)^{1/2}\mathcal{E}_\alpha(u,u)^{1/2},$$

the inequality (4.1) holds for $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ by letting $\alpha \downarrow 0$, and is extended to $u \in \mathcal{D}_e(\mathcal{E})$. ∎

By [5, Theorem 2.2.3], the measure $\nu \in \mathcal{S}_0^{(0)}$ charges no set of zero $\mathcal{E}$-capacity. Define

$$\mathcal{E}^\mu(\nu) = \iint_{E\times E} r^\mu(x,y)\,\nu(dx)\,\nu(dy).$$

Noting that $\mathcal{E}^{\mu,g}(g\nu) = \mathcal{E}^\mu(\nu) < \infty$ by (3.9), we have the next lemma.

Lemma 4.2. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$ and g be the gauge function in (1.2). If $\nu \in \mathcal{M}$ satisfies $\mathcal{E}^\mu(\nu) < \infty$, then $g\nu \in \mathcal{S}_0^{(0)}(\mathcal{E}^{\mu,g})$, $R^{\mu,g}(g\nu) \in \mathcal{D}_e(\mathcal{E}^{\mu,g})$ and $gR^{\mu,g}(g\nu) = R^\mu\nu \in \mathcal{D}_e(\mathcal{E}^\mu)$.*

We set

$$\mathcal{S}_0^{(0)}(\mathcal{E}^\mu) = \{\nu \in \mathcal{M} \mid \mathcal{E}^\mu(\nu) < \infty\}. \tag{4.2}$$

Lemma 4.3. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. For $\nu \in \mathcal{S}_0^{(0)}(\mathcal{E}^\mu)$,*

$$\mathcal{E}^\mu(R^\mu\nu, \varphi) = \langle \nu, \varphi\rangle, \qquad \varphi \in \mathcal{D}_e(\mathcal{E}^\mu).$$

Proof. We compute

$$\begin{aligned}\mathcal{E}^\mu(R^\mu\nu,\varphi) &= \mathcal{E}^{\mu,g}(R^\mu\nu/g, \varphi/g) = \mathcal{E}^{\mu,g}(gR^{\mu,g}(g\nu)/g, \varphi/g)\\ &= \mathcal{E}^{\mu,g}(R^{\mu,g}(g\nu), \varphi/g) = \langle g\nu, \varphi/g\rangle = \langle \nu,\varphi\rangle.\ ∎\end{aligned}$$

Lemma 4.4. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. Then for $\nu \in \mathcal{S}_1$,*

$$R^\mu\nu(x) - p_t^\mu R^\mu\nu(x) = \mathbb{E}_x\Big[\int_0^{t\wedge\zeta} e^{A_s^\mu}\,dA_s^\nu\Big]. \tag{4.3}$$

Proof. By the Markov property,

$$\begin{aligned}p_t^\mu R^\mu\nu(x) &= \mathbb{E}_x\Big[e^{A_t^\mu}\mathbb{E}_{X_t}\Big[\int_0^\zeta e^{A_s^\mu}\,dA_s^\nu\Big]; t<\zeta\Big]\\ &= \mathbb{E}_x\Big[e^{A_t^\mu}\int_0^{\zeta(\theta_t)} e^{A_s^\mu(\theta_t)}\,dA_s^\nu(\theta_t); t<\zeta\Big].\end{aligned} \tag{4.4}$$

Since $A_t^\mu + A_s^\mu(\theta_t) = A_{t+s}^\mu$ and $dA_s^\nu(\theta_t) = dA_{t+s}^\nu$, the right-hand side of (4.4) equals

$$\mathbb{E}_x\Big[\int_0^{\zeta(\theta_t)} e^{A_{t+s}^\mu}\, dA_{t+s}^\nu;\, t<\zeta\Big] = \mathbb{E}_x\Big[\int_t^{t+\zeta(\theta_t)} e^{A_s^\mu}\, dA_s^\nu;\, t<\zeta\Big]$$
$$= \mathbb{E}_x\Big[\int_t^{\zeta} e^{A_s^\mu}\, dA_s^\nu;\, t<\zeta\Big],$$

and thus

$$R^\mu\nu(x) - p_t^\mu R^\mu\nu(x) = \mathbb{E}_x\Big[\int_0^{\zeta} e^{A_s^\mu}\, dA_s^\nu\Big] - \mathbb{E}_x\Big[\int_t^{\zeta} e^{A_s^\mu}\, dA_s^\nu;\, t<\zeta\Big]$$
$$= \mathbb{E}_x\Big[\int_0^{t} e^{A_s^\mu}\, dA_s^\nu;\, t<\zeta\Big] + \mathbb{E}_x\Big[\int_0^{\zeta} e^{A_s^\mu}\, dA_s^\nu;\, t\ge\zeta\Big]$$
$$= \mathbb{E}_x\Big[\int_0^{t\wedge\zeta} e^{A_s^\mu}\, dA_s^\nu\Big]. \ \blacksquare$$

Denote by S_ν the topological support of $\nu \in \mathcal{M}$.

Lemma 4.5. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. For $\nu \in \mathcal{S}_1$,*

$$\sup_{x\in E} \mathbb{E}_x\Big[\int_0^{t\wedge\zeta} e^{A_s^\mu}\, dA_s^\nu\Big] \le \sup_{x\in E} g^\mu(x) \cdot \sup_{x\in S_\nu} \mathbb{E}_x\Big[\int_0^{t\wedge\zeta} e^{A_s^\mu}\, dA_s^\nu\Big]. \tag{4.5}$$

Proof. Let $\sigma = \inf\{t>0 \mid X_t \in S_\nu\}$. Since $A_s^\nu = 0$ on $\{s \le \sigma\}$, we see from the strong Markov property that

$$\mathbb{E}_x\Big[\int_0^{t\wedge\zeta} e^{A_s^\mu}\, dA_s^\nu\Big] = \mathbb{E}_x\Big[\int_\sigma^{t\wedge\zeta} e^{A_s^\mu}\, dA_s^\nu;\, \sigma < t\wedge\zeta\Big]$$
$$= \mathbb{E}_x\Big[\int_0^{t\wedge\zeta-\sigma} e^{A_{\sigma+s}^\mu}\, dA_{\sigma+s}^\nu;\, \sigma < t\wedge\zeta\Big]$$
$$= \mathbb{E}_x\Big[e^{A_\sigma^\mu}\int_0^{t\wedge\zeta(\theta_\sigma)} e^{A_s^\mu(\theta_\sigma)}\, dA_s^\nu(\theta_\sigma);\, \sigma < t\wedge\zeta\Big]$$
$$= \mathbb{E}_x\Big[e^{A_\sigma^\mu}\mathbb{E}_{X_\sigma}\Big[\int_0^{t\wedge\zeta} e^{A_s^\mu}\, dA_s^\nu\Big];\, \sigma < t\wedge\zeta\Big].$$

Since $X_\sigma \in S_\nu$ on $\{\sigma < t\wedge\zeta\}$, the right-hand side is less than or equal to

$$\mathbb{E}_x[e^{A_\sigma^\mu};\, \sigma < t\wedge\zeta] \cdot \sup_{x\in S_\nu} \mathbb{E}_x\Big[\int_0^{t\wedge\zeta} e^{A_s^\mu}\, dA_s^\nu\Big]. \ \blacksquare$$

LEMMA 4.6. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. If a Radon measure ν satisfies $\sup_{x\in S_\nu} R^\mu\nu(x) < \infty$, then ν belongs to $\mathcal{S}_1$.*

Proof. For a compact set K, let $\nu_K = 1_K \cdot \nu$. Since

$$\sup_{x\in K} R\nu_K(x) \le \sup_{x\in S_\nu} R\nu_K(x) \le \sup_{x\in S_\nu} R\nu(x) < \infty,$$

we have $\int_E R\nu_K \, d\nu_K < \infty$ and $\nu_K \in \mathcal{S}_0^{(0)}$. Denote by $U\nu_K \in \mathcal{D}_e(\mathcal{E})$ the 0-potential. Then

$$\int_E r(x,y) 1_K(y)\, d\nu(y) = U\nu_K = \mathbb{E}_x[A_\zeta^{\nu_K}], \quad \text{q.e.} \tag{4.6}$$

Note that the exceptional set is allowed in the correspondence between ν_K and A^{ν_K}. Let $\sigma = \inf\{t > 0 \mid X_t \in S_\nu\}$. Since $dA_s^\nu = 0$ on $\{s < \sigma\}$, the right-hand side of (4.6) equals

$$\begin{aligned}\mathbb{E}_x[A_\zeta^{\nu_K} - A_\sigma^{\nu_K}] &= \mathbb{E}_x[A_\zeta^{\nu_K}(\theta_\sigma)] = \mathbb{E}_x[\mathbb{E}_{X_\sigma}[A_\zeta^{\nu_K}]] = \mathbb{E}_x[R\nu_K(X_\sigma)] \\ &\le \sup_{x\in S_\nu} R\nu_K(x) \le \sup_{x\in S_\nu} R\nu(x), \quad \text{q.e.}\end{aligned}$$

Hence, for an increasing sequence $\{K_n\}$ of compact sets with $K_n \uparrow E$, there exists N with $\mathrm{Cap}(N) = 0$ such that

$$\begin{aligned}\int_E r(x,y)\, d\nu(y) &= \lim_{n\to\infty} \int_E r(x,y) 1_{K_n}(y)\, d\nu(y) \\ &\le \sup_{x\in S_\nu} R\nu(x) < \infty, \qquad x \in E \setminus N.\end{aligned}$$

Noting that $r(\cdot, y)$ is lower-semicontinuous by the strong Feller property, we have for $x_0 \in N$ such that $\{x_n\} \subset E \setminus N$ and $x_n \to x_0$,

$$\begin{aligned}R\nu(x_0) &= \int_E r(x_0,y)\, d\nu \le \int_E \liminf_{n\to\infty} r(x_n,y)\, d\nu \le \liminf_{n\to\infty} \int_E r(x_n,y)\, d\nu \\ &\le \sup_{x\in S_\nu} R\nu(x) < \infty.\end{aligned}$$

Hence $\sup_{x\in E} R\nu(x) < \infty$ and so $1_K \cdot \nu \in \mathcal{S}_{00}$ for any compact set K, which implies $\nu \in \mathcal{S}_1$. ∎

THEOREM 4.7. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. For a Radon measure ν with compact support S_ν,*

$$\sup_{x\in E} R^\mu\nu(x) \le \sup_{x\in E} g^\mu(x) \cdot \sup_{x\in S_\nu} R^\mu\nu(x). \tag{4.7}$$

Proof. By Lemma 4.6, it is sufficient to prove this for $\nu \in \mathcal{S}_1$, in which case it follows from (4.5) by letting $t \to \infty$. ∎

Theorem 4.7 says that the kernel $r^\mu(x,y)$ satisfies Ugaheri's maximum principle.

REMARK 4.8. (i) If $\mu = 0$, Theorem 4.7 yields Frostman's maximum principle,

$$\sup_{x\in E} R\nu(x) \le \sup_{x\in S_\nu} R\nu(x). \tag{4.8}$$

(ii) Using (3.9), we have another proof of Theorem 4.7. Put $g = g^\mu$. Since $R^\mu\nu = gR^{\mu,g}(\nu/g)$ and $1 \le g(x) \le \sup_{x\in E} g(x)$, we have

$$\begin{aligned}\sup_{x\in E} R^\mu\nu(x) &\le \sup_{x\in E} g(x)\cdot \sup_{x\in E} R^{\mu,g}(\nu/g)(x) = \sup_{x\in E} g(x)\cdot \sup_{x\in S_\nu} R^{\mu,g}(\nu/g)(x)\\ &= \sup_{x\in E} g(x)\cdot \sup_{x\in S_\nu}\left(\frac{1}{g(x)}R^\mu\nu(x)\right) \le \frac{\sup_{x\in E} g(x)}{\inf_{x\in S_\nu} g(x)}\cdot \sup_{x\in S_\nu} R^\mu\nu(x).\end{aligned}$$

Here, we employ Frostman's maximum principle (4.8) for $r^{\mu,g}(x,y)$.

(iii) If $\lambda(\mu) = 1$, then there exists a function $h \in \mathcal{H}^\mu_+$ (see (3.2)), which is unique up to a positive constant factor, and $X^{\mu,h}$ is an h^2m-symmetric recurrent Markov process. Hence, for any open set O,

$$\mathbb{P}^{\mu,h}_x(\sigma_O\circ\theta_n < \infty,\ \forall n > 0) = 1 \quad \text{for q.e. } x \in \mathbb{R}^d \tag{4.9}$$

by [5, Theorem 4.7.1]. Here σ_O is the first hitting time of O, that is, $\sigma_O = \inf\{t > 0 \mid X_t \in O\}$. Moreover, since the Markov process $X^{\mu,h}$ has the transition density function $p^{\mu,h}_t(x,y) = p^\mu_t(x,y)/(h(x)h(y))$ with respect to h^2m, (4.9) holds for all $x \in E$ by [5, Exercise 4.7.1]. Using the strong Feller property and [10, Chapter X, proof of Proposition (3.11)], we see from (4.9) that $X^{\mu,h}$ is Harris recurrent. Hence, by [10, Chapter X, Proposition (3.11)],

$$R^\mu\nu(x) = \mathbb{E}_x\left[\int_0^\zeta e^{A^\mu_t}\, dA^\nu_t\right] = h(x)\mathbb{E}^{\mu,h}_x\left[\int_0^\infty \frac{1}{h(X_t)}\, dA^\nu_t\right] = \infty, \qquad \forall x \in E.$$

COROLLARY 4.9. *Let $\mu \in \mathcal{K}_\infty$ satisfy $\lambda(\mu) > 1$ and ν be a Radon measure with compact support S_ν. If $R^\mu\nu$ is continuous on S_ν, then it is continuous on E.*

Proof. By assumptions in this corollary, $\sup_{x\in S_\nu} R^\mu\nu(x) < \infty$, and thus $R^\mu\nu$ is bounded on E by Theorem 4.7. Hence $p^\mu_t R^\mu\nu$ is bounded continuous on E by the strong Feller property. Put $I_t(x) = \mathbb{E}_x[\int_0^{t\wedge\zeta} e^{A^\mu_s}\, dA^\nu_s]$. Then I_t is continuous on S_ν by Lemma 4.4.

Since $I_t(x)$ monotonically decreases and converges pointwise to 0 as $t \downarrow 0$, its convergence is uniform on E by Dini's theorem and Lemma 4.5. Therefore, $p^\mu_t R^\mu\nu(x)$ converges to $R^\mu\nu$ uniformly on E as $t \downarrow 0$, and thus $R^\mu\nu$ is continuous on E. ∎

Denote $p^{\mu,\alpha}_t = e^{-\alpha t}p^\mu_t$ for $\alpha \ge 0$. Then $p^{\mu,0}_t$ equals p^μ_t. A non-negative function f is said to be a $p^{\mu,\alpha}_t$-*excessive function* if $p^{\mu,\alpha}_t f(x) \le f(x)$ and $p^{\mu,\alpha}_t f(x) \uparrow f(x)$ as $t \downarrow 0$.

LEMMA 4.10. *The function g^μ is p_t^μ-excessive, $p_t^\mu g^\mu(x) \uparrow g^\mu(x)$ as $t \downarrow 0$.*

Proof. By the Markov property of X,

$$\begin{aligned}\mathbb{E}_x(e^{A_t^\mu} g^\mu(X_t)) &= \mathbb{E}_x(e^{A_t^\mu} g^\mu(X_t);\, t<\zeta)\\ &= \mathbb{E}_x(e^{A_t^\mu}\mathbb{E}_{X_t}(e^{A_\zeta^\mu});\, t<\zeta)\\ &= \mathbb{E}_x\big(\mathbb{E}_x(e^{A_t^\mu + A_\zeta^\mu(\theta_t)} 1_{\{t<\zeta\}} \mid \mathscr{F}_t)\big).\end{aligned}$$

The right-hand side equals $\mathbb{E}_x(e^{A_\zeta^\mu};\, t<\zeta)$ because $A_t^\mu + A_\zeta^\mu(\theta_t) = A_\zeta^\mu$ on $\{t<\zeta\}$. Therefore

$$p_t^\mu g^\mu(x) = \mathbb{E}_x(e^{A_\zeta^\mu};\, t<\zeta) \uparrow \mathbb{E}_x(e^{A_\zeta^\mu}) = g^\mu(x), \qquad t\downarrow 0. \ \blacksquare$$

LEMMA 4.11. *Suppose that $\lambda(\mu) > 1$. If f is a $p_t^{\mu,\alpha}$-excessive function, then there exists a sequence $\{g_n\}$ of bounded non-negative functions such that $R_\alpha^\mu g_n$, $\alpha>0$, increasingly converges to f.*

Proof. The argument is similar to that in [4, p. 85, Theorem 9]. Let g^μ be the gauge function and define $h_n = f \wedge n g^\mu$. Then h_n is $p_t^{\mu,\alpha}$-excessive because g^μ is also $p_t^{\mu,\alpha}$-excessive by Lemma 4.10. Moreover $h_n(x)$ is bounded and $h_n(x) \uparrow f(x)$ for any $x \in E$ because g^μ is a bounded, strictly positive function. By the resolvent equation,

$$nR_{n+\alpha}^\mu h_n = R_\alpha^\mu(n(h_n - nR_{n+\alpha}^\mu h_n)).$$

Define $g_n = n(h_n - nR_{n+\alpha}^\mu h_n)$. As h_n is $p_t^{\mu,\alpha}$-excessive, g_n is non-negative and

$$f = \lim_{n\to\infty} nR_{n+\alpha}^\mu h_n(x) = \lim_{n\to\infty} R_\alpha^\mu g_n. \ \blacksquare$$

LEMMA 4.12. *Suppose that $\lambda(\mu) > 1$. If f is a p_t^μ-excessive function, then for any stopping time T,*

$$f(x) \ge \mathbb{E}_x[e^{A_T^\mu} f(X_T)].$$

Proof. Let $\{g_n\}$ be a sequence as in Lemma 4.11. Then

$$\begin{aligned}R_\alpha^\mu g_n(x) &= \mathbb{E}_x\Big[\int_0^\zeta e^{-\alpha t + A_t^\mu} g_n(X_t)\,dt\Big]\\ &= \mathbb{E}_x\Big[\int_T^{T+\zeta(\theta_T)} e^{-\alpha t + A_t^\mu} g_n(X_t)\,dt;\, T<\zeta\Big]\\ &= \mathbb{E}_x\Big[e^{-\alpha T + A_T^\mu}\int_0^{\zeta(\theta_T)} e^{-\alpha t + A_t^\mu(\theta_T)} g_n(X_t(\theta_T))\,dt;\, T<\zeta\Big].\end{aligned}$$

By the strong Markov property, the right-hand side is equal to

$$\mathbb{E}_x\Big[e^{-\alpha T+A_T^\mu}\mathbb{E}_{X_T}\Big[\int_0^\zeta e^{-\alpha t+A_t^\mu} g_n(X_t)\,dt\Big];\,T<\zeta\Big] = \mathbb{E}_x\Big[e^{-\alpha T+A_T^\mu} R_\alpha^\mu g_n(X_T);\,T<\zeta\Big].$$

By letting $n\to\infty$ and $\alpha\to 0$, we have this lemma. ■

Suppose that the semigroup p_t^μ admits an integral kernel $p_t^\mu(x,y)$ such that

(i) $p_t^\mu(x,y)=p_t^\mu(y,x)$, $t>0$, $(x,y)\in E\times E$;
(ii) $p_{s+t}^\mu(x,y)=\int_E p_s^\mu(x,z)p_t^\mu(z,y)\,dm(z)$, $s,t>0$, $(x,y)\in E\times E$.

For a sufficient condition on p_t for the properties above, refer to [1]. Then for a positive Radon measure ν,

$$p_t^\mu R^\mu\nu(x)=\int_t^\infty\Big(\int_E p_s^\mu(x,y)\,d\nu(y)\Big)\,ds\uparrow R^\mu\nu(x),\qquad t\downarrow 0,$$

i.e., $R^\mu\nu$ is p_t^μ-excessive.

We would like to remark that Cartan's maximum principle can be proved by using the strong Markov property.

THEOREM 4.13. *If $\nu_1\in\mathcal{S}_0(\mathcal{E}^\mu)$ and $\nu_2\in\mathcal{M}$ satisfy*

$$R^\mu\nu_1(x)\le R^\mu\nu_2(x),\qquad \forall x\in S_{\nu_1},\tag{4.10}$$

then the inequality holds on E.

Proof. First assume $\nu_1\in\mathcal{S}_1$. Let $\sigma_1=\inf\{t>0\mid X_t\in S_{\nu_1}\}$. Since $dA_t^{\nu_1}=0$ on $\{t<\sigma_1\}\cup\{t\ge\zeta\}$, we have

$$R^\mu\nu_1(x)=\mathbb{E}_x\Big[\int_0^\zeta e^{A_t^\mu}\,dA_t^{\nu_1}\Big]=\mathbb{E}_x\Big[\int_{\sigma_1}^\zeta e^{A_t^\mu}\,dA_t^{\nu_1};\,\sigma_1<\zeta\Big].$$

Noting that $\zeta=\sigma_1+\zeta(\theta_{\sigma_1})$ and $dA_{\sigma_1+t}^{\nu_1}=dA_t^{\nu_1}(\theta_{\sigma_1})$ on $\{\sigma_1<\zeta\}$, we see that the right-hand side equals

$$\begin{aligned}\mathbb{E}_x\Big[\int_{\sigma_1}^{\sigma_1+\zeta(\theta_{\sigma_1})} e^{A_t^\mu}\,dA_t^{\nu_1};\,\sigma_1<\zeta\Big]&=\mathbb{E}_x\Big[\int_0^{\zeta(\theta_{\sigma_1})} e^{A_{\sigma_1+t}^\mu}\,dA_{\sigma_1+t}^{\nu_1};\,\sigma_1<\zeta\Big]\\&=\mathbb{E}_x\Big[\int_0^{\zeta(\theta_{\sigma_1})} e^{A_{\sigma_1}^\mu+A_t^\mu(\theta_{\sigma_1})}\,dA_t^{\nu_1}(\theta_{\sigma_1});\,\sigma_1<\zeta\Big]\\&=\mathbb{E}_x\Big[e^{A_{\sigma_1}^\mu}\mathbb{E}_{X_{\sigma_1}}\Big[\int_0^\zeta e^{A_t^\mu}\,dA_t^{\nu_1}\Big];\,\sigma_1<\zeta\Big]=\mathbb{E}_x\Big[e^{A_{\sigma_1}^\mu}R^\mu\nu_1(X_{\sigma_1});\,\sigma_1<\zeta\Big]\end{aligned}$$

by the strong Markov property. Since $X_{\sigma_1} \in S_{\nu_1}$, we see from assumption (4.10) that the right-hand side is less than or equal to

$$\mathbb{E}_x[e^{A^\mu_{\sigma_1}} R^\mu \nu_2(X_{\sigma_1}); \sigma_1 < \zeta],$$

which is less than or equal to $R^\mu\nu_2(x)$ by Lemma 4.12 because $R^\mu\nu_2$ is p^μ_t-excessive.

For a general $\nu_1 \in \mathcal{S}_0(\mathcal{E}^\mu)$, let $\{K_n\}$ be a general compact nest such that $\nu_n := \nu_1(K_n \cap \cdot) \in S^{(0)}_{00}(\mathcal{E}) \subset \mathcal{S}_1$. We then see that

$$R^\mu\nu_n(x) = R^\mu\nu_2(x), \quad \forall x \in E.$$

Since $\nu_1(E \setminus \bigcup_{n=1}^\infty K_n) = 0$ we conclude that

$$R^\mu\nu_1(x) = \lim_{n\to\infty} R^\mu\nu_n(x) \le R^\mu\nu_2(x), \quad \forall x \in E. \ ■$$

For $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$, the kernel r^μ satisfies the *energy principle*: For a signed measure $\nu = \nu^+ - \nu^- \in \mathcal{S}_0(\mathcal{E}^\mu) - \mathcal{S}_0(\mathcal{E}^\mu)$, its energy integral

$$\mathcal{E}^\mu(\nu) = \int_E R^\mu\nu\, d\nu = \int_E R^\mu(\nu^+ - \nu^-)\, d(\nu^+ - \nu^-)$$

is non-negative, $\mathcal{E}^\mu(\nu) \ge 0$, and if $\mathcal{E}^\mu(\nu) = 0$, then ν is trivial. Indeed,

$$\begin{aligned}\mathcal{E}^\mu(\nu) &= \int_E R^\mu\nu^+\, d\nu^+ - 2\int_E R^\mu\nu^+\, d\nu^- + \int_E R^\mu\nu^-\, d\nu^- \\ &= \mathcal{E}^\mu(R^\mu\nu^+ - R^\mu\nu^-, R^\mu\nu^+ - R^\mu\nu^-) \ge 0.\end{aligned}$$

Furthermore, if $\mathcal{E}^\mu(\nu) = 0$, then $R^\mu\nu^+ = R^\mu\nu^-$ by Lemma 3.3 and

$$\langle \nu^+, \varphi\rangle = \mathcal{E}^\mu(R^\mu\nu^+, \varphi) = \mathcal{E}^\mu(R^\mu\nu^-, \varphi) = \langle \nu^-, \varphi\rangle, \quad \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E),$$

by Lemma 4.3, and thus $\nu \equiv 0$.

For $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$, the kernel r^μ satisfies the *balayage principle*: If K is a compact subset of E, then for every measure $\nu \in \mathcal{S}_0(\mathcal{E}^\mu)$ there exists a measure μ_K, with $S_{\mu_K} \subset K$, such that $R^\mu\nu_K = R^\mu\nu$ q.e. and $R^\mu\nu_K \le R^\mu\nu$ everywhere. Indeed, for $\nu \in \mathcal{S}_0(\mathcal{E}^\mu)$ define

$$\mathscr{F}^\mu_{\nu,K} = \{u \in \mathcal{D}_e(\mathcal{E}^\mu) \mid u \ge R^\mu\nu \text{ q.e. on } K\}.$$

Then

$$\begin{aligned}\inf\{\mathcal{E}^\mu(u,u) \mid u \in \mathscr{F}^\mu_{\nu,K}\} &= \inf\{\mathcal{E}^{\mu,g}(u/g, u/g) \mid u \in \mathscr{F}^\mu_{\nu,K}\} \\ &= \inf\{\mathcal{E}^{\mu,g}(v,v) \mid v \in \mathcal{D}_e(\mathcal{E}^{\mu,g}),\ v \ge R^\mu\nu/g \text{ q.e. on } K\} \\ &= \inf\{\mathcal{E}^{\mu,g}(v,v) \mid v \in \mathcal{D}_e(\mathcal{E}^{\mu,g}),\ v \ge R^{\mu,g}(g\nu) \text{ q.e. on } K\}.\end{aligned} \tag{4.11}$$

We put

$$\mathcal{L}_{R^{\mu,g}(g\nu),K} = \{v \in \mathcal{D}_e(\mathcal{E}^{\mu,g}) \mid v \ge R^{\mu,g}(g\nu) \text{ q.e. on } K\}.$$

Then the space $\mathcal{L}_{R^{\mu,g}(g\nu),K}$ is a closed convex set in $(\mathcal{E}^{\mu,g}, \mathcal{D}_e(\mathcal{E}^{\mu,g}))$, and there is a unique $v_K \in \mathcal{L}_{R^{\mu,g}(g\nu),K}$ minimizing the right-hand side of (4.11).

This v_K is again a 0-potential of some measure in $\mathcal{S}_0(\mathcal{E}^{\mu,g})$ supported by K, $v_K = R^{\mu,g}\widetilde{\nu}_K$ (see [5, Section 2.3]). We know that $v_K = R^{\mu,g}(g\nu)$ q.e. on K and $0 \le v_K \le R^{\mu,g}(g\nu)$ m-a.e. on E. Define $u_K = gv_K$ and $\nu_K = \widetilde{\nu}_K/g$. Then $u_K \in \mathcal{D}_e(\mathcal{E}^\mu)$ and

$$u_K = gR^{\mu,g}(\widetilde{\nu_K}) = gR^{\mu,g}(g\nu_K) = R^\mu\nu_K \quad m\text{-a.e. on } E.$$

Hence, $R^\mu\nu_K = R^\mu\nu$ q.e. on K and $R^\mu\nu_K \le R^\mu\nu$ m-a.e. on E. Since for any $\alpha > 0$, $\alpha R^\mu_\alpha R^\mu\nu_K \le \alpha R^\mu_\alpha R^\mu\nu$ everywhere, by letting $\alpha \to \infty$ we deduce that $R^\mu\nu_K \le R^\mu\nu$ everwhere. We say that u_K and ν_K are the 0-*reduced function of $R^\mu\nu$ on K* and the *sweeping out of ν on K*, respectively.

REMARK 4.14. Defining $\mathcal{E}^\mu$-Cap by replacing $\mathcal{E}$ with $\mathcal{E}^\mu$, we have

$$\mathcal{E}^\mu\text{-Cap}(A) \le \mathcal{E}^{\mu,g}\text{-Cap}(A) \le M^2 \cdot \mathcal{E}^\mu\text{-Cap}(A), \qquad M = \sup_{x\in E} g(x).$$

Let us treat the case when μ is a signed measure written as $\mu = \mu^+ - \mu^- \in \mathcal{K} - \mathcal{K}_{\mathrm{loc}}$. We define the regular Dirichlet form $(\mathcal{E}^{\mu^-}, \mathcal{D}(\mathcal{E}^{\mu^-}))$ and the Schrödinger form by

$$\mathcal{E}^{\mu^-}(u,u) = \mathcal{E}(u,u) + \int_E u^2\,d\mu^-, \qquad \mathcal{D}(\mathcal{E}^{\mu^-}) = \mathcal{D}(\mathcal{E}) \cap L^2(E;\mu^-),$$
$$\mathcal{E}^{\mu}(u,u) = \mathcal{E}(u,u) - \int_E u^2\,d\mu, \qquad \mathcal{D}(\mathcal{E}^{\mu}) = \mathcal{D}(\mathcal{E}^{\mu^-}).$$

The symmetric Markov process generated by $(\mathcal{E}^{\mu^-}, \mathcal{D}(\mathcal{E}^{\mu^-}))$ is the subprocess $X^{\mu^-} = (\mathbb{P}^{\mu^-}_x, X_t, \zeta)$ of X generated by the multiplicative functional $\exp(-A^{\mu^-}_t)$. If X^{μ^-} is also strong Feller (for this condition, refer to [3]), then we can obtain the statements corresponding to those above by replacing X by X^{μ^-} and $\mathcal{K}_\infty(X)$ by $\mathcal{K}_\infty(X^{\mu^-})$.

For a relatively compact open set D, the measure $\mu^-_D(\cdot) := \mu^-(D \cap \cdot)$ is Green-tight and $\|R^D\mu^-\|_\infty \le \|R\mu^-_D\|_\infty < \infty$, where R^D is the 0-resolvent of the part process on D. Hence, by the Stollmann–Voigt inequality,

$$\int_D u^2\,d\mu^- \le \|R\mu^-_D\|_\infty \cdot \mathcal{E}(u,u), \qquad u \in \mathcal{D}(\mathcal{E}) \cap C_0(D),$$

and thus for $A \subset D$,

$$\mathcal{E}^{\mu^-,D}\text{-Cap}(A) = 0 \iff \mathcal{E}^D\text{-Cap}(A) = 0,$$

where $\mathcal{E}^{\mu^-,D}$ (resp. $\mathcal{E}^D$) is the part of the Dirichlet form $\mathcal{E}^{\mu^-}$ (resp. $\mathcal{E}$) on D. On account of [5, (2.16) and Theorem 4.4.3], we have the next lemma.

LEMMA 4.15. *Let $\mu^- \in \mathcal{K}_{\mathrm{loc}}$. For $B \in \mathscr{B}(E)$,*

$$\mathcal{E}\text{-Cap}(B) = 0 \iff \mathcal{E}^{\mu^-}\text{-Cap}(B) = 0.$$

Lemma 4.16. *If a Radon measure ν satisfies*

$$\sup_{x\in S_\nu} R^\mu\nu(x) < \infty,$$

then $\nu \in \mathcal{S}_1(\mathcal{E}^{\mu^-})$.

Proof. Since $\sup_{x\in S_\nu} R^{\mu^-}\nu \le \sup_{x\in S_\nu} R^\mu\nu < \infty$, the measure ν is in $\mathcal{S}_1(\mathcal{E}^{\mu^-})$ by the same argument as in Lemma 4.6. ∎

Lemma 4.17. *For $\nu \in \mathcal{S}$,*

$$\mathbb{E}_x\Big[\int_0^t e^{A_s^\mu}\, dA_s^\nu\Big] = \mathbb{E}_x^{\mu^-}\Big[\int_0^t e^{A_s^{\mu^+}}\, dA_s^\nu\Big] \quad \textit{for q.e. } x.$$

Proof. Let N be an exceptional set of A^ν. It follows from Sharpe [11, Section 62] that for $x \in E\setminus N$,

$$\begin{aligned}\mathbb{E}_x^{\mu^-}\Big[\int_0^t e^{A_s^{\mu^+}}\, dA_s^\nu\Big] &= \mathbb{E}_x\Big[\int_0^t\Big(\int_0^s e^{A_u^{\mu^+}}\, dA_u^\nu\Big)\, d(-e^{-A_s^{\mu^-}}) + \int_0^t e^{A_s^{\mu^+}}\, dA_s^\nu\cdot e^{-A_t^{\mu^-}}\Big]\\ &= \mathbb{E}_x\Big[\int_0^t e^{A_s^\mu}\, dA_s^\nu\Big]. \ ∎\end{aligned}$$

We see from Lemma 4.17 that for all x,

$$R^\mu\nu(x) = \mathbb{E}_x^{\mu^-}\Big[\int_0^\zeta e^{A_s^{\mu^+}}\, dA_s^\nu\Big].$$

We can prove Lemma 4.18, Lemma 4.19 and Theorem 4.20 below by the same arguments as above; we omit the details.

Lemma 4.18. *Let $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty(X^{\mu^-}) - \mathcal{K}_{\mathrm{loc}}$ and $\nu \in \mathcal{S}_1$. Then*

$$R^\mu\nu(x) - p_t^\mu R^\mu\nu(x) = \mathbb{E}_x^{\mu^-}\Big[\int_0^{t\wedge\zeta} e^{A_s^{\mu^+}}\, dA_s^\nu\Big].$$

Lemma 4.19. *Let $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty(X^{\mu^-}) - \mathcal{K}_{\mathrm{loc}}$ and $\nu \in \mathcal{S}_1$. Then*

$$\sup_{x\in E}\mathbb{E}_x^{\mu^-}\Big[\int_0^{t\wedge\zeta} e^{A_s^{\mu^+}}\, dA_s^\nu\Big] \le \sup_{x\in E} g^\mu(x)\cdot\sup_{x\in S_\nu}\mathbb{E}_x^{\mu^-}\Big[\int_0^{t\wedge\zeta} e^{A_s^{\mu^+}}\, dA_s^\nu\Big]. \tag{4.12}$$

Theorem 4.20. *If $\lambda(\mu) > 1$, then for a Radon measure ν,*

$$\sup_{x\in E} R^\mu\nu(x) \le \sup_{x\in E} g^\mu(x)\cdot\sup_{x\in S_\nu} R^\mu\nu(x).$$

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