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A RIGIDITY THEOREM FOR CENTROAFFINE CHEBYSHEV HYPEROVALOIDS

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Abstract. In this note, we investigate centroaffine hyperovaloids. We first establish an integral formula under the additional *Chebyshev* condition. Then, combining the integral formula with our recent classification of locally strongly convex centroaffine hypersurfaces with parallel traceless difference tensor [J. Geom. Anal. 28 (2018), 643–655], we obtain a rigidity theorem which shows that if a centroaffine Chebyshev hyperovaloid has nonnegative centroaffine sectional curvatures then it must be an ellipsoid.

1. Introduction. In equiaffine differential geometry, the classical theorem of Blaschke and Deicke states that if a *hyperovaloid* (a connected compact locally strongly convex hypersurface without boundary in the $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$) is an affine hypersphere then it is an ellipsoid (cf. [LS$^+$, Theorem 3.35, p. 145]).

A natural problem in affine differential geometry is to extend the above Blaschke–Deicke theorem from affine hyperspheres to centroaffine hypersurfaces under suitable conditions. In this respect one notices that *Chebyshev hypersurfaces*, a notion introduced by Liu and Wang [LW1] in centroaffine differential geometry (cf. Definition 2.1), have very similar properties to those of affine hyperspheres in equiaffine differential geometry. Therefore, the following problem seems interesting and reasonable:

Problem 1.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Chebyshev hyperovaloid. Must $x(M^n)$ then be an ellipsoid containing the origin $O \in \mathbb{R}^{n+1}$?*

It turns out that the above problem is difficult and still unresolved, although many related results have been obtained in the past two decades.

Before stating our result, we will summarize the progress achieved on Problem 1.1. We first recall the earliest result due to Liu and Wang [LW1] which solved the problem for $n = 2$:

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THEOREM 1.2 ([LW1]). *Let $x : M^2 \to \mathbb{R}^3$ be a centroaffine Chebyshev ovaloid. Then $x(M^2)$ is an ellipsoid containing the origin $O \in \mathbb{R}^3$.*

Moreover, Liu and Wang [LW2] generalized Theorem 1.2 as follows:

THEOREM 1.3 ([LW2]). *Let $x : M^2 \to \mathbb{R}^3$ be a relative Chebyshev ovaloid. Then $x(M^2)$ is an ellipsoid.*

Extending Theorem 1.3 to higher dimensions in relative affine differential geometry, A.-M. Li et al. [LL$^+$] established the following result:

THEOREM 1.4 ([LL$^+$, Theorem 6.2]). *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \geq 2)$ be a relative Chebyshev hyperovaloid. If the normalized scalar curvature κ and the Ricci tensor Ric of the relative metric h are related by the Ricci-pinched condition*

$$Ric - \frac{2n}{n+2}\kappa h \geq 0, \tag{1.1}$$

and strict inequality holds at least at one point in M^n, then $x(M^n)$ is an ellipsoid.

Recently, by calculating the Laplacian of the squared norm of the cubic Simon form, M. Li [L] proved several rigidity theorems on relative Chebyshev hypersurfaces. One of his results, pertaining to hyperovaloids, can be formulated as follows:

THEOREM 1.5 ([L, Theorem 5]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be a relative Chebyshev hyperovaloid. Denote by T, S, $\tilde{K}$ and $\|\cdot\|$ the Chebyshev vector field, the affine shape operator, the traceless difference tensor and the norm with respect to the relative metric, respectively. Then, letting $\lambda_{\min}(S)$ denote the minimal eigenvalue of S, if*

$$\|T\|^2 < \frac{4(n+2)}{n^2(n+10)}\left((n+1)\lambda_{\min}(S) + \frac{1}{n}\|\tilde{K}\|^2\right), \tag{1.2}$$

then $x(M^n)$ is an ellipsoid.

Finally, trying to extend Theorem 1.2 to higher dimensions, Liu, Simon and Wang [LSW] obtained the following result under an additional nondegeneracy condition:

THEOREM 1.6 ([LSW]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Chebyshev hyperovaloid with nondegenerate equiaffine Gauss map. Then $x(M^n)$ is an ellipsoid.*

Here, according to [LSW] or [LS$^+$, p. 34], the condition of “nondegenerate equiaffine Gauss map” is equivalent to the Blaschke normal $Y : M^n \to \mathbb{R}^{n+1}$ of the corresponding centroaffine immersion $x : M^n \to \mathbb{R}^{n+1}$ being an immersion.

In this paper, we adopt a different approach. First, we establish an integral formula for centroaffine Chebyshev hyperovaloids by applying Ros's technique [R] and an argument of Urbano [U] to our situation. Then, combining the integral formula with our recent classification of locally strongly convex centroaffine hypersurfaces with parallel traceless difference tensor [CH, CHM], we show that if a centroaffine Chebyshev hyperovaloid is of nonnegative centroaffine sectional curvatures then it is an ellipsoid.

More precisely, our main result is as follows:

THEOREM 1.7. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Chebyshev hyperovaloid. Denote by UM, $\hat{\nabla}$, $\hat{R}$ and $d\sigma$ the unit tangent bundle, the Levi-Civita connection, the Riemannian curvature tensor of M^n and the canonical measure on UM with respect to the centroaffine metric h, respectively. Then*

$$\int_{UM} \left(\|(\hat{\nabla}\tilde{K})(v,v,v)\|^2 + 3h(\hat{R}(\tilde{K}(v,v),v)v,\tilde{K}(v,v))\right) d\sigma = 0. \tag{1.3}$$

Moreover, if (M^n,h) has all sectional curvatures nonnegative, then $\hat{\nabla}\tilde{K} = 0$ and $x(M^n)$ is an ellipsoid containing the origin $O \in \mathbb{R}^{n+1}$.

REMARK 1.8. To better understand the independence of Theorem 1.7 from Theorem 1.6, we state the following:

FACT. *The condition of having nonnegative centroaffine sectional curvatures does not imply that the equiaffine Gauss map is nondegenerate.*

To verify this fact, we define M to be the elliptic paraboloid

$$x_{n+1} = (x_1)^2 + \cdots + (x_n)^2 \tag{1.4}$$

with the origin removed, where $(x_1,\ldots,x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$. From [LSW, Corollary 4.2.5], we know that M is a Chebyshev hypersurface. Moreover, the Blaschke normal Y of M is constant (cf. [LS$^+$, Example 2.1]) and thus its equiaffine Gauss map is degenerate.

Now, if one treats M as a centroaffine hypersurface with $-x$ being the centroaffine normal and takes $(x_1,\ldots,x_n)$ as local coordinates, direct calculations give the following relations:

$$\begin{cases} T = \dfrac{n+2}{2n}\displaystyle\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}, \quad h\left(\dfrac{\partial}{\partial x_i},\dfrac{\partial}{\partial x_j}\right) = \dfrac{2}{x_{n+1}}\delta_{ij}, \\ K\left(\dfrac{\partial}{\partial x_i},\dfrac{\partial}{\partial x_j}\right) = \displaystyle\sum_{k=1}^n \frac{x_i\delta_{jk}+x_j\delta_{ik}+x_k\delta_{ij}}{x_{n+1}}\frac{\partial}{\partial x_k}, \end{cases} \quad 1 \le i,j \le n. \tag{1.5}$$

Using these calculations, we obtain

CLAIM. *There exists a local h-orthonormal frame field $\{e_1,\ldots,e_n\}$ on M such that the difference tensor takes the following form:*

$$(1.6)\qquad \begin{cases} K(e_1,e_1)=\dfrac{3\sqrt{2}}{2}e_1, \quad K(e_1,e_i)=\dfrac{\sqrt{2}}{2}e_i, \quad 2\le i\le n,\\ K(e_i,e_j)=\dfrac{\sqrt{2}}{2}\delta_{ij}e_1, \quad 2\le i,j\le n. \end{cases}$$

To verify the claim, we first choose $e_1=T/\|T\|$, where $\|T\|=\frac{n+2}{\sqrt{2}\,n}$. Then it is easy to show that $K(e_1,e_1)=(3\sqrt{2}/2)e_1$. Next, we choose an arbitrary h-orthonormal basis $\{e_1,\dots,e_n\}$ and we assume that $e_i=\sum_{k=1}^n E_{ik}\frac{\partial}{\partial x_k}$ for $2\le i\le n$. By orthogonality and the second equation in (1.5) we have

$$(1.7)\qquad \frac{2}{x_{n+1}}\sum_{k=1}^n E_{ik}E_{jk}=\delta_{ij}, \quad \sum_{k=1}^n E_{ik}x_k=0, \quad i,j=2,\dots,n.$$

The Claim is now immediate from (1.5) and (1.7).

Finally, combining the Claim and the following Gauss equation for centroaffine hypersurfaces (cf. [LLS, V]; see also [CH, (2.2)], or [W, (1.14)], corresponding to $\varepsilon=1$):

$$\hat{R}(X,Y)Z=h(Y,Z)X-h(X,Z)Y-[K_X,K_Y]Z,$$

we can easily see that all the centroaffine sectional curvatures of the centroaffine hypersurface M are nonnegative, and actually take values in the interval $[0,1/2]$.

This completes the proof of the Fact. ■

Remark 1.9. Problem 1.1 is equivalent to asking whether, in the second part of Theorem 1.7, the assumption that (M^n,h) has all sectional curvatures nonnegative is redundant.

2. Preliminaries. In this section, we briefly recall basic facts about centroaffine hypersurfaces (cf. [LS$^+$, NS, SSV]) and Ros's integral formula on Riemannian manifolds.

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional affine space equipped with its canonical flat connection D. Let M^n be an n-dimensional smooth manifold. An immersion $x: M^n\to\mathbb{R}^{n+1}$ is said to be a *centroaffine hypersurface* if, for each point $x\in M^n$, the position vector x (from O) is transversal to the tangent space T_xM^n of M^n at x. In that situation, the position vector x defines the *centroaffine normalization* modulo orientation. For any vector fields X and Y tangent to M^n, we have the centroaffine formula of Gauss:

$$(2.1)\qquad D_X x_*(Y)=x_*(\nabla_X Y)+h(X,Y)(-\varepsilon x),$$

where $\varepsilon=1$ or -1. Moreover, in (2.1), we will call $-\varepsilon x$, ∇ and h the *centroaffine normal*, the *induced* (centroaffine) *connection* and the *centroaffine metric*, respectively. In this paper, we will consider only locally strongly convex centroaffine hypersurfaces, in the sense that the bilinear 2-form h

defined by (2.1) is definite; then we will choose ε such that the centroaffine metric h is positive definite.

Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface and $\hat{\nabla}$ be the Levi-Civita connection of its centroaffine metric h. Then its *difference tensor* K is defined by $K(X,Y) := K_X Y := \nabla_X Y - \hat{\nabla}_X Y$; it is symmetric as both connections are torsion-free. Then, the *Chebyshev vector field* T is defined by

$$h(T,X) = \frac{1}{n}\,\mathrm{trace}\,(K_X). \tag{2.2}$$

Using the difference tensor K and the Chebyshev vector field T one can define a traceless symmetric tensor $\tilde{K}$, the *traceless difference tensor*, by

$$\tilde{K}(X,Y) := K(X,Y) - \frac{n}{n+2}[h(X,Y)T + h(X,T)Y + h(Y,T)X]. \tag{2.3}$$

It is well-known that $\tilde{K}$ vanishes if and only if $x(M^n)$ lies in a hyperquadric (cf. [SSV, Section 7.1] and [LL$^+$, Lemma 2.1 and Remark 2.2]).

The *centroaffine shape operator* $\mathcal{T} : TM^n \to TM^n$ of $x : M^n \to \mathbb{R}^{n+1}$, also called the *Chebyshev operator*, is defined by (cf. [W])

$$\mathcal{T}(v) := \hat{\nabla}_v T, \qquad v \in TM^n. \tag{2.4}$$

It was shown by Wang [W] that $\mathcal{T}$ is self-adjoint with respect to the centroaffine metric, and that M^n is extremal for the centroaffine volume functional if and only if the trace of the Chebyshev operator vanishes identically. As stated in the introduction, an important subclass of centroaffine hypersurfaces consists of the *Chebyshev hypersurfaces* as defined below, which were first introduced by Liu and Wang [LW1] and have been extensively studied in [LSW, LW1, LW2, STV, L].

Definition 2.1 ([LW1]). A centroaffine hypersurface M^n is called a *Chebyshev hypersurface* if its Chebyshev operator $\mathcal{T}$ is proportional to the identity isomorphism $\mathrm{id} : TM^n \to TM^n$, i.e., $\mathcal{T} = \lambda \cdot \mathrm{id}$ for a function λ on M^n.

For a locally strongly convex centroaffine Chebyshev hypersurface $x : M^n \to \mathbb{R}^{n+1}$, just as for affine hyperspheres, we have (cf. [LSW])

$$h(\tilde{K}(X,Y),Z) = h(\tilde{K}(X,Z),Y), \tag{2.5}$$

$$(\hat{\nabla}_Z \tilde{K})(X,Y) = (\hat{\nabla}_Y \tilde{K})(X,Z). \tag{2.6}$$

As usual we define the covariant derivatives $\hat{\nabla}\tilde{K}$ and $\hat{\nabla}^2\tilde{K}$ by

$$\begin{aligned}(\hat{\nabla}\tilde{K})(Z,X,Y) &:= (\hat{\nabla}_Z \tilde{K})(X,Y),\\ (\hat{\nabla}^2\tilde{K})(W,Z,X,Y) &:= \hat{\nabla}_W((\hat{\nabla}\tilde{K})(Z,X,Y)) - (\hat{\nabla}\tilde{K})(\hat{\nabla}_W Z,X,Y)\\ &\quad - (\hat{\nabla}\tilde{K})(Z,\hat{\nabla}_W X,Y) - (\hat{\nabla}\tilde{K})(Z,X,\hat{\nabla}_W Y).\end{aligned}$$

Then we have the following Ricci identity:

$$(\hat{\nabla}^2\tilde{K})(W,Z,X,Y) - (\hat{\nabla}^2\tilde{K})(Z,W,X,Y) = \hat{R}(W,Z)\tilde{K}(X,Y) - \tilde{K}(\hat{R}(W,Z)X,Y) - \tilde{K}(X,\hat{R}(W,Z)Y). \tag{2.7}$$

Next, we review an important integral formula due to A. Ros [R] for Riemannian manifolds, which will be used in our proof of Theorem 1.7.

Let (M,g) be a compact Riemannian manifold with Levi-Civita connection $\tilde{\nabla}$. Let $\Pi : UM \to M$ be the unit tangent bundle of M, and UM_p its fiber over $p \in M$. We denote the canonical measures on M, UM_p and UM by dp, $d\sigma_p$ and $d\sigma$, respectively. Then, for any continuous function $f : UM \to \mathbb{R}$, we have

$$\int_{UM} f\, d\sigma = \int_M \Big\{ \int_{UM_p} f\, d\sigma_p \Big\}\, dp. \tag{2.8}$$

Let A be a k-covariant tensor on M, and $\tilde{\nabla}A$ the covariant derivative of A, which is a $(k+1)$-covariant tensor. Then Ros's integral formula states that for an orthonormal basis $\{e_1,\dots,e_n\}$ of TM, one has

$$\int_{UM} \sum_{i=1}^{n} (\tilde{\nabla}A)(e_i,e_i,v,\dots,v)\, d\sigma = 0. \tag{2.9}$$

3. Proof of Theorem 1.7. To prove Theorem 1.7, we partly follow the idea of F. Urbano [U], who showed that an n-dimensional compact totally real submanifold immersed in an n-dimensional complex space form with parallel mean curvature vector and nonnegative sectional curvature has parallel second fundamental form.

Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Chebyshev hyperovaloid. First of all, we define a $(0,7)$-type tensor A on M^n by

$$A(v_1,v_2,v_3,v_4,v_5,v_6,v_7) = h((\hat{\nabla}\tilde{K})(v_1,v_2,v_3),v_4)\, h(\tilde{K}(v_5,v_6),v_7).$$

Using (2.5)–(2.7), and assuming that $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_pM^n, $p \in M^n$, from direct calculations we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} (\hat{\nabla}A)(e_i,e_i,v,v,v,v,v,v) \\
&= \sum_{i=1}^{n} \big(h((\hat{\nabla}\tilde{K})(e_i,v,v),v)\big)^2 + h(\tilde{K}(v,v),v) \sum_{i=1}^{n} h((\hat{\nabla}^2\tilde{K})(e_i,v,e_i,v),v) \\
&= \|(\hat{\nabla}\tilde{K})(v,v,v)\|^2 + h(\tilde{K}(v,v),v) \sum_{i=1}^{n} h((\hat{\nabla}^2\tilde{K})(v,e_i,e_i,v),v) \\
&\quad - h(\tilde{K}(v,v),v) \sum_{i=1}^{n} \big(2h(\hat{R}(e_i,v)v,\tilde{K}(e_i,v)) + h(\hat{R}(e_i,v)e_i,\tilde{K}(v,v))\big).
\end{aligned} \tag{3.1}$$

Integrating (3.1) over UM, then using (2.9) and the relation $\mathrm{trace}_h(\tilde{K})=0$, we get

$$
\begin{aligned}
0 &= \int_{UM} \sum_{i=1}^{n} (\hat{\nabla} A)(e_i, e_i, v, v, v, v, v, v)\, d\sigma \\
&= \int_{UM} \|(\hat{\nabla}\tilde{K})(v, v, v)\|^2\, d\sigma - \int_{UM} h(\tilde{K}(v, v), v) \\
&\quad \cdot \sum_{i=1}^{n} \big(h(\hat{R}(e_i, v)e_i, \tilde{K}(v, v)) + 2h(\hat{R}(e_i, v)v, \tilde{K}(e_i, v)) \big)\, d\sigma.
\end{aligned}
\tag{3.2}
$$

Next, let α be the 1-form over UM_p defined by

$$\alpha_v(e) = h(\tilde{K}(v, v), v)\, h(\hat{R}(e, v)v, \tilde{K}(v, v))$$

for $v \in UM_p$ and $e \in T_v(UM_p)$. Take an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $T_v(UM_p)$. We extend this basis, by parallel translation along geodesics through v, to a normal neighbourhood of v and denote it by $\{E_1, \ldots, E_{n-1}\}$. For $i = 1, \ldots, n-1$, we put

$$\gamma_i(t) = v \cos t + e_i \sin t \in UM_p.$$

Then $\gamma_i(t)$ are geodesics in UM_p through v and they satisfy

$$
\begin{cases}
\gamma_i'(t) = -v \sin t + e_i \cos t, \quad \gamma_i'(0) = e_i, \\
\bar{\nabla}_{\gamma_i'(t)} \gamma_i'(t) = 0,
\end{cases}
\tag{3.3}
$$

where $\bar{\nabla}$ is the Levi-Civita connection on UM_p.

It follows that

$$\gamma_i'(t) = E_i(\gamma_i(t)), \qquad i = 1, \ldots, n-1. \tag{3.4}$$

Let $e_n = v$ and denote by δ the co-differential operator on UM_p. Then a straightforward computation, with the use of (3.3) and (3.4), yields

$$
\begin{aligned}
-(\delta\alpha)(v) &= \sum_{i=1}^{n-1} (\bar{\nabla}_{E_i}\alpha)E_i\Big|_v = \sum_{i=1}^{n-1} E_i\alpha(E_i)\Big|_v - \sum_{i=1}^{n-1} \alpha(\bar{\nabla}_{E_i}E_i)\Big|_v \\
&= \sum_{i=1}^{n-1} E_i\alpha(E_i)\Big|_v = \sum_{i=1}^{n-1} \frac{d}{dt}\alpha_{\gamma_i(t)}(E_i(\gamma_i(t)))\Big|_{t=0} \\
&= \sum_{i=1}^{n-1} \frac{d}{dt}\alpha_{\gamma_i(t)}(\gamma_i'(t))\Big|_{t=0} \\
&= 3h(\hat{R}(\tilde{K}(v, v), v)v, \tilde{K}(v, v)) + h(\tilde{K}(v, v), v) \\
&\quad \cdot \sum_{i=1}^{n} \big(h(\hat{R}(e_i, v)e_i, \tilde{K}(v, v)) + 2h(\hat{R}(e_i, v)v, \tilde{K}(e_i, v)) \big).
\end{aligned}
\tag{3.5}
$$

Now, integrating over UM_p and noting that $\int_{UM_p}(\delta\alpha)(v)d\sigma_p = 0$, we obtain

$$\begin{aligned}(3.6)\qquad &\int_{UM_p} 3h(\hat{R}(\tilde{K}(v,v),v)v,\tilde{K}(v,v))\,d\sigma_p\\ &= -\int_{UM_p} h(\tilde{K}(v,v),v)\sum_{i=1}^{n}\big(h(\hat{R}(e_i,v)e_i,\tilde{K}(v,v))\\ &\qquad\qquad + 2h(\hat{R}(e_i,v)v,\tilde{K}(e_i,v))\big)\,d\sigma_p.\end{aligned}$$

Combining (2.8), (3.2) and (3.6), we immediately see that

$$(3.7)\qquad 0 = \int_{UM}\big(\|(\hat{\nabla}\tilde{K})(v,v,v)\|^2 + 3h(\hat{R}(\tilde{K}(v,v),v)v,\tilde{K}(v,v))\big)\,d\sigma.$$

By assumption, $x(M^n)$ has all centroaffine sectional curvatures nonnegative, so $h(\hat{R}(\tilde{K}(v,v),v)v,\tilde{K}(v,v)) \geq 0$. It follows from (3.7) and (2.6) that $\hat{\nabla}\tilde{K} = 0$. Then, we can apply the recent classification of locally strongly convex centroaffine hypersurfaces with $\hat{\nabla}\tilde{K} = 0$, i.e., [CH, Theorem 1.1] (cf. also [CHM] for its preliminary version) to conclude that $x(M^n)$ is an ellipsoid due to compactness of $x(M^n)$. Finally, since M^n is a centroaffine hypersurface, the origin $O \in \mathbb{R}^{n+1}$ must be in the interior of the ellipsoid.

This completes the proof of Theorem 1.7. ■

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