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GORENSTEIN HOMOLOGICAL PROPERTIES FOR HOPF–GALOIS EXTENSIONS

BY

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Abstract. Let H be a semisimple Hopf algebra over a field k and A/B be a right H-Galois extension. We establish a relationship between several Gorenstein homological dimensions of A and B and we characterize when the category of finitely generated Gorenstein projective A-modules is abelian. As applications, we obtain some relations for crossed products and smash products.

1. Introduction. The definition of Hopf–Galois extension has its roots in the Chase–Harrison–Rosenberg approach to Galois theory for groups acting on commutative rings (see [7]). In 1969 Chase and Sweedler extended these ideas to coactions of a Hopf algebra H on a commutative k-algebra, for k a commutative ring (see [8]); the general definition appears in the 1981 paper [22]. Hopf–Galois extensions also generalize strongly graded algebras (here H is a group algebra) and certain inseparable field extensions (here the Hopf algebra is the restricted enveloping algebra of a restricted Lie algebra), twisted group rings $R * G$ of a group G acting on a ring R, crossed products (cleft extensions) and smash products, and others.

Gorenstein homological algebra has been developed to an advanced level in recent twenty years. The main idea is to replace projectives and injectives respectively by Gorenstein projectives and Gorenstein injectives, introduced by Enochs and Jenda [13]. This concept also goes back to the work of Auslander and Bridger [2]. Now it is widely used in singularity theory, Tate cohomology, representation theory, triangulated categories, etc. (see e.g. [3], [6], [9], [10], [15], [18], [20], [23], [26], [29], [30], [31]).

The aim of this note is to study the relations of Gorenstein homological dimensions of Hopf–Galois extensions.

This paper is organized as follows.

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In Section 2, we recall some definitions and basic properties related to Hopf–Galois extensions and Gorenstein homological dimensions.

In Section 3, we prove that if A/B is a right H-Galois extension over a semisimple Hopf algebra H, then the global Gorenstein projective [injective] dimension and the finitistic Gorenstein projective [injective] dimension of A is less than or equal to that of B. We also study the global dimensions of the coherent functors over the categories A-Gproj and B-Gproj of Gorenstein projective modules. Finally, we characterize when the category of finitely generated Gorenstein projective A-modules is abelian.

In Section 4, as applications, we obtain some relations for crossed products and smash products by using the results of Section 3.

2. Preliminaries. Throughout this paper, k denotes a fixed field, and we will always work over k. The tensor product $\otimes = \otimes_k$ and Hom are always assumed to be over k. For an algebra A, denote by A-Mod and A-mod the categories of left A-modules and of finitely generated left A-modules, respectively. For a left A-module M, let $\operatorname{add}(M)$ denote the full subcategory of A-mod whose objects are direct summands of finite sums of copies of M. The reader is referred to [25] and [27] as general references about Hopf algebras. If C is a coalgebra, we use the Sweedler-type notation for comultiplication: $\Delta(c) = c_1 \otimes c_2$ for all $c \in C$.

2.1. Hopf–Galois extensions. Let H be a Hopf algebra over a field k and A a right H-comodule algebra, i.e., A is a k-algebra together with an H-comodule structure $\rho_A : A \to A\otimes H$ (with notation $a \mapsto a_0\otimes a_1$) such that ρ_A is an algebra map. Let B be the subalgebra of H-coinvariant elements, $B := A^{\mathrm{co}H} := \{a \in A \mid \rho_A(a) = a \otimes 1\}$. Then the extension A/B is *right H-Galois* if the map

$$\beta : A \otimes_B A \to A \otimes H, \quad a \otimes_B b \mapsto (a \otimes 1)\rho(b) = ab_0 \otimes b_1,$$

is bijective.

2.2. Gorenstein homological dimensions. Following [14], an A-module M is said to be *Gorenstein projective* in A-Mod (resp. in A-mod) if there is a *complete projective resolution*, that is, an exact sequence of projective modules in A-Mod (resp. in A-mod)

$$\mathcal{P}^\bullet : \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \to \cdots$$

with $\operatorname{Hom}_A(\mathcal{P}^\bullet, Q)$ exact for any projective module Q in A-Mod (resp. in A-mod), such that $M \cong \operatorname{Im} d^{-1}$. Denote by A-GProj (resp. A-Gproj) the full subcategory of Gorenstein projective modules in A-Mod (resp. in A-mod). Gorenstein injective modules are defined dually.

Also following [14], if A is an algebra, the *Gorenstein projective dimension* of a left A-module M, denoted $\operatorname{Gpd} M$, is defined as the smallest integer $n \geq 0$ such that M has a GProj-resolution of length n. The *global Gorenstein projective dimension* of A is $\mathrm{gl.Gpd}(A) = \sup\{\operatorname{Gpd} M \mid M \in A\text{-Mod}\}$. One can define the *global Gorenstein injective dimension* of A, denoted $\mathrm{gl.Gid}(A)$, in a similar way. Note that in general they are not equal.

Following [18], the (left) *finitistic Gorenstein projective dimension* of an algebra A is defined to be

$$\mathrm{FGPD}\,(A) = \sup\{\operatorname{Gpd}{}_A M \mid M \text{ is a (left) } A\text{-module with } \operatorname{Gpd}{}_A M < \infty\}.$$

Similarly, one can define the *finitistic Gorenstein injective dimension* of A, denoted $\mathrm{FGID}(A)$.

3. Gorenstein homological dimensions for Hopf–Galois extensions. Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. In this section, we prove that the global Gorenstein projective [injective] dimension and the finitistic Gorenstein projective [injective] dimension of A are less than or equal to the corresponding dimensions of B. We also study the global dimensions of coherent functors over the categories A-Gproj and B-Gproj of Gorenstein projective modules.

Let A/B be a right H-Galois extension. Consider the following two functors:

$$A \otimes_B - : B\text{-Mod} \to A\text{-Mod}, \quad M \mapsto A \otimes_B M,$$
$${}_B(-) : A\text{-Mod} \to B\text{-Mod}, \quad M \mapsto M,$$

where ${}_B(-)$ is the restriction functor. Obviously, ${}_B(-) = \operatorname{Hom}_A(A, -) = A \otimes_A -$.

Lemma 3.1. *Let A/B be a right H-Galois extension over a finite-dimensional Hopf algebra H. Then $(A \otimes_B -, {}_B(-))$ and $({}_B(-), A \otimes_B -)$ are both adjoint pairs.*

Proof. By the adjoint isomorphism theorem, $(A \otimes_B -, {}_B(-))$ and $({}_B(-), \operatorname{Hom}_B(A, -))$ are both adjoint pairs. By [11, Theorem 5], $A \otimes_B -$ is naturally isomorphic to $\operatorname{Hom}_B(A, -)$. Thus $({}_B(-), A \otimes_B -)$ is also an adjoint pair. ∎

Remark. Let (F, G) be an adjoint pair of functors of abelian categories. If G is exact, then F preserves projective objects; if F is exact, then G preserves injective objects. If A/B is a right H-Galois extension over a finite-dimensional Hopf algebra H, then A is projective as a left and right B-module (see [22]). It follows that the above functors $A \otimes_B -$ and ${}_B(-)$ are both exact, and so they preserve projective objects and injective objects.

The following lemma generalizes Lemma 3.1 of [24] and the proof is similar. We give a detailed proof for completeness.

Lemma 3.2. *Let A/B be a right H-Galois extension over a finite-dimensional Hopf algebra H.*

(i) *Let M be a left A-module. If M is Gorenstein projective* (*resp. Gorenstein injective*) *as an A-module, then it is also Gorenstein projective* (*resp. Gorenstein injective*) *as a B-module.*
(ii) *Let M be a left B-module. If M is a Gorenstein projective* (*resp. Gorenstein injective*) *left B-module, then $A \otimes_B M$ is a Gorenstein projective* (*resp. Gorenstein injective*) *left A-module.*

Proof. (i) First, by Lemma 3.1, $(A \otimes_B -, {}_B(-))$ and $({}_B(-), A \otimes_B -)$ are both adjoint pairs. Since $M \in A$-Mod is Gorenstein projective, there exists a complete projective resolution $\mathcal{P}^\bullet : \cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$ such that $M \cong \mathrm{Im}(P^{-1} \to P^0)$. Because ${}_B(-)$ is exact, $\mathcal{P}^\bullet$ is an exact sequence in B-Mod and $M \cong \mathrm{Im}(P^{-1} \to P^0)$ as B-modules. Also, P^i is projective as a left B-module for every i. Finally, assume that $P \in B$-Mod is projective. Then

$$\mathrm{Hom}_B({}_B(\mathcal{P}^\bullet), P) \cong \mathrm{Hom}_A(\mathcal{P}^\bullet, A \otimes_B P).$$

Since $A \otimes_B P$ is projective as a left A-module, $\mathrm{Hom}_A(\mathcal{P}^\bullet, A \otimes_B P)$ is exact. From the above natural isomorphism, it follows that $\mathrm{Hom}_B({}_B(\mathcal{P}^\bullet), P)$ is exact. Thus ${}_B(\mathcal{P}^\bullet)$ is a complete projective resolution in B-Mod. Hence M is Gorenstein projective as a B-module.

(ii) If $M \in B$-Mod is Gorenstein projective, then there is a complete projective resolution $\mathcal{P}^\bullet : \cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$ such that $M \cong \mathrm{Im}(P^{-1} \to P^0)$. Since A is projective as a right B-module, $A \otimes_B \mathcal{P}^\bullet$ is exact in A-Mod and $A \otimes_B M \cong \mathrm{Im}(A \otimes_B P^{-1} \to A \otimes_B P^0)$. Also, $A \otimes_B P^i$ is projective as an A-module for each i. Then for any projective A-module P, we have

$$\mathrm{Hom}_A(A \otimes_B \mathcal{P}^\bullet, P) \cong \mathrm{Hom}_B(\mathcal{P}^\bullet, {}_B(P)).$$

But ${}_B(P)$ is a projective B-module, and so $\mathrm{Hom}_B(\mathcal{P}^\bullet, {}_B(P))$ is exact. By the above, $\mathrm{Hom}_A(A \otimes_B \mathcal{P}^\bullet, P)$ is exact. Therefore $A \otimes_B M$ is a Gorenstein projective left A-module.

When M is a Gorenstein injective module, statements (i) and (ii) can be proved in a similar way. ■

The following lemma given in [32] will be needed.

Lemma 3.3. *Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. Then any A-module M is an A-direct summand of $A \otimes_B M$.*

The following proposition gives a relation between A-Gproj and B-Gproj.

Lemma 3.4. *Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. Then* A-Gproj $= \mathrm{add}(A \otimes_B (B\text{-Gproj}))$.

Proof. Let G be a Gorenstein projective left B-module. Then by Lemma 3.2, $A \otimes_B G$ is a Gorenstein projective A-module.

Let G' be a Gorenstein projective left A-module. Then by Lemma 3.2 it is also a Gorenstein projective B-module. By Lemma 3.3, G' is a direct summand of $A \otimes_B G'$ as a left A-module.

Hence A-Gproj $= \mathrm{add}(A \otimes_B (B\text{-Gproj}))$. ∎

LEMMA 3.5. *Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. Then for each A-module M,*

$$\mathrm{Gpd}_A M = \mathrm{Gpd}_B M \quad \textit{and} \quad \mathrm{Gid}_A M = \mathrm{Gid}_B M.$$

Proof. For any A-module M, we have $\mathrm{Gpd}_B M \le \mathrm{Gpd}_A M$, since any GProj-resolution of ${}_AM$ is a GProj-resolution of ${}_BM$ by Lemma 3.2.

Conversely, we may assume $\mathrm{Gpd}_B M = n < \infty$, so there exists a GProj-resolution $\mathcal{G}^\bullet$ of ${}_BM$ of length n. Then $A \otimes_B \mathcal{G}^\bullet$ is a GProj-resolution of $A \otimes_B M$ as an A-module by Lemma 3.2 and the exactness of $A \otimes_B -$. This means $\mathrm{Gpd}\, A \otimes_B M \le n$. By Lemma 3.3, M is an A-direct summand of $A \otimes_B M$, so $\mathrm{Gpd}\ {}_AM \le \mathrm{Gpd}\ {}_A(A \otimes_B M) \le n$. Thus $\mathrm{Gpd}_A M \le \mathrm{Gpd}_B M$.

The equality $\mathrm{Gid}_A M = \mathrm{Gid}_B M$ follows in a similar way. ∎

In view of Lemma 3.5, we immediately obtain the following main result of this section.

THEOREM 3.6. *Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. Then:*

(i) $\mathrm{gl.Gpd}(A) \le \mathrm{gl.Gpd}(B)$ *and* $\mathrm{gl.Gid}(A) \le \mathrm{gl.Gid}(B)$.
(ii) $\mathrm{FGPD}(A) \le \mathrm{FGPD}(B)$ *and* $\mathrm{FGID}(A) \le \mathrm{FGID}(B)$.

We recall from [12] that a Grothendieck category $\mathcal{A}$ with enough projectives is Gorenstein if and only if $\mathrm{gl.Gpd}(\mathcal{A})$ and $\mathrm{gl.Gid}(\mathcal{A})$ are both finite. Therefore the above theorem can produce new examples of Gorenstein categories from a given one.

It is shown in [18] that there is equality between the classical (left) finitistic projective dimension, $\mathrm{FPD}(A)$, and the (left) finitistic Gorenstein projective dimension, $\mathrm{FGPD}(A)$, and also between the classical (left) finitistic injective dimension, $\mathrm{FID}(A)$, and the (left) finitistic Gorenstein injective dimension, $\mathrm{FGID}(A)$, of an algebra A. So we obtain the following corollary.

COROLLARY 3.7. *Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. Then* $\mathrm{FPD}(A) \le \mathrm{FPD}(B)$ *and* $\mathrm{FID}(A) \le \mathrm{FID}(B)$.

Next we study the global dimensions of the coherent functors over two categories: A-Gproj and B-Gproj.

We recall from [17] the well-known definition of coherent functors which have been used to solve problems on algebraic space curves.

Definition 3.8. Given an additive k-category $\mathcal{A}$, we say that a functor $F : \mathcal{A}^{\mathrm{op}} \to Ab$ is *coherent* if there is an exact sequence $(-, A_1) \to (-, A_0) \to F \to 0$ with $A_i \in \mathcal{A}$ for $i = 0, 1$. Here Ab is the category of all abelian groups, and $(-, A_i)$ stands for the functor $\mathrm{Hom}_{\mathcal{A}}(-, A_i)$. The full subcategory of the functor category $(\mathcal{A}^{\mathrm{op}}, Ab)$ consisting of all coherent functors is denoted by $\widehat{\mathcal{A}}$.

By Yoneda's lemma, the projective objects in $\widehat{\mathcal{A}}$ are of the form $(-, X)$ with X an object in $\mathcal{A}$, and each coherent functor $F : \mathcal{A}^{\mathrm{op}} \to Ab$ can be determined by a morphism $f : A_1 \to A_0$, that is, there is an exact sequence $(-, A_1) \xrightarrow{(-,f)} (-, A_0) \to F \to 0$ in $\widehat{\mathcal{A}}$. As in the case of a module category, we may define the global dimension of the category $\widehat{\mathcal{A}}$ to be the supremum of the projective dimensions of all functors in $\widehat{\mathcal{A}}$. For more information on coherent functors we refer to [1] and [17].

Lemma 3.9 ([28, Theorem 3.6]). *Suppose $\mathcal{C}$ and $\mathcal{D}$ are additive k-categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ an adjoint pair of additive functors with F a left adjoint and G a right adjoint. If there is an endofunctor $E : \mathcal{C} \to \mathcal{C}$ such that GF is naturally equivalent to* $\mathrm{Id}_{\mathcal{C}} \oplus E$*, then* $\mathrm{gl.dim}(\widehat{\mathcal{C}}) \le \mathrm{gl.dim}(\widehat{\mathcal{D}})$.

Now we obtain another main result of this section.

Theorem 3.10. *Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. Then* $\mathrm{gl.dim}(\widehat{A\text{-Gproj}}) \le \mathrm{gl.dim}(\widehat{B\text{-Gproj}})$.

Proof. By Lemmas 3.1 and 3.2, $({}_B(-), A \otimes_B -)$ is an adjoint pair between the categories A-Gproj and B-Gproj. Since H is semisimple, the extension A/B is separable, i.e., the functor ${}_B(-)$ is separable. Then by [16, Proposition 5] the unit $\eta : \mathrm{Id}_{A\text{-Gproj}} \to (A \otimes_B -) \circ {}_B(-)$ is, as a natural transformation, a splitting monomorphism. It follows from Lemma 3.9 that $\mathrm{gl.dim}(\widehat{A\text{-Gproj}}) \le \mathrm{gl.dim}(\widehat{B\text{-Gproj}})$. ■

Let A be a finite-dimensional algebra. It is well-known that the category of finitely generated Gorenstein projective A-modules, denoted $\mathcal{G}P(A)$, is a Frobenius exact category [19]. Thus its stable category $\underline{\mathcal{G}P(A)}$ is a triangulated category. In [21], it is characterized when $\overline{\mathcal{G}P(A)}$ is an abelian category. Let A/B be a right H-Galois extension over a finite-dimensional Hopf algebra H. We will characterize when $\mathcal{G}P(A)$ is an abelian category.

Lemma 3.11 ([21, Theorem 2.1]). *Let A be a finite-dimensional algebra. Then the following are equivalent:*

(i) *$\mathcal{G}P(A)$ is an abelian category.*
(ii) *A is a quasi-Auslander algebra, i.e., ${}_AA$ has an injective resolution*

$$0 \to {}_AA \to I_0 \to I_1 \to I_2 \to 0$$

such that I_0, I_1 are projective-injective modules.

THEOREM 3.12. *Let A/B be a right H-Galois extension over a finite-dimensional Hopf algebra H. If $\mathcal{GP}(B)$ is an abelian category, then so is $\mathcal{GP}(A)$.*

Proof. Since $\mathcal{GP}(B)$ is an abelian category, ${}_BB$ has an injective resolution

$$0 \to {}_BB \to I_0 \to I_1 \to I_2 \to 0$$

such that I_0, I_1 are projective-injective modules. By applying the functor $A \otimes_B -$ to the above sequence, we obtain

$$0 \to {}_A(A \otimes_B B) \to A \otimes_B I_0 \to A \otimes_B I_1 \to A \otimes_B I_2 \to 0.$$

Obviously, ${}_A(A \otimes_B B) \cong {}_AA$, and by the Remark following Lemma 3.1, $A \otimes_B I_0, A \otimes_B I_1$ are also projective-injective modules. By Lemma 3.11, $\mathcal{GP}(A)$ is an abelian category. ■

4. Applications. In this section, as applications of the results of Section 3, we obtain some relations for crossed products and smash products.

First we recall some notations for crossed products (see [4]). A Hopf algebra H is said to *measure* an algebra A if there is a k-linear map $H \otimes A \to A$ given by $h \otimes a \mapsto h \cdot a$ such that $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1 = \varepsilon(h)1$, for all $a, b \in A$ and $h \in H$. A map σ in $\mathrm{Hom}(H \otimes H, A)$ is said to be *convolution invertible* if there exists a map τ in $\mathrm{Hom}(H \otimes H, A)$ such that $(\sigma * \tau)(h \otimes g) = \sigma(h_1, g_1)\tau(h_2, g_2) = \varepsilon(h)\varepsilon(g)1_A$ and $(\tau * \sigma)(h \otimes g) = \tau(h_1, g_1)\sigma(h_2, g_2) = \varepsilon(h)\varepsilon(g)1_A$, for all $h, g \in H$.

Let H be a Hopf algebra and A an algebra. Assume that H measures A, and σ is a convolution invertible map in $\mathrm{Hom}(H \otimes H, A)$. The *crossed product* $A \#_\sigma H$ of A and H is the set $A \otimes H$ as a vector space, with multiplication

$$(a \#_\sigma h)(b \#_\sigma k) = a(h_1 \cdot b)\sigma(h_2, k_1) \#_\sigma h_3k_2$$

for $h, k \in H$ and $a, b \in A$. Here we write $a \#_\sigma h$ for the tensor product $a \otimes h$. Then $A \#_\sigma H$ is an associative algebra with identity element $1 \#_\sigma 1$ if and only if the following two conditions are satisfied:

(i) A is a twisted H-module: $1 \cdot a = a$ for all $a \in A$, and

$$h \cdot (k \cdot a) = \sigma(h_1, k_1)(h_2k_2 \cdot a)\sigma^{-1}(h_3, k_3) \quad \text{for all } h, k \in H \text{ and } a \in A.$$

(ii) σ is a cocycle: $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1$ for all $h \in H$, and

$$(h_1 \cdot \sigma(k_1, m_1))\sigma(h_2, k_2m_2) = \sigma(h_1, k_1)\sigma(h_2k_2, m) \quad \text{for all } h, k, m \in H.$$

Note that if σ is trivial, that is, $\sigma(h, k) = \varepsilon(h)\varepsilon(k)1$ for $h, k \in H$, then (i) simply says that A is an H-module, and (ii) is trivial. Thus A is a left H-module algebra. Moreover, the definition of multiplication above then reduces to multiplication in a smash product, and so $A \#_\sigma H = A \# H$ is just the smash product (see [25, Definition 4.1.3]).

Let $A\,\#_\sigma\, H$ be a crossed product. Then $A\,\#_\sigma\, H/A$ is a right H-Galois extension (in fact, $A\,\#_\sigma\, H/A$ is a right H-Galois extension and has the normal basis property) (see [5, Theorem 1.18]).

Corollary 4.1. *Let H be a Hopf algebra which is semisimple together with its dual H^*, and let $A\,\#_\sigma\, H$ be a crossed product. Then:*

(i) $\mathrm{gl.Gpd}(A\,\#_\sigma\, H) = \mathrm{gl.Gpd}(A)$ *and* $\mathrm{gl.Gid}(A\,\#_\sigma\, H) = \mathrm{gl.Gid}(A)$.
(ii) $\mathrm{FGPD}(A\,\#_\sigma\, H) = \mathrm{FGPD}(A)$ *and* $\mathrm{FGID}(A\,\#_\sigma\, H) = \mathrm{FGID}(A)$.

Proof. By Theorem 3.6 and the fact that $A\,\#_\sigma\, H/A$ is a right H-Galois extension, we get $\mathrm{gl.Gpd}(A\,\#_\sigma\, H) \le \mathrm{gl.Gpd}(A)$.

Conversely, note that $A\#_\sigma H$ is a left H^*-module algebra via $f\cdot(a\#_\sigma h) = a\#_\sigma(f \rightharpoonup h) = \langle f, h_2\rangle a\#_\sigma h_1$ for $a\#_\sigma h \in A\#_\sigma H$ and $f \in H^*$. Thus we may form the smash product algebra $(A\,\#_\sigma\, H)\,\#\, H^*$. It is well-known that the smash product is a special case of crossed products. So $(A\#_\sigma H)\#H^*/A\#_\sigma H$ is also a right H^*-Galois extension. By the above conclusion and semisimplicity of H^*, we have $\mathrm{gl.Gpd}((A\#_\sigma H)\#H^*) \le \mathrm{gl.Gpd}(A\#_\sigma H)$. By [4, Theorem 2.2], $(A\#_\sigma H)\#H^*$ is isomorphic to $M_n(A)$, where $n = \dim H$, so it is Morita equivalent to A. It follows that $\mathrm{gl.Gpd}(A) = \mathrm{gl.Gpd}((A\#_\sigma H)\#H^*)$. Thus

$$\mathrm{gl.Gpd}(A) = \mathrm{gl.Gpd}((A\,\#_\sigma\, H)\,\#\, H^*) \le \mathrm{gl.Gpd}(A\,\#_\sigma\, H) \le \mathrm{gl.Gpd}(A).$$

Therefore $\mathrm{gl.Gpd}(A\,\#_\sigma\, H) = \mathrm{gl.Gpd}(A)$.

The remaining equalities can be proved in a similar way. ∎

By applying the arguments used in the proof of Corollary 4.1, we obtain the following two useful corollaries.

Corollary 4.2. *Let H be a Hopf algebra which is semisimple together with its dual H^*, and $A\,\#_\sigma\, H$ be a crossed product. Then*

$$\mathrm{gl.dim}(\widehat{A\,\#_\sigma\, H\text{-Gproj}}) = \mathrm{gl.dim}(\widehat{A\text{-Gproj}}).$$

Corollary 4.3. *Let $A\,\#_\sigma\, H$ be a crossed product over a finite-dimensional Hopf algebra H. Then $\mathcal{G}P(A\,\#_\sigma\, H)$ is an abelian category if and only if so is $\mathcal{G}P(A)$.*

Analogously, we have the following two corollaries describing relationships between the global dimensions and the weak dimensions of A and B.

Corollary 4.4. *Let A/B be a right H-Galois extension over a semisimple Hopf algebra H. Then*

$$\mathrm{gl.dim}(A) \le \mathrm{gl.dim}(B) \quad \textit{and} \quad \mathrm{w.dim}(A) \le \mathrm{w.dim}(B).$$

Corollary 4.5. *Let H be a Hopf algebra which is semisimple together with its dual H^*, and $A\,\#_\sigma\, H$ be a crossed product. Then*

$$\mathrm{gl.dim}(A\,\#_\sigma\, H) = \mathrm{gl.dim}(A) \quad \textit{and} \quad \mathrm{w.dim}(A\,\#_\sigma\, H) = \mathrm{w.dim}(A).$$

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