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THE CONTINUOUS WAVELET TRANSFORM ON ULTRADISTRIBUTION SPACES

BY

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Abstract. The wavelet transform of ultradistributions in $\mathscr{Z}'_\omega$ is defined. The corresponding inversion formula is established by interpreting convergence in the weak distributional sense.

1. Introduction. Let $\psi \in L^2(\mathbb{R})$. Define

$$\psi_{b,a}(t) := \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R},\, b \in \mathbb{R},\, a \in \mathbb{R}_0 = \mathbb{R}\setminus\{0\}. \tag{1.1}$$

The *wavelet transform* $W(b,a)$ of $f \in L^2(\mathbb{R})$ with respect to the wavelet $\psi \in L^2(\mathbb{R})$ is defined by

$$W(b,a) := \int_{\mathbb{R}} f(t)\overline{\psi_{b,a}(t)}\,dt, \tag{1.2}$$

and the corresponding wavelet inversion formula is given by

$$\frac{1}{C_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}_0} \frac{1}{\sqrt{|a|}} W(b,a)\psi\left(\frac{x-b}{a}\right)\frac{db\,da}{a^2} = f(x), \tag{1.3}$$

where

$$C_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|}\,dw < \infty \quad [2, p. 9].$$

Now, we recall definitions and properties of the needed test function and ultradistribution spaces from [3]. Let $\mathfrak{M}$ be the set of all real valued functions ω on $\mathbb{R}$ which can be represented as $\omega(x) = \sigma(|x|)$ where σ is an increasing continuous concave function on $[0,\infty)$ and satisfying the following conditions [1, p. 363]:

$$0 = \omega(0) \le \omega(\xi+\eta) \le \omega(\xi) + \omega(\eta), \quad \forall \xi,\eta \in \mathbb{R}; \tag{1.4}$$

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$$\int_{\mathbb{R}} \frac{\omega(\xi)}{(1+|\xi|)^2}\, d\xi < \infty; \tag{1.5}$$

there exists a real number p and a positive real number q such that for ξ large enough,

$$\omega(\xi) \geq p + q\log(1+|\xi|); \tag{1.6}$$

and ω is concave down on $[0,\infty)$ and is even. An example is $\omega(\xi) = \sqrt{|\xi|}$.

Let $\omega \in \mathfrak{M}$. We denote by $\mathscr{Z}_\omega$ the set of all functions $\psi(t) \in \mathscr{Z}_\omega$ on $-\infty < t < \infty$ which satisfy

$$P_{k,\lambda}(\psi) = \sup_{t\in\mathbb{R}} e^{\lambda\omega(|t|)} |D^k\psi(t)| < \infty \tag{1.7}$$

for all non-negative integers k and all non-negative real λ. The topology on $\mathscr{Z}_\omega$ is defined by the seminorms $\{P_{k,\lambda}\}$. It can be readily seen that $\mathscr{Z}_\omega$ is a vector space. A sequence $\{\psi_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence in $\mathscr{Z}_\omega$ if for each non-negative integer k, $P_{k,\lambda}(\psi_\mu - \psi_\nu) \to 0$ as $\mu,\nu \to \infty$ independently of each other. The space $\mathscr{Z}_\omega$ is a sequentially complete space and therefore it is a complete countably multinormed space and so a Fréchet space. The dual of $\mathscr{Z}_\omega$ is denoted by $\mathscr{Z}'_\omega$; it is a distribution space [3, p. 170]. The space $\mathcal{D}(\mathbb{R})$ consisting of C^∞-functions of compact support is a subspace of $\mathscr{Z}_\omega(\mathbb{R})$ and $\mathscr{Z}'_\omega \subset \mathcal{D}'$.

Suppose that the Fourier transform of $\psi \in \mathscr{Z}_\omega$, defined by

$$\hat{\psi}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi}\psi(x)\, dx,$$

satisfies

$$\pi_{k,\lambda}(\psi) = \sup_{\xi\in\mathbb{R}} e^{\lambda\omega(|\xi|)} |D^k\hat{\psi}(\xi)| < \infty, \qquad \lambda \geq 0,\ k \in \mathbb{N}_0. \tag{1.8}$$

Then the space of all functions $\psi \in L^1(\mathbb{R})$ such that $\psi, \hat{\psi} \in C^\infty(\mathbb{R})$ and (1.7), (1.8) hold is denoted by $\mathscr{S}_\omega$; the topology of $\mathscr{S}_\omega$ is defined by the seminorms $P_{k,\lambda}$ and $\pi_{k,\lambda}$ [1, p. 377].

Let K be a compact subset of $\mathbb{R}$. The space $\mathcal{D}_\omega(K)$ is the set of all ψ in $L^1(\mathbb{R})$ such that ψ has support in K and

$$\|\psi\|_\lambda := \int_{\mathbb{R}} |\hat{\psi}(\xi)| e^{\lambda\omega(|\xi|)}\, d\xi < \infty, \qquad \forall\lambda > 0. \tag{1.9}$$

Let $\{K_n\}$ be a sequence of compact sets in $\mathbb{R}$ such that $\bigcup_{n=1}^\infty K_n = \mathbb{R}$ and K_n is contained in K_{n+1} for all n. Then $\mathcal{D}_\omega(\mathbb{R}) = \lim\operatorname{ind} \mathcal{D}_\omega(K_n)$, where lim ind denotes inductive limit. It is clear that $\mathcal{D}_\omega$ is dense in $\mathscr{S}_\omega$ and $\mathscr{Z}_\omega$ ($\omega \in \mathfrak{M}$). Since the topology of $\mathcal{D}_\omega$ is stronger than that induced on $\mathcal{D}_\omega$ by $\mathscr{S}_\omega$, it follows that the restriction of any $f \in \mathscr{S}'_\omega$ to $\mathcal{D}_\omega$ is in $\mathcal{D}'_\omega$. The elements of $\mathcal{D}'_\omega$ are called *ultradistributions* [3]. Since the topology of $\mathcal{D}_\omega$ is finer than the topology on $\mathcal{D}'_\omega$ induced by $\mathscr{L}_\omega$, we have $\mathscr{L}'_\omega \subset \mathcal{D}'_\omega$.

Recently, the wavelet transform (1.2) has been extended to distributions, and inversion formulae have been established in the distribution setting by Pathak [5] and Pathak et al. [6, 7, 8].

Band-limited wavelets with subexponential decay were studied by Dziubański et al. [4] and extended by Pathak et al. [9]. Using these wavelets in the present work we shall investigate properties of the wavelet $\psi_{b,a} \in \mathscr{L}_\omega(\mathbb{R})$. The wavelet transform of $f \in \mathscr{L}'_\omega$ will be studied and the inversion formula (1.3) will be extended to the distribution space $\mathscr{L}'_\omega$.

2. Wavelet transform on $\mathscr{L}'_\omega$. In this section, certain basic properties of the wavelets in $\mathscr{L}_\omega$ and the wavelet transform of $f \in \mathscr{L}'_\omega$ are obtained.

LEMMA 2.1. *If $\psi \in \mathscr{L}_\omega$, then $\psi_{b,a} \in \mathscr{L}_\omega$ for fixed $b, a \in \mathbb{R}, a \neq 0$.*

Proof. Let $a \neq 0$ and b be fixed real numbers. Then for $k = 0, 1, 2, \ldots,$

$$\sup_{-\infty<t<\infty} \left| e^{\lambda\omega(|t|)} D^k \psi\left(\frac{t-b}{a}\right)\right| = \sup_{-\infty<t<\infty} \left| e^{\lambda\omega(|\frac{t-b}{a}|)} \psi^{(k)}\left(\frac{t-b}{a}\right)\left(\frac{1}{a^k}\right)\right| \left|\frac{e^{\lambda\omega(|t|)}}{e^{\lambda\omega(|\frac{t-b}{a}|)}}\right|.$$

Assume that there exists a positive integer n such that $|a| \le n$. Then by (1.4) it follows that $\omega(|t|) \le \omega(n|t/a|) \le n\omega(|t/a|)$, so that for $n\lambda = \mu$ we get

$$\begin{aligned}\sup_{-\infty<t<\infty} \left| e^{\lambda\omega(|t|)} D^k \psi\left(\frac{t-b}{a}\right)\right| &\le \frac{1}{|a|^k} P_{k,\mu}(\psi) \sup_{-\infty<t<\infty} \left|\frac{e^{\mu\omega(|\frac{t-b}{a}|+|\frac{b}{a}|)}}{e^{\mu\omega(|\frac{t-b}{a}|)}}\right| \quad \text{(by (1.7))}\\ &\le P_{k,\mu}(\psi)\left(\frac{1}{|a|^k}\right) e^{\mu\omega(\frac{|b|}{|a|})} < \infty \quad \text{(by (1.4))}\end{aligned}$$

for all real numbers b and $a \neq 0$. ∎

In what follows we shall assume that $\psi \in \mathscr{L}_\omega(\mathbb{R})$ is the basic function generating the wavelet $\psi_{b,a}$ given by (1.1), such that $C_\psi < \infty$. Since $\psi_{b,a}$ belongs to $\mathscr{L}_\omega$ for fixed b and $a \neq 0$ as a function of t under the conditions of Lemma 2.1, for $f \in \mathscr{L}'_\omega$ the wavelet transform $W(b,a)$ of f is defined by

$$W(b,a) = \langle f(t), \overline{\psi_{b,a}}\rangle, \quad a \neq 0,\ a, b \in \mathbb{R}. \tag{2.1}$$

THEOREM 2.2. *Let $f \in \mathscr{L}'_\omega$, $\psi \in \mathscr{L}_\omega$ and $W(b,a)$ be defined by* (2.1). *Then $W(b,a)$ is smooth.*

Proof. The proof is similar to one in [8] and is omitted. ∎

THEOREM 2.3. *For real b and $a \neq 0$ let $W(b,a)$ be defined by* (2.1). *Then under the conditions of Lemma* 2.1,

$$W(b,a) = O\big(|a|^{-r-1/2}\exp(\lambda\omega(|b/a|))\big), \qquad |a| \to 0, \text{ for some } r \in \mathbb{N}.$$

Proof. By the boundedness property of generalized functions there exist a constant $C > 0$ and a non-negative integer r depending on ψ such that

$$\begin{aligned}
|W(b,a)| &= \left| \frac{1}{\sqrt{|a|}} \left\langle f(t), \overline{\psi\left(\frac{t-b}{a}\right)} \right\rangle \right| \\
&\leq \frac{C}{\sqrt{|a|}} \max_{0\leq k\leq r} \sup_{b,t\in\mathbb{R}} \left| e^{\lambda\omega(\frac{|t|}{|a|})} D_t^k \overline{\psi\left(\frac{t-b}{a}\right)} \right| \\
&= \frac{C}{\sqrt{|a|}} \left[\sup_{b,t\in\mathbb{R}} \left| \frac{e^{\lambda\omega(|\frac{t}{a}|)}}{e^{\lambda\omega(|\frac{t-b}{a}|)}} \right| \right] \frac{1}{|a|^r} \max_{0\leq k\leq r} P_{k,\lambda}(\psi) \qquad \text{for } |a| < 1 \\
&\leq C \max_{0\leq k\leq r} P_{k,\lambda}(\psi) \frac{e^{\lambda\omega(|b/a|)}}{|a|^{r+1/2}}
\end{aligned}$$

by using (1.4), as in Lemma 2.1. This gives the required result. ∎

3. Inversion of the wavelet transform on $\mathscr{L}'_\omega$. Now, motivated by the inversion formula (1.3), we define a function $H_{N,R}(t,\xi)$ as follows:

$$H_{N,R}(t,\xi) := \frac{1}{C_\psi} \int_{-N}^{N} \int_{-R}^{R} \psi\left(\frac{t-b}{a}\right) \overline{\psi\left(\frac{\xi-b}{a}\right)} \frac{db\,da}{a^2|a|}, \qquad a \neq 0, \tag{3.1}$$

where $N, R, t, \xi \in \mathbb{R}$. This leads to a new inversion formula for the wavelet transform.

LEMMA 3.1. *Let $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ and let*

$$\psi_{N,R}(t,\xi) := \frac{1}{C_\psi} \int_{-N}^{N} \int_{-R}^{R} \psi_{b,a}(t)\psi_{b,a}^1(\xi) \frac{db\,da}{a^2}, \tag{3.2}$$

where $\psi_{b,a}^1(\xi) = \int_\Delta^\xi \psi_{b,a}(x)\,dx$ for some $\Delta \in \mathbb{R}$. Then under the conditions of Lemma 2.1,

$$\lim_{\substack{N\to\infty \\ R\to\infty}} \psi_{N,R}(t,\xi) = \begin{cases} 1, & \Delta < t < \xi, \\ 0, & t \geq \xi \text{ or } t \leq \Delta. \end{cases} \tag{3.3}$$

Proof. From (1.3) we have

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}_0} W(b,a)\psi_{b,a}(t)\, \frac{db\, da}{a^2} \tag{3.4}$$
$$= \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \Big(\int_{\mathbb{R}} f(x)\overline{\psi_{b,a}(x)}\, dx \Big) \psi_{b,a}(t)\, \frac{db\, da}{a^2}$$
$$= \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \psi_{b,a}(t)\, \frac{db\, da}{a^2} \int_{\mathbb{R}} f(x)\overline{\psi_{b,a}(x)}\, dx.$$

If we put

$$f(t) = \begin{cases} 1, & \Delta < t < \xi, \\ 0, & t \geq \xi \text{ or } t \leq \Delta, \end{cases}$$

in (3.4), we obtain

$$1 = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \psi_{b,a}(t)\, \frac{db\, da}{a^2} \int_{\Delta}^{\xi} \overline{\psi_{b,a}(x)}\, dx$$
$$= \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \psi_{b,a}(t)\psi^1_{b,a}(\xi)\, \frac{db\, da}{a^2} = \lim_{\substack{N\to\infty \\ R\to\infty}} \psi_{N,R}(t,\xi).$$

Thus we arrive at (3.3). ∎

LEMMA 3.2. *Let $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. Then for $c, d \in \mathbb{R}$ such that $\Delta < c < d$, we have*

$$\lim_{\substack{N\to\infty \\ R\to\infty}} \int_c^d H_{N,R}(t,\xi)\, d\xi = \begin{cases} 1 & \text{if } c < t < d, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Proof. For $\Delta < c < d$, we have

$$\lim_{\substack{N\to\infty \\ R\to\infty}} \int_c^d H_{N,R}(t,\xi)\, d\xi = \lim_{\substack{N\to\infty \\ R\to\infty}} \Big[\int_\Delta^d H_{N,R}(t,\xi)\, d\xi - \int_\Delta^c H_{N,R}(t,\xi)\, d\xi \Big]$$
$$= \lim_{\substack{N\to\infty \\ R\to\infty}} \Big[\int_{-N}^{N} \int_{-R}^{R} \psi_{b,a}(t)\, \frac{db\, da}{a^2} \int_\Delta^d \overline{\psi_{b,a}(x)}\, dx - \int_{-N}^{N} \int_{-R}^{R} \psi_{b,a}(t)\, \frac{db\, da}{a^2} \int_\Delta^c \overline{\psi_{b,a}(x)}\, dx \Big]$$
$$= \lim_{\substack{N\to\infty \\ R\to\infty}} \Big[\int_{-N}^{N} \int_{-R}^{R} \psi_{b,a}(t)\psi^1_{b,a}(d)\, \frac{db\, da}{a^2} - \int_{-N}^{N} \int_{-R}^{R} \psi_{b,a}(t)\psi^1_{b,a}(c)\, \frac{db\, da}{a^2} \Big]$$
$$= \lim_{\substack{N\to\infty \\ R\to\infty}} [\psi_{N,R}(t,d) - \psi_{N,R}(t,c)] = \begin{cases} 0 & \text{if } t < c \text{ or } t > d, \\ 1 & \text{if } c < t < d. \end{cases}$$

This yields (3.5). ∎

Lemma 3.3. *Assume that the wavelet $\psi_{b,a}$ satisfies the conditions of Lemma 2.1. Then the function $H_{N,R}(t,\xi)$ is uniformly bounded for all real numbers t and ξ.*

Proof. We have

$$|H_{N,R}(t,\xi)| \le \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\psi\left(\frac{t-b}{a}\right)\overline{\psi\left(\frac{\xi-b}{a}\right)}\right|\frac{db\,da}{a^2|a|}, \qquad a\neq 0$$

$$= \frac{1}{C_\psi}\Big(\int_0^{\infty}\int_0^{\infty} + \int_{-\infty}^{0}\int_{-\infty}^{0}\Big)\left|\psi\left(\frac{t-\xi+u}{a}\right)\overline{\psi\left(\frac{u}{a}\right)}\right|\frac{du\,da}{a^2|a|}$$

(putting $b=\xi-u$)

$$= I_1 + I_2 \quad \text{(say)}.$$

Since $\psi\in\mathscr{Z}_\omega$, we have

$$\text{(3.6)}\quad I_1 = \frac{1}{C_\psi}\int_0^{\infty}\int_0^{\infty}\left|\psi\left(\frac{t-\xi+u}{a}\right)\overline{\psi\left(\frac{u}{a}\right)}\right|\frac{du\,da}{a^3}$$

$$\le C\int_0^{\infty}\int_0^{\infty} e^{-\lambda\omega(|\frac{t-\xi+u}{a}|)}e^{-\mu\omega(|\frac{u}{a}|)}\frac{du\,da}{a^3} \qquad \text{for some } C>0$$

$$\le C_1\int_0^{\infty}\int_0^{\infty}\left(1+\left|\frac{t-\xi+u}{a}\right|\right)^{-\lambda p}\left(1+\frac{u}{a}\right)^{-\mu q}\frac{du\,da}{a^3} \qquad \text{(by (1.6))}$$

$$\le C_1\Big(\int_0^{1}\int_0^{1} + \int_1^{\infty}\int_1^{\infty}\Big)(1+|t-\xi+u|x)^{-\lambda p}(1+ux)^{-\mu q}x\,du\,dx$$

(putting $1/a=x$)

$$= T_1 + T_2 \quad \text{(say)}.$$

Then

$$T_1 = C_1\int_0^{1}\int_0^{1}(1+|t-\xi+u|x)^{-\lambda p}(1+ux)^{-\mu q}x\,du\,dx \le C_1\int_0^{1}\int_0^{1}du\,dx \le C_1.$$

Also,

$$T_2 = C_1\int_1^{\infty}\int_1^{\infty}(1+|t-\xi+u|x)^{-\lambda p}(1+ux)^{-\mu q}x\,du\,dx \le \int_1^{\infty}\int_1^{\infty}(ux)^{-\mu q}x\,du\,dx$$

$$\le \int_1^{\infty}u^{-\mu q}\,du\int_1^{\infty}x^{1-\mu q}\,dx < \infty \qquad \text{for } \mu > 2/q\,.$$

Similarly, we can prove the boundedness of I_2. ∎

LEMMA 3.4. *For* $\delta > 0$,

$$\int_c^{t-\delta} H_{N,R}(t,\xi)\,d\xi \to 0$$

as $N, R \to \infty$ *uniformly for all* $t \in (c+\delta, L]$ *for some* $L \in \mathbb{R}$.

Proof. We have

$$\int_c^{t-\delta} H_{N,R}(t,\xi)\,d\xi = \frac{1}{C_\psi}\int_{-N}^{N}\int_{-R}^{R}\psi_{b,a}(t)\,\frac{db\,da}{a^2}\int_c^{t-\delta}\overline{\psi_{b,a}(\xi)}, \qquad a \neq 0$$

$$= \frac{1}{C_\psi}\int_{-N}^{N}\int_{-R}^{R}\psi_{b,a}(t)\,\frac{db\,da}{a^2}\int_{\Delta}^{t-\delta}\overline{\psi_{b,a}(\xi)} - \frac{1}{C_\psi}\int_{-N}^{N}\int_{-R}^{R}\psi_{b,a}(t)\,\frac{db\,da}{a^2}\int_{\Delta}^{c}\overline{\psi_{b,a}(\xi)}$$

$$= \psi_{N,R}(t, t-\delta) - \psi_{N,R}(t,c)$$

$$\to 0 - 0 = 0 \quad \text{(using Lemma 3.1).} \ \blacksquare$$

LEMMA 3.5. *For* $\delta > 0$,

$$\int_{t+\delta}^{d} H_{N,R}(t,\xi)\,d\xi \to 0$$

as $N, R \to \infty$ *uniformly for all* $t \in [-L, d-\delta)$ *for some* $L \in \mathbb{R}$.

Proof. We have

$$\int_{t+\delta}^{d} H_{N,R}(t,\xi)\,d\xi = \frac{1}{C_\psi}\int_{-N}^{N}\int_{-R}^{R}\psi_{b,a}(t)\,\frac{db\,da}{a^2}\int_{t+\delta}^{d}\overline{\psi_{b,a}(\xi)}, \qquad a \neq 0$$

$$= \frac{1}{C_\psi}\int_{-N}^{N}\int_{-R}^{R}\psi_{b,a}(t)\,\frac{db\,da}{a^2}\int_{\Delta}^{d}\overline{\psi_{b,a}(\xi)} - \frac{1}{C_\psi}\int_{-N}^{N}\int_{-R}^{R}\psi_{b,a}(t)\,\frac{db\,da}{a^2}\int_{\Delta}^{t+\delta}\overline{\psi_{b,a}(\xi)}$$

$$= \psi_{N,R}(t,d) - \psi_{N,R}(t, t+\delta)$$

$$\to 1 - 1 = 0 \quad \text{(using Lemma 3.1).} \ \blacksquare$$

LEMMA 3.6. *Let* $\phi \in \mathcal{D}_\omega$ *with support in* $[c,d]$, *where* $\Delta < c < d$. *Let* $c+\delta \le t \le L$, $\delta > 0$. *Then*

$$\int_c^{t-\delta} H_{N,R}(t,\xi)\phi(\xi)\,d\xi \to 0 \qquad \text{as } N \to \infty, R \to \infty \tag{3.7}$$

uniformly for all $t \in [c+\delta, L]$.

Proof. By Lemma 3.3, there exists a constant M such that $|H_{N,R}(t,\xi)| \le M$ uniformly for all $x \in [c,d]$, $t \in [c+\delta, L]$ and $N, R > 0$. In view of the uniform continuity of $\phi(\xi)$ in $c \le \xi \le d$, for a given $\varepsilon > 0$, we can find a continuous function $\chi(\xi)$ such that

$$\int_{c}^{t-\delta} |\phi(\xi)-\chi(\xi)|\,d\xi \le \int_{c}^{L-\delta} |\phi(\xi)-\chi(\xi)|\,d\xi < \frac{\varepsilon}{M}.$$

The interval $(c, t-\delta)$ may be divided into subintervals $(c,\xi_1), (\xi_1,\xi_2), \ldots, (\xi_{n-1}, t-\delta)$ such that the fluctuation of $\chi(\xi)$ in each of these subintervals is less than $\varepsilon/(K(L-\delta-c))$. Let $\theta(\xi)$ be a function which, in the interior of each part (ξ_{r-1},ξ_r), where $r=1,\ldots,n$, has the constant value $c_r = \chi(\xi_r+\xi_{r-1})/2$. At the extremities of the parts, we take $\theta(\xi)$ to have the value zero. Thus, $\theta(\xi)$ has the finite set of values $c_1,\ldots,c_n,0$.

Since $|\chi(\xi)-\theta(\xi)| < \varepsilon/(K(L-\delta-c))$ everywhere expect at the end points of n subintervals of $(c,t-\delta)$, we have

$$\int_{c}^{t-\delta} |\chi(\xi)-\theta(\xi)|\,d\xi < \frac{\varepsilon}{K}, \quad \text{and therefore} \quad \int_{c}^{t-\delta} |\phi(\xi)-\theta(\xi)|\,d\xi < \frac{2\varepsilon}{M}.$$

Now,

$$\begin{aligned}
&\Big|\int_{c}^{t-\delta} \phi(\xi)H_{N,R}(t,\xi)\,d\xi\Big| \\
&\qquad \le \Big|\int_{c}^{t-\delta} \{\phi(\xi)-\theta(\xi)\}H_{N,R}(t,\xi)\,d\xi\Big| + \Big|\sum_{r=1}^{n} c_r \int_{\xi_{r-1}}^{\xi_r} H_{N,R}(t,\xi)\,d\xi\Big| \\
&\qquad \le \int_{c}^{t-\delta} |\phi(\xi)-\theta(\xi)|\,|H_{N,R}(t,\xi)|\,d\xi + \sum_{r=1}^{n} |c_r| \Big|\int_{\xi_{r-1}}^{\xi_r} H_{N,R}(t,\xi)\,d\xi\Big| \\
&\qquad < 2\varepsilon + \sum_{r=1}^{n} |c_r| \Big|\int_{\xi_{r-1}}^{\xi_r} H_{N,R}(t,\xi)\,d\xi\Big|.
\end{aligned}$$

Since t lies outside the interval $[\xi_{r-1},\xi_r]$ for each $r=1,2,\ldots,$ in view of Lemma 3.5,

$$\Big|\int_{\xi_{r-1}}^{\xi_r} H_{N,R}(t,\xi)\,d\xi\Big| \to 0$$

independently of t for all $t\in[c+\delta,L]$ as $N\to\infty$ and $R\to\infty$. A positive number N_ε (not depending on ξ) can be so chosen that

$$\Big|\int_{\xi_{r-1}}^{\xi_r} H_{N,R}(t,\xi)\,d\xi\Big| < \frac{\varepsilon}{\sum_{r=1}^{n}|c_r|} \qquad \text{for } r=1,2,\ldots,$$

and for all values of $t\in[c+\delta,L]$. Thus, $|\int_c^{t-\delta}\phi(\xi)H_{N,R}(t,\xi)\,d\xi| < 3\varepsilon$, provided $N\ge N_\varepsilon$, for all values of $t\in[c+\delta,L]$. ∎

LEMMA 3.7. *Let $\phi \in \mathcal{D}_\omega$ with support in $[c,d]$, where $\Delta < c < d$. Then*

$$\int_{t+\delta}^{d} H_{N,R}(t,\xi)\phi(\xi)\,d\xi \to 0 \qquad \text{as } N\to\infty, R\to\infty \tag{3.8}$$

uniformly for all $t \in (\Delta, d-\delta)$.

Proof. Assume at first that $\phi(\xi)$ is an infinitely differentiable real valued function defined on $[t+\delta, d]$, $\Delta < t < d-\delta$. Then $\phi(\xi)$ is of bounded variation on $[c, t-\delta]$ [10, Ex. b, p. 118]. Consequently, there exist increasing functions $\mathfrak{p}(\xi)$ and $\mathfrak{q}(\xi)$ on $[t+\delta, d]$, with $\mathfrak{p}(t+\delta) = \mathfrak{q}(d) = 0$, such that

$$\phi(\xi) = \mathfrak{p}(t+\delta) + \mathfrak{p}(\xi) - \mathfrak{q}(\xi) \qquad (t+\delta \le \xi \le d)$$

(see [10, Theorem 6.27, p. 120]). Hence

$$\int_{t+\delta}^{d} H_{N,R}(t,\xi)\phi(\xi)\,d\xi = \mathfrak{p}(t+\delta)\int_{t+\delta}^{d} H_{N,R}(t,\xi)\,d\xi + \int_{t+\delta}^{d} \mathfrak{p}(\xi)H_{N,R}(t,\xi)\,d\xi - \int_{t+\delta}^{d} \mathfrak{q}(\xi)H_{N,R}(t,\xi)\,d\xi.$$

The result can now be proved by using the mean value theorem of integral calculus followed by a variation of the technique used in the proof of Lemma 3.5. ■

LEMMA 3.8. *Let the support of $\phi \in \mathcal{D}_\omega$ be contained in $[c,d] \subset \mathbb{R}$, where $d > c$ and $H_{N,R}(t,\xi)$ be the function defined in* (3.1). *Let $\psi_{b,a} \in \mathscr{Z}_\omega(\mathbb{R})$ and $a, b \in \mathbb{R}, a \neq 0$. Then*

$$\int_{c}^{d} H_{N,R}(t,\xi)\phi(\xi)\,d\xi \to \phi(t) \qquad \text{in } \mathcal{D}_\omega \text{ as } N\to\infty, R\to\infty. \tag{3.9}$$

Proof. It is easy to verify that

$$D_t^k H_{N,R}(t,\xi) = (-1)^k D_\xi^k H_{N,R}(t,\xi), \qquad k \in \mathbb{N}_0. \tag{3.10}$$

Hence,

$$D_t^k \int_{c}^{d} H_{N,R}(t,\xi)\phi(\xi)\,d\xi = \int_{c}^{d} (-1)^k D_\xi^k H_{N,R}(t,\xi)\phi(\xi)\,d\xi = \int_{c}^{d} H_{N,R}(t,\xi)[D_\xi^k \phi(\xi)]\,d\xi.$$

Then using (3.5) we get

$$\lim_{\substack{N\to\infty\\R\to\infty}} e^{\lambda\omega(|t|)} D_t^k\Big[\int_c^d H_{N,R}(t,\xi)\phi(\xi)\,d\xi - \phi(t)\Big] \tag{3.11}$$
$$= \lim_{\substack{N\to\infty\\R\to\infty}} e^{\lambda\omega(|t|)}\Big[\int_c^d H_{N,R}(t,\xi)\{\phi^{(k)}(\xi)\}\,d\xi - \phi^{(k)}(t)\int_c^d H_{N,R}(t,\xi)\,d\xi\Big].$$

Therefore, this problem is reduced to proving the following result:

$$\lim_{\substack{N\to\infty\\R\to\infty}} e^{\lambda\omega(|t|)}\int_c^d H_{N,R}(t,\xi)[\Phi(\xi)-\Phi(t)]\,d\xi = 0$$

uniformly for all $t\in\mathbb{R}$, where $\Phi(\xi)=\phi^{(k)}(\xi)$. For $t\in[c,d]$, we write

$$I = e^{\lambda\omega(|t|)}\int_c^d [\Phi(\xi)-\Phi(t)]H_{N,R}(t,\xi)\,d\xi.$$

Let δ be a positive number less than 1. Then

$$I = e^{\lambda\omega(|t|)}\Big(\int_c^{t-\delta} + \int_{t-\delta}^{t+\delta} + \int_{t+\delta}^{d}\Big)[\Phi(\xi)-\Phi(t)]H_{N,R}(t,\xi)\,d\xi$$
$$= I_1 + I_2 + I_3 \quad \text{(say)}.$$

Now, $I_2 = 0$ for $t > d+\delta$ and $t < c-\delta$. Using Lemma 3.3 for $t\in[c-\delta,d+\delta]$ we can find a positive constant M independent of t such that

$$|I_2| = \Big|e^{\lambda\omega(|t|)}\int_{t-\delta}^{t+\delta}[\Phi(\xi)-\Phi(t)]H_{N,R}(t,\xi)\,d\xi\Big|$$
$$\le e^{\lambda\omega(|t|)}\sup_{t-\delta<\eta<t+\delta}|\Phi'(\eta)|\int_{t-\delta}^{t+\delta}|H_{N,R}(t,\xi)|\,d\xi \le \delta M.$$

Now, for arbitrary $\varepsilon>0$ choose $\delta>0$ such that $\delta M<\varepsilon$. Then

$$|I_2| < \varepsilon \quad \text{for } \delta = \min(1,\varepsilon/M). \tag{3.12}$$

Next we consider

$$I_1 = e^{\lambda\omega(|t|)}\int_c^{t-\delta} H_{N,R}(t,\xi)[\Phi(\xi)-\Phi(t)]\,d\xi = I_{1,1} - I_{1,2}.$$

Then $I_{1,2}=0$ if $t<c$ or $t>d$. For $t\in(c,d)$,

$$|I_{1,2}| \le P_{0,\lambda}(\Phi)\Big|\int_c^{t-\delta} H_{N,R}(t,\xi)\,d\xi\Big| \to 0 \quad \text{as } N\to\infty, R\to\infty \tag{3.13}$$

in view of Lemma 3.4. Therefore,

$$\lim_{\substack{N\to\infty\\ R\to\infty}} I_{1,2}=0 \quad \text{uniformly } \forall t\in(c+\delta,L]. \tag{3.14}$$

Now, applying the technique used in the proof of Lemma 2.1, we have

$$\begin{aligned}
|I_{11}| &\le e^{\lambda\omega(|t|)}\int_c^{t-\delta}|\Phi(\xi)|\,|H_{N,R}(t,\xi)|\,d\xi\\
&\le Ce^{(\lambda+1)\omega(|t|)}e^{-\omega(|t|)}\int_c^{t-\delta}|\Phi(\xi)|\,d\xi\int_{-N}^{N}\int_{-R}^{R}\psi\left(\left|\frac{t-b}{a}\right|\right)\overline{\psi\left(\left|\frac{\xi-b}{a}\right|\right)}\,\frac{db\,da}{a^2|a|}\\
&\le Ce^{-\omega(|t|)}\int_c^{t-\delta}|\Phi(\xi)|\,d\xi\int_{-N}^{N}\int_{-R}^{R}\left|e^{\mu\omega(|\frac{t-b}{a}|)}\psi\left(\left|\frac{t-b}{a}\right|\right)\right|e^{(\lambda+1)\omega(|t|)}\\
&\quad\cdot e^{-\mu\omega(|\frac{t-b}{a}|)}e^{-\nu\omega(|\frac{\xi-b}{a}|)}\,\frac{db\,da}{a^2|a|}\qquad(\text{for some }\mu,\nu>0)\\
&\le C_1e^{-\omega(|t|)}\int_c^{t-\delta}|\Phi(\xi)|\,d\xi\int_{-N}^{N}\int_{-R}^{R}e^{(\lambda+1+n)\omega(|\frac{t}{a}|)}e^{-\mu\omega(|\frac{t-b}{a}|)}e^{-\nu\omega(|\frac{\xi-b}{a}|)}\,\frac{db\,da}{a^2|a|}\\
&\hspace{25em}(\text{assuming }|a|\le n)\\
&\le C_1e^{-\omega(|t|)}\int_c^{t-\delta}|\Phi(\xi)|\,d\xi\int_{-N}^{N}\int_{-R}^{R}e^{\lambda_1\omega(|\frac{t-b}{a}|)+\lambda_1\omega(|\frac{b}{a})|}e^{-\mu_1\omega(|\frac{t-b}{a}|)}\\
&\quad\cdot e^{-\mu_2\omega(|\frac{t-b}{a}|)}e^{-\nu\omega(|\frac{\xi-b}{a}|)}\,\frac{db\,da}{a^2|a|}\qquad(\text{for }\lambda_1=\lambda+1+n,\ \mu=\mu_1+\mu_2)\\
&\le C_1e^{-\omega(|t|)}\int_c^{t-\delta}|\Phi(\xi)|\,d\xi\int_{-N}^{N}\int_{-R}^{R}e^{\lambda_1\omega(|\frac{b}{a}|)}e^{-\mu_2\omega(|\frac{t-b}{a}|)}e^{-\nu\omega(|\frac{\xi-b}{a}|)}\,\frac{db\,da}{a^2|a|}\\
&\hspace{25em}(\text{choosing }\mu_1=\lambda_1)\\
&\le C_1e^{-\omega(|t|)}\int_c^{t-\delta}|\Phi(\xi)|\,d\xi\int_{-N}^{N}\int_{-R}^{R}e^{\lambda_1\omega(|\frac{\xi-b}{a}|)+\lambda_1\omega(|\frac{\xi}{a}|)}e^{-\nu_1\omega(|\frac{\xi-b}{a}|)}\\
&\quad\cdot e^{-\mu_2\omega(|\frac{t-b}{a}|)}e^{-\nu_2\omega(|\frac{\xi-b}{a}|)}\,\frac{db\,da}{a^2|a|}\qquad(\text{for }\nu=\nu_1+\nu_2)
\end{aligned}$$

so

(3.15)

$$|I_{11}|\le C_1e^{-\omega(|t|)}\int_c^{d}|\Phi(\xi)|\,d\xi\int_{-N}^{N}\int_{-R}^{R}e^{\lambda_1\omega(|\frac{\xi}{a}|)}e^{-\mu_2\omega(|\frac{t-b}{a}|)}e^{-\nu_2\omega(|\frac{\xi-b}{a}|)}\,\frac{db\,da}{a^2|a|}$$

(if $\nu_1=\lambda_1$)

Now, $\Phi(\xi) \in \mathcal{D}_\omega \subset \mathscr{Z}_\omega$. Hence

$$\begin{aligned} P_{\lambda_1,0}(\Phi) &= \sup_\xi |\Phi(\xi) e^{\lambda_1 \omega(|\frac{\xi}{a}|)}| \le \sup_\xi |\Phi(\xi) e^{\lambda_1 \omega(|\xi|)}| < \infty \quad (\text{if } |a| \ge 1) \\ &\le \sup_\xi |\Phi(\xi) e^{\lambda_1 \omega(m|\xi|)}| < \infty \quad (\text{if } |a| < 1 \text{ and } 1/|a| \ge m \text{ for some } m \in \mathbb{N}_0) \\ &= \sup_\xi |\Phi(\xi) e^{m\lambda_1 \omega(|\xi|)}| < \infty. \end{aligned}$$

Thus, (3.15) is estimated by

$$|I_{11}| \le C_1 e^{-\omega(|t|)} P_{\lambda_1,0}(\Phi) \int_c^d d\xi \int_{-N}^{N} \int_{-R}^{R} e^{-\mu_2 \omega(|\frac{t-b}{a}|)} e^{-\nu_2 \omega(|\frac{\xi-b}{a}|)} \frac{db\,da}{a^2|a|}. \tag{3.16}$$

Here, the a, b-integrals are similar to those in (3.6) as $N, R \to \infty$. Therefore, these integrals are bounded uniformly for all real t and ξ (by Lemma 3.3).

We then have

$$|I_{11}| \le C_2 e^{-\omega(|t|)} P_{\lambda,0}(\Phi) \int_c^d d\xi \to 0 \quad \text{as } |t| \to \infty.$$

This proves the uniform convergence of $I_{1,1}$ (and therefore of I_1) to zero on $-\infty < t < \infty$.

A similar argument shows that I_3 also converges uniformly to zero on $-\infty < t < \infty$ as $N \to \infty, R \to \infty$. Altogether this proves that the limit superior of (3.11) is uniformly bounded by ε for $N, R \to \infty$. Since $\varepsilon > 0$ is arbitrary, our proof is complete. ∎

Theorem 3.9. *Assume that the wavelet transform $W(b,a)$ of $f \in \mathscr{Z}'_\omega$ is given by (2.1). Then*

$$\lim_{\substack{N\to\infty \\ R\to\infty}} \left\langle \frac{1}{C_\psi} \int_{-R}^{R} \int_{-N}^{N} W(b,a) \psi_{b,a} \frac{db\,da}{a^2}, \phi \right\rangle = \langle f, \phi \rangle \tag{3.17}$$

for each $\phi \in \mathcal{D}_\omega(\mathbb{R})$ with support in $[c,d] \subset \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$.

Proof. Assume that $\operatorname{supp} \phi \subset [c,d]$. Then

$$\begin{aligned} &\left\langle \frac{1}{C_\psi} \int_{-N}^{N} \int_{-R}^{R} W(b,a) \psi_{b,a}(t) \frac{db\,da}{a^2}, \phi(t) \right\rangle \\ &\qquad = \frac{1}{C_\psi} \int_c^d \phi(t)\,dt \int_{-N}^{N} \int_{-R}^{R} \langle f(\xi), \overline{\psi_{b,a}(\xi)} \rangle \psi_{b,a}(t) \frac{db\,da}{a^2} \end{aligned}$$

$$= \left\langle f(\xi), \int_c^d \phi(t)\, dt \, \frac{1}{C_\psi} \int_{-N}^{N} \int_{-R}^{R} \overline{\psi_{b,a}(\xi)} \psi_{b,a}(t) \, \frac{db\, da}{a^2} \right\rangle$$

$$= \left\langle f(\xi), \int_c^d H_{N,R}(t,\xi)\phi(t)\, dt \right\rangle \to \langle f, \phi \rangle$$

by Lemma 3.8. ∎

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