Invariant universality for quandles and fields

by

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Abstract. We show that the embeddability relations for countable quandles and for countable fields of any given characteristic other than 2 are maximally complex in a strong sense: they are invariantly universal. This notion from the theory of Borel reducibility states that any analytic quasi-order on a standard Borel space essentially appears as the restriction of the embeddability relation to an isomorphism-invariant Borel set. As an intermediate step we show that the embeddability relation of countable quandles is a complete analytic quasi-order.

1. Introduction. The comparison of different equivalence relations in terms of Borel reducibility has proven to be an extremely fruitful area of research, with implications in diverse areas of mathematics, most notably in showing that various classification programmes are impossible to complete satisfactorily. For an introduction to the area see, for example, [Hjo00]; note however that all necessary preliminaries for this paper will be provided in Section 2. The area was initiated by the pioneering papers of H. Friedman and L. Stanley [FS89] and of Harrington, Kechris and Louveau [HKL90], with the former paper in particular focused on the equivalence relation of isomorphism between countable structures. Indeed, the set of all structures of a given type with underlying set the natural numbers may be endowed with the topology of a complete separable metric space, and in this framework the results of descriptive set theory have been brought to bear on questions about equivalence relations to great effect.

In the underlying descriptive-set-theoretic machinery, there is nothing that requires us to constrain investigation to equivalence relations, and recently attention in this field has expanded to include quasi-orders (reflexive
and transitive binary relations), beginning with the work of Louveau and Rosendal [LR05]. A central example of a quasi-order is the embeddability relation between countable structures of a given type. This also fits with previous work in category theory studying the complexity of different categories, as for example in [PT80]. Indeed, there is a kind of “Church’s thesis for real mathematics” that states that, assuming the objects in question are reasonably encoded as members of a standard Borel space, hands-on constructions will invariably be Borel. Thus, from the functors between categories that demonstrate universality one can expect to derive Borel reductions that respect embeddings. For example, building on work of Przeździecki [Prz14] in a category-theoretic context, the second author [Cal18] has shown that, when $\kappa$ is an uncountable cardinal satisfying certain assumptions, the embeddability relation between $\kappa$-sized graphs Borel reduces in a generalized sense suitable for $\kappa$ to embeddability between $\kappa$-sized torsion-free abelian groups.

Louveau and Rosendal [LR05] showed that within the class of analytic quasi-orders (see Section 2 for definitions) there are quasi-orders that are maximal with respect to Borel reducibility—so-called complete analytic quasi-orders. Louveau and Rosendal furnish a number of examples, including the embeddability relation between graphs. In fact, the restriction of the graph embeddability relation to connected acyclic graphs—combinatorial trees—is already complete analytic, a fact that we will make use of below. We prove in Section 4 that the embeddability relation on quandles is complete analytic. We also observe in Section 5 that an old result of Fried and Kollár [FK82], when expressed in these terms, states that the embeddability relation of fields is complete analytic.

When restricting to subclasses of structures, it is reasonable to consider the case when the subclass is closed under isomorphism. Thus arises the notion of invariant universality (Definition 3.1), first introduced by Camerlo, Marcone and Motto Ros [CMM13] building on fundamental observations of S.-D. Friedman and Motto Ros [FM11]. Whilst invariant universality imposes significant requirements making it stronger than complete analyticity, a general trend observed in [CMM13, CM18] is that in practice, whenever the relation of embeddability on some space of countable structures is a complete analytic quasi-order, it is moreover invariantly universal with respect to isomorphism.

In Section 3 of this paper we give the formal definition of invariant universality, and recall a special case of Theorem 4.2 of [CMM13], which will be our main tool for proving invariant universality. In Section 4 we first show that the embedding relation on countable quandles is a complete analytic quasi-order, and then use this fact to show that the relation is invariantly
universal. We further observe that arguing similarly we obtain invariant universality of the embedding relations of related classes of countable structures such as kei as LD-monoids. In Section 5 we turn to the embedding relation on fields of a given characteristic other than 2. In this case, the fact that the embeddability relation is complete analytic was essentially shown by Fried and Kollár [FK82], and using their construction we are able to show that the relation is invariantly universal. Our results all add weight to the trend mentioned above, and hint that in the search for a natural example of a complete analytic quasi-order that is not invariantly universal, it might be best to focus on relations other than embeddability.

2. Preliminaries. A standard Borel space is a pair \((X, \mathcal{B})\) such that \(\mathcal{B}\) is the \(\sigma\)-algebra of Borel subsets of \(X\) with respect to some Polish topology on \(X\). The class of standard Borel spaces is closed under countable products, and a Borel subset of a standard Borel space is standard Borel when viewed as a subspace. Every uncountable standard Borel space is in fact isomorphic to the Baire space \(\mathbb{N}^\mathbb{N}\) of all functions from \(\mathbb{N}\) to \(\mathbb{N}\), with the Borel structure generated by the product topology. We recall that this topology is generated by all sets \([s] = \{g \in \mathbb{N}^\mathbb{N} \mid g \supseteq s\}\) of end extensions of a given finite string \(s\). We also define the set \((\mathbb{N})^\mathbb{N}\) as \(\{x \in \mathbb{N}^\mathbb{N} \mid x\) is injective\}, which is a closed subset of the Baire space \(\mathbb{N}^\mathbb{N}\) and therefore a Polish space with the induced topology. Given any Polish space \(X\), the set \(F(X)\) of closed subsets of \(X\) is a standard Borel space when equipped with the Effros Borel structure, namely, the \(\sigma\)-algebra generated by the sets

\[
\{C \in F(X) \mid C \cap U \neq \emptyset\},
\]

where \(U\) is an open subset of \(X\) (see [Hjo00, Example 2.4] or [Kec95, Section 12.C]). A Polish group is a topological group whose topology is Polish. A well-known example of a Polish group is \(S_\infty\), the group of all bijections from \(\mathbb{N}\) to \(\mathbb{N}\). In fact, \(S_\infty\) is a \(G_\delta\) subset of the Baire space \(\mathbb{N}^\mathbb{N}\) and a topological group under the induced topology. We define \(N_s\) as \(\{s\} \cap S_\infty\). Note that the set \(\{N_s \mid s \in (\mathbb{N})^{<\mathbb{N}}\}\) is a basis for \(S_\infty\), where \((\mathbb{N})^{<\mathbb{N}}\) denotes the set of finite sequences of distinct natural numbers.

A subset \(A\) of a standard Borel space \(X\) is analytic, or \(\Sigma^1_1\), if there is a Polish space \(Y\) and some Borel set \(B \subseteq X \times Y\) such that \(A\) is the projection

\[
p(B) = \{x \in X \mid \exists y \in Y \ ((x, y) \in B)\}.
\]

A subset of a standard Borel space whose complement is analytic is called co-analytic, or \(\Pi^1_1\). Souslin’s Theorem (see [Kec95, Theorem 14.11]) states that the Borel sets of a standard Borel space are precisely the sets that are both \(\Sigma^1_1\) and \(\Pi^1_1\).
A function \( f : X \to Y \) between standard Borel spaces \( X \) and \( Y \) is Borel if the inverse image under \( f \) of any Borel set is Borel. A corollary of Souslin’s Theorem is that a function \( f : X \to Y \) between standard Borel spaces is Borel if and only if \( \{(x, f(x)) \in X \times Y \mid x \in X\} \) is an analytic subset of \( X \times Y \) (see [Kec95, Theorem 14.12]).

A quasi-order is a reflexive and transitive binary relation. Any quasi-order \( Q \) on a set \( X \) naturally induces an equivalence relation \( E_Q \) on \( X \) which is given by defining \( x E_Q y \) if and only if \( x Q y \) and \( y Q x \). In the cases considered in this paper, \( Q \) will be the relation of embeddability between structures, in which case \( E_Q \) will be bi-embeddability, a coarsening of the equivalence relation of isomorphism between structures.

A quasi-order \( Q \) on a standard Borel space \( X \) is a subset of \( X \times X \) so we say that the quasi-order \( Q \) is analytic (resp. Borel) if \( Q \) is analytic (resp. Borel) as a subset of \( X \times X \) equipped with the product Borel structure. If \( Q \) is analytic (or Borel), then so is \( E_Q \).

If \( G \) is a Polish group and there is a Borel action \( a \) of \( G \) on a standard Borel space \( X \), then we say that \( X \) is a standard Borel \( G \)-space and we denote by \( E_a \) the orbit equivalence relation induced by that action. When the action is clear from the context we shall write \( E_G \) instead of \( E_a \). Such equivalence relations are often called \( G \)-equivalence relations. Every \( G \)-equivalence relation is analytic by definition and it is well-known that all the classes of any \( G \)-equivalence relation are Borel (see [BK96, 2.3.3]). The stabilizer of a point \( x \) in \( X \) is the subgroup \( \text{Stab}(x) := \{ g \in G \mid g \cdot x = x \} \), where \( g \cdot x \) denotes the value of the action on the pair \((g, x)\). We will use the fact that each stabilizer is a closed subgroup of \( G \) (see [Kec95, 9.17]), and that the set \( \text{Subg}(G) \) of closed subgroups of \( G \) is a Borel subset of \( \mathcal{F}(G) \). Thus \( \text{Subg}(G) \) is a standard Borel space with the induced Borel structure.

In this paper we focus mainly on standard Borel spaces of countable structures. If \( L \) is a countable (relational) language we denote by \( X_L \) the space of \( L \)-structures with domain \( \mathbb{N} \), whose topology is defined by taking as basic open sets those of the form

\[
\{ \mathcal{M} \in X_L \mid \mathcal{M} \models R(n_0, \ldots, n_{k-1}) \}, \quad \{ \mathcal{M} \in X_L \mid \mathcal{M} \models \neg R(n_0, \ldots, n_{k-1}) \},
\]

for any \( k \)-tuples \((n_0, \ldots, n_{k-1})\) of natural numbers and any relation \( R \) in \( L \) of arity \( k = a(R) \). Such a space is Polish because it is homeomorphic to \( \prod_{R \in L} 2^{a(R)} \). (An analogous definition can also be given for languages with function symbols; see [BK96, Section 2.5].) Let \( S_\infty \) act on \( X_L \) continuously by the so-called logic action: for every \( g \) in \( S_\infty \) and \( \mathcal{M}, \mathcal{N} \in X_L \) we set \( g \cdot \mathcal{M} = \mathcal{N} \) if for all \( k \)-ary relations \( R \) in \( L \) and all \( k \)-tuples of natural numbers \((n_0, \ldots, n_{k-1})\), we have

\[
\mathcal{N} \models R(n_0, \ldots, n_{k-1}) \iff \mathcal{M} \models R(g^{-1}(n_0), \ldots, g^{-1}(n_{k-1})).
\]
In other words, the structure \( g \cdot M \) is obtained by interpreting each relation symbol as in \( M \) up to \( g \), which is a permutation of natural numbers. Thus, for any countable \( L \), the space \( X_L \) is a standard Borel \( S_\infty \)-space; and the isomorphism relation on \( X_L \), usually denoted by \( \cong_L \), coincides with the orbit equivalence relation \( E_{S_\infty}^X \). Moreover notice that, for every \( M \) in \( X_L \), \( \text{Stab}(M) \) and the group of automorphisms of \( M \), \( \text{Aut}(M) \), coincide.

Given quasi-orders \( P \) and \( R \) on the standard Borel spaces \( X \) and \( Y \), respectively, we say that \( P \) Borel reduces (or is Borel reducible) to \( R \), written \( P \leq_B R \), if there is a Borel function \( f: X \to Y \) such that for every \( x, y \) in \( X \),

\[
x P y \iff f(x) R f(y).
\]

Such an \( f \) is called a Borel reduction. We say that \( P \) is essentially \( R \), denoted \( P \sim_B R \), whenever \( P \leq_B R \) and \( R \leq_B P \).

Louveau and Rosendal proved in [LR05] that among all \( \Sigma_1^1 \) quasi-orders there are \( \leq_B \)-maximum elements called complete \( \Sigma_1^1 \) quasi-orders. One of the most prominent examples of such a maximum element is the quasi-order of embeddability between combinatorial trees. By a graph we mean a structure for an irreflexive and symmetric binary relation symbol called the edge relation. A combinatorial tree is a connected acyclic graph.

Let \( X_{Gr} \) be the space of graphs on \( \mathbb{N} \). If we identify each graph with the characteristic function of its edge relation as above, \( X_{Gr} \) becomes a closed subset of \( 2^{\mathbb{N}^2} \), and thus a Polish space. We denote by \( X_{CT} \) the set of combinatorial trees with vertex set \( \mathbb{N} \), and note that \( X_{CT} \) is a \( G_\delta \) subset of \( X_{Gr} \) (towards this, first observe that the set of graphs with a path from \( m \) to \( n \) is open for all \( m \) and \( n \) in \( \mathbb{N} \)). Hence, \( X_{CT} \) is a Polish space with the induced topology (see for example [Kec95, Theorem 3.11]). For graphs \( S, T \) in \( X_{Gr} \), we say that \( S \) embeds, or \( S \) is embeddable into \( T \), and write \( S \sqsubseteq_{Gr} T \), if there is a one-to-one function \( f: \mathbb{N} \to \mathbb{N} \) which realizes an isomorphism between \( S \) and \( T \upharpoonright \text{Im}(f) \). The quasi-order \( \sqsubseteq_{Gr} \) is analytic because it is the set

\[
\{(S, T) \in (X_{Gr})^2 | \exists f \in (\mathbb{N})^\mathbb{N} \land (\forall n, m \in \mathbb{N} ((n, m) \in S \leftrightarrow (f(n), f(m)) \in T))\},
\]

which is a projection of a closed subset of \( \mathbb{N}^\mathbb{N} \times X_{Gr} \times X_{Gr} \). We denote by \( \sqsubseteq_{CT} \) the restriction of the quasi-order \( \sqsubseteq_{Gr} \) to \( X_{CT} \).

**Theorem 2.1 ([LR05, Theorem 3.1]).** The relation \( \sqsubseteq_{CT} \) of embeddability between countable combinatorial trees is a complete \( \Sigma_1^1 \) quasi-order.

All trees built in the proof of Theorem 2.1 satisfy the further property that there are no complete vertices, expressible by the formula

\[
(\sqcup) \quad \forall x \exists y (x \neq y \land (x, y) \notin T).
\]
We denote by $X_{CT\sqcup}$ the standard Borel space of combinatorial trees satisfying (\[\square\]). In [FM11, Section 2] and [CMM13, Section 3], the authors modified the proof of Theorem 2.1 to prove the following proposition.

**Proposition 2.2.** There is a Borel $X \subseteq X_{CT\sqcup}$ such that:

(i) the equality and isomorphism relations restricted to $X$, denoted respectively by $=_X$ and $\cong_X$, coincide;
(ii) each graph in $X$ is rigid; that is, it has no nontrivial automorphism;
(iii) for every $\Sigma_1^1$ quasi-order $P$ on $2^\omega$, there exists an injective Borel reduction $\alpha \mapsto T_\alpha$ from $P$ to $\sqsubseteq_X$.

This result is a strengthening of Theorem 2.1. A closer look into [CMM13] shows that the map $\alpha \mapsto T_\alpha$ in Proposition 2.2(iii) is constructed by first reducing $P$ to a quasi-order, which is denoted by $\leq_{\text{max}}$ and is defined on the standard Borel space $T$ of normal trees (\[\footnote{1}\]) on $2 \times \omega$, and then reducing $\leq_{\text{max}}$ to $\sqsubseteq_{CT\sqcup}$. Both those reductions are injective. Next, one defines $X$ as the image of the whole of $T$ through the second map. Clearly, $X$ is a Borel subset of $X_{CT\sqcup}$ as it is the injective image of a standard Borel space through a Borel map [Kec95, Corollary 15.2]. Moreover, since $\leq_{\text{max}}$ is known to be a complete $\Sigma_1^1$ quasi-order (see [LR05, Theorem 2.5]), so is the quasi-order $\sqsubseteq_X$. Therefore in contrast to (i) and (ii) the bi-embeddability relation on $X$ will be highly nontrivial, and the graphs in $X$ will have many nontrivial endomorphisms.

### 3. Invariant universality.

The property of invariant universality (Definition 3.1) was first observed in [CMM13] for embeddability between countable combinatorial trees when the equivalence relation is isomorphism.

**Definition 3.1 ([CMM13]).** Let $P$ be a $\Sigma_1^1$ quasi-order on some standard Borel space $X$ and let $E$ be a $\Sigma_1^1$ equivalence subrelation of $P$. We say that $(P,E)$ is invariants universally (or $P$ is invariantly universal with respect to $E$) if for every $\Sigma_1^1$ quasi-order $R$ there is a Borel subset $B \subseteq X$ which is invariant with respect to $E$ and such that $P|B$ is essentially $R$.

When we look at relations defined on a space of countable structures, if $P,E$ are as in Definition 3.1 and $E$ is the relation of isomorphism, we simply say that $P$ is invariantly universal. By a classical result of López-Escobar (see [Kec95, Theorem 16.8]), a subset of a space of countable structures is closed under isomorphism if and only if it is definable in the logic $L_{\omega_1\omega}$. Examples of invariantly universal quasi-orders found in [CMM13, CM18, CMM18] include: linear isometric embeddability between separable Banach spaces; em-

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\(\footnote{1}\) The precise definition of $\leq_{\text{max}}$ is not relevant to the results of this paper. We refer the interested reader to [LR05, Definition 2.3].
beddability between countable groups; and isometric embeddability on ultrametric Polish spaces with any prescribed ill-founded set of distances.

The standard Borel space $X$ defined in Section 2 is used to test whether a pair $(Q, E)$ satisfying the hypotheses of Definition 3.1 is invariantly universal. The following result, which is essentially a particular case of [CMM13, Theorem 4.2], gives a sufficient condition for the invariant universality of a pair.

**Theorem 3.2 (CMM13).** Let $P$ be a $\Sigma_1^1$ quasi-order on a space $X_L$ of $L$-structures with domain $\mathbb{N}$ such that $\cong_L \subseteq P$. Suppose that the following conditions hold:

(i) there is a Borel reduction $f: X \to X_L$ of $\sqsubseteq_X$ to $P$;
(ii) $f$ is also a Borel reduction of $=_{X}$ (equivalently, of $\cong_X$) to $\cong_L$;
(iii) the map $X \to \text{Subg}(S_\infty), T \mapsto \text{Stab}(f(T)) = \text{Aut}(f(T))$, is Borel.

Then, for every $\Sigma_1^1$ quasi-order $R$ there is a Borel $B \subseteq X_L$ such that $R$ is essentially $P|B$.

One of the open questions about invariant universality in the paper by Camerlo, Marcone, and Motto Ros is the following.

**Question 3.3 (CMM13 Question 6.3).** Is there a natural pair $(P, E)$ which is not invariantly universal but for which $P$ is a complete analytic quasi-order?

We stress the word “natural”—although examples of such pairs are known, none of them consists of relations that arise in other contexts defined over a space of mathematical objects. Our results show that the specific examples of quandle embedding and of field embedding for fields of characteristic not equal to 2 (each with the equivalence relation of isomorphism) do not furnish examples for an affirmative answer to Question 3.3.

4. Quandles and related structures. In this section we use the reduction from graphs to quandles defined in [BTM20] to prove that embeddability between countable quandles is a complete $\Sigma_1^1$ quasi-order. Recall that a set $Q$ with a binary relation $*$ is a quandle if:

(a) $\forall x, y, z \in Q \ (x * (y * z) = (x * y) * (x * z));$
(b) $\forall x, z \in Q \ \exists! y \in Q \ (x * y = z);$
(c) $\forall x \in Q \ (x * x = x).$

For an introduction to the theory of quandles, see for example [EN15].

We now recall the reduction appearing in [BTM20]. For any $T$ in $X_{Gr}$, let $Q_T$ be the quandle with underlying set $\mathbb{N} \times \{0, 1\}$ and the binary operation
be \(*_T\) defined as follows:

\[
(\ast) \quad (u, i) *_T (v, j) = \begin{cases} 
(v, j) & \text{if } u = v \text{ or } (u, v) \in T, \\
(v, 1 - j) & \text{otherwise}.
\end{cases}
\]

It is straightforward to check that \((Q_T, *_T)\) satisfies (a)–(c). In what follows, we denote the space of quandles with domain \(\mathbb{N}\) by \(X_{Qdl}\), which is a \(G_δ\) subset of \(2^{\mathbb{N}^3}\) and thus a Polish space. For every graph \(T\) in \(X_{Gr}\), the quandle \(Q_T\) can be easily coded as an isomorphic structure \(Q_T\) with domain \(\mathbb{N}\), for example use the bijection \(\mathbb{N} \times 2 \to \mathbb{N}, (n, i) \mapsto 2n + i\). Clearly the map \(T \mapsto Q_T\) is Borel; in fact, it is continuous. Recall the following definition.

**Definition 4.1.** Suppose that there is a Borel action \(a\) of \(S_\infty\) on some standard Borel space and \(E = E_a\). We say that \(E\) is \(S_\infty\)-complete if every equivalence relation induced by a Borel action of \(S_\infty\) on some standard Borel space Borel reduces to \(E\).

The main theorem of [BTM20] is the following.

**Theorem 4.2 ([BTM20, Theorem 3]).** For all graphs \(S, T\) in \(X_{Gr}\), we have

\[
S \cong_{Gr} T \iff Q_S \cong_{Qdl} Q_T.
\]

Thus, the equivalence relation of isomorphism on the space of countable quandles is \(S_\infty\)-complete.

Proving that \(S \cong_{Gr} T\) implies \(Q_S \cong_{Qdl} Q_T\) is straightforward but the converse is considerably more involved. In the proof of Theorem 4.2, whenever \(S\) contains complete vertices and \(\rho\) is an isomorphism from \(Q_S\) to \(Q_T\), the surjectivity of \(\rho\) is used substantially to recover an isomorphism of graphs between \(S\) and \(T\). Since embeddings do not need to be surjective, we cannot prove an analog of Theorem 4.2 in the same way. However, if we restrict our attention to \(X_{CT^\omega}\), a simpler argument allows us to prove Theorem 4.6.

Towards this we now analyze quandle embeddings. We recall the following fact from [BTM20].

**Lemma 4.3 ([BTM20, Lemma 1]).** For every \(T\) in \(X_{Gr}\) and every \(A \subseteq \mathbb{N}\), the function \(I_A\): \(Q_T \to Q_T\) defined by

\[
I_A(v, j) = \begin{cases} 
(v, j) & \text{if } v \in A, \\
(v, 1 - j) & \text{otherwise},
\end{cases}
\]

is an involution of \(Q_T\).

For any quandle homomorphism \(\rho\): \(Q_S \to Q_T\) between quandles derived from graphs, let us denote by \(\rho_V(v, i)\) and \(\rho_I(v, i)\) the first and the second component of \(\rho(v, i)\), respectively.
Lemma 4.4. For every graph $S$ satisfying $[\square]$ and every graph $T$, every vertex $v$ of $S$, and every quandle homomorphism $\rho: Q_S \to Q_T$, 
$$\rho_V(v, 0) = \rho_V(v, 1).$$

Proof. Since $S$ satisfies $[\square]$, for every vertex $v$ of $S$ there is another vertex $v^+$ such that $v$ and $v^+$ are not adjacent in $S$. Then, by applying $\rho$ to both sides of 
$$(v^+, 0) \ast_S (v, 0) = (v, 1)$$
we get $\rho(v^+, 0) \ast_T \rho(v, 0) = \rho(v, 1)$, which implies that $\rho_V(v, 0) = \rho_V(v, 1)$ by definition (see $[\ast]$).

With these ingredients we can present a factorization lemma for the quandle homomorphisms we shall be interested in. We thank the anonymous referee for their suggestion to streamline our results in this way.

Lemma 4.5. Let $S$ and $T$ be graphs satisfying $[\square]$. Every embedding $\rho: Q_S \to Q_T$ is obtained in the following manner: there is some graph embedding $h: S \to T$ and some $A \subseteq \mathbb{N}$ such that 
$$\rho(v, j) = I_A(h(v), j).$$

Proof. Assume that $S$ and $T$ are graphs satisfying $[\square]$ and $\rho: Q_S \to Q_T$ is a quandle embedding. We define 
$$h: S \to T, \quad v \mapsto \rho_V(v, 0) = \rho_V(v, 1).$$
Lemma 4.4 shows that $h$ is injective. Next we show that $h$ is injective. The equality $h(v) = h(w)$ implies that 
$$\rho_V(v, 0) = \rho_V(v, 1) = \rho_V(w, 0) = \rho_V(w, 1),$$
which implies in turn that $\rho(v, 0) = \rho(w, i)$ for either $i = 0$ or $i = 1$. By injectivity of $\rho$, we get $i = 0$ and $v = w$. It remains to show that $h$ is a graph embedding. Pick any two adjacent vertices $u$ and $v$ in $S$. Notice that $u$ and $v$ are necessarily distinct and $\rho(u, 0) \ast_T \rho(v, 0) = \rho(v, 0)$. So either $\rho_V(u, 0) = \rho_V(v, 0)$ or $(\rho_V(u, 0), \rho_V(v, 0)) \in T$. The former cannot hold by the injectivity of $h$ just shown. Thus 
$$(h(u), h(v)) = (\rho_V(u, 0), \rho_V(v, 0)) \in T.$$ 
On the other hand, if $(u, v) \notin S$ then $(v, j) \ast_S (u, 0) = (u, 1)$. By applying $\rho$ to both terms, we get $\rho(v, j) \ast_T \rho(u, 0) = \rho(u, 1)$. By Lemma 4.4, $\rho_V(u, 0)$ equals $\rho_V(u, 1)$, so necessarily $\rho_I(u, 0) \neq \rho_I(u, 1)$ because $\rho$ is injective. Then, by the definition of $\ast_T$ we have 
$$(h(u), h(v)) = (\rho_V(u, 0), \rho_V(v, j)) \notin T.$$ 
So $h$ is a graph embedding.

To complete the proof of Lemma 4.5 let 
$$A = \{n \in \mathbb{N} \mid \rho_I(n, i) \neq i\}.$$ 
By construction we obtain $\rho(v, j) = I_A(h(v), j)$ for every $(v, i) \in Q_S$. ■
THEOREM 4.6. The relation $\sqsubseteq_{Qdl}$ of embeddability on the space of countable quandles is a complete $\Sigma^1_1$ quasi-order.

Proof. It suffices to prove that $\sqsubseteq_{CT}$ Borel reduces to $\sqsubseteq_{Qdl}$. We show that the map from $X_{CT}$ to $X_{Qdl}$ taking $T$ to $Q_T$ is a reduction. Assume that $f: S \to T$ is a graph embedding, then consider the function $\theta: Q_S \to Q_T$ such that $(v,i) \mapsto (f(v),i)$. Injectivity of $\theta$ is immediate. Moreover, for all $(u,i)$ and $(v,j)$ in $Q_S$,

$$\theta((u,i)*S(v,j)) = \theta(u,i)*T\theta(v,j).$$

In fact, by applying the definitions of $\theta$ and $*_S$, we have

$$\theta((u,i)*S(v,j)) = \begin{cases} (f(v),j) & \text{if } u = v \text{ or } (u,v) \in T, \\ (f(v),1-j) & \text{otherwise}, \end{cases}$$

and the first condition is equivalent to $f(u) = f(v)$ or $(f(u),f(v)) \in T$ because $f$ is a graph embedding. Therefore, $\theta$ witnesses that $Q_S$ is embeddable into $Q_T$.

Now the converse is a straightforward consequence of Lemma 4.5. Whenever $\rho: Q_S \to Q_T$ is a quandle embedding we recover a graph embedding $h: S \to T$ such that $\rho(v,j) = I_A(h(v),j)$. Therefore, $S \sqsubseteq_{Gr} T$, as desired. $\blacksquare$

Before proving the main result of this section we isolate a particular case of Lemma 4.5.

LEMMA 4.7. Let $T$ be a graph satisfying $[□]$. Every $\rho$ in $\text{Aut}(Q_T)$ is obtained from some graph automorphism $h$ in $\text{Aut}(T)$ in the following manner: there is an $h$ in $\text{Aut}(T)$ and some $A \subseteq \mathbb{N}$ such that

$$\rho(v,j) = I_A(h(v),j).$$

Proof. Every automorphism $\rho$ of $Q_T$ is in particular a self-embedding of $Q_T$. Hence by Lemma 4.5 there is an embedding $h: T \to T$ such that $\rho(v,j) = I_A(h(v),j)$. By construction $h$ is surjective. $\blacksquare$

We recall that $Q_T$ is defined as the quandle with domain $\mathbb{N}$ which is isomorphic to $Q_T$ via the bijection $\mathbb{N} \times 2 \to \mathbb{N}$ taking $(n,i)$ to $2n + i$.

THEOREM 4.8. The relation $\sqsubseteq_{Qdl}$ of embeddability between countable quandles is an invariantly universal $\Sigma^1_1$ quasi-order.

Proof. By Theorem 3.2 it suffices to prove that $\sqsubseteq_{Qdl}$ and $\equiv_{Qdl}$ together satisfy $[i],[ii]$. Let $f$ be the map from $\mathcal{X}$ to $X_{Qdl}$ taking $T$ to $Q_T$. By Theorem 4.6, $f$ Borel reduces $\sqsubseteq_{\mathcal{X}}$ to $\sqsubseteq_{Qdl}$, and by Theorem 4.2 $\equiv_{\mathcal{X}}$ Borel reduces to $\equiv_{Qdl}$ via the same map, hence $[i]$ and $[ii]$ hold.

By Lemma 4.7 whenever $\rho$ is in $\text{Aut}(Q_T)$ there exist $h$ in $\text{Aut}(T)$ and $A \subseteq \mathbb{N}$ such that $\rho(v,j) = I_A(h(v),j)$. Further, since each $T$ in $\mathcal{X}$ is rigid, we have $h = \text{id}$ and consequently $\rho = I_A$ for some $A \subseteq \mathbb{N}$. Thus for every $T$
In $\mathbb{X}$, $g$ is an automorphism of $Q_T$ if and only if there is some $A \subseteq \mathbb{N}$ such that for $i \in \{0, 1\}$,

$$g(2v + i) = \begin{cases} 2v + i & \text{if } v \in A, \\ 2v + 1 - i & \text{otherwise}. \end{cases}$$

Note in particular that this depends only on $A$, not on $T$.

To see that $T \mapsto \text{Aut}(Q_T)$ is Borel it suffices to show that the preimage of every basic open set is Borel. For every fixed $s$ in $(\mathbb{N})^{<\mathbb{N}}$, the preimage of

$$\{G \in \text{Subg}(S_\infty) \mid G \cap N_s \neq \emptyset\}$$

through the map $T \mapsto \text{Aut}(Q_T)$ is the set

$$\{T \in \mathbb{X} \mid \text{Aut}(Q_T) \cap N_s \neq \emptyset\} = \begin{cases} \mathbb{X} & \text{if every } n \text{ in dom } s \text{ is either sent to itself or, if not, swapped with its successor if } n \text{ is even and predecessor if } n \text{ is odd}, \\ \emptyset & \text{otherwise}, \end{cases}$$

which is certainly a Borel set.

**Corollary 4.9.** For every $\Sigma^1_1$ quasi-order $R$ there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $B$ of countable quandles such that the embeddability relation on $B$ is Borel bi-reducible with $R$.

In [BTM20] other quandle-like structures are considered. A quandle is a kei if and only if it satisfies

$$\forall x \forall y (x \ast (x \ast y) = y).$$

It is easy to check that for every $T$ in $X_{\text{Gr}}$, $Q_T$ defined as in Section 3 is a kei. Therefore, arguing as in Theorem 4.8 one can prove the following.

**Theorem 4.10.** The embeddability relation between countable kei is invariantly universal.

**Definition 4.11.** An LD-monoid, or algebra satisfying $\Sigma$, is a structure over the language $\{\ast, \circ\}$ consisting of two binary operational symbols satisfying for all $a, b, c$ the following identities:

$$a \circ (b \circ c) = (a \circ b) \circ c, \quad a \ast (b \circ c) = (a \ast b) \circ (a \ast c),$$

$$a \circ (b \ast c) = (a \circ b) \ast c, \quad (a \ast b) \circ a = a \circ b.$$
In [BTM20, Theorem 4] the authors observed that the equivalence relation of isomorphism between LD-monoids is $S^\infty_\infty$-complete.

**Theorem 4.12.** The quasi-order of embeddability between countable LD-monoids is invariantly universal.

**Proof.** In [Wil14] Williams defined a Borel reduction $h: X_{Gr} \to X_{Gp}$ from $\sqsubseteq_{Gr}$ to $\sqsubseteq_{Gp}$. Then in [CM18, Theorem 3.5] the second author and Motto Ros observed that

(a) $h|X$ is a Borel reduction from $=_X$ to $\cong_{Gp}$, and
(b) the map $X \to \text{Subg}(S^\infty_\infty)$ sending $T$ to $\text{Aut}(h(T))$ is Borel.

For any $G = (N, \circ_G)$ in $X_{Gp}$, let $M(G) = (N, \circ_G, *_G)$ be the LD-monoid over $N$ such that $*_G$ is interpreted as the conjugation operation in $(N, \circ_G)$. That is, we define $*_G: N \times N \to N$ by

$$k *_G m = k \circ_G m \circ_G k^{-1}.$$ 

The LD-monoid $M(G)$ is thus $G$ with enriched structure—in model-theoretic terms, it is a definitional expansion of $G$. Now we observe that this map $M: X_{Gp} \to X_{LD-m}$ to the space $X_{LD-m}$ of LD-monoids is a Borel reduction, reducing group embeddability to the embeddability relation between LD-monoids, and reducing group isomorphism to LD-monoid isomorphism. Indeed, any homomorphism $\phi: G \to H$ (resp. embedding) realizes a homomorphism (resp. embedding) between the corresponding $M(G)$ and $M(H)$ as

$$k *_G m = n \iff k \circ_G m = n \circ_G k \iff \phi(k) \circ_H \phi(m) = \phi(n) \circ_H \phi(k) \iff \phi(k) *_H \phi(m) = \phi(n).$$

Conversely, it is clear that any homomorphism (resp. embedding) from $M(G)$ to $M(H)$ gives a group homomorphism (resp. embedding) $G \to H$ between the underlying group structures by simply “forgetting” the $*$ operation.

If we let $f: X \to X_{LD-m}$ be the composition $M \circ h$, verifying conditions [i],[iii] of Theorem 3.2 will give the invariant universality of bi-embeddability on LD-monoids, as desired. Since $h$ Borel reduces $\sqsubseteq_{Gr}$ to $\sqsubseteq_{Gp}$, it follows that $f$ Borel reduces $\sqsubseteq_X$ to embeddability on LD-monoids; hence [i] holds. Condition [a] implies that $f$ is a reduction from $\cong_X$ to isomorphism on LD-monoids; hence we get [ii]. Finally, since $\text{Aut}(M(h(T))) = \text{Aut}(h(T))$, condition (b) ensures that the map $T \mapsto \text{Aut}(M(h(T)))$ is Borel, which gives condition (iii) of Theorem 3.2.

**5. Fields.** We denote by $X_{Fld,p}$ the standard Borel space of fields of fixed characteristic $p$. The relation of isomorphism on $X_{Fld,p}$ is an $S^\infty_\infty$-complete equivalence relation for every characteristic $p$—see [FS89, Theo-
rem 10] and [Sha90]. In this section we study the quasi-order of embeddability on $X_{\text{Fld},p}$, which we denote by $\sqsubseteq_{\text{Fld},p}$. Recall that, since any field has only trivial ideals, every field homomorphism is one-to-one, and thus the notions of embeddability and homomorphism coincide. Therefore we adopt the usual terminology from algebra that if $f : F \to L$ is a homomorphism of fields we say that $F$ is a subfield of $L$, or that $L$ is a field extension of $F$.

If $F$ is a field and $S$ is a set of algebraically independent elements over $F$, we denote by $F(S)$ the purely transcendental extension of $F$ by $S$. If $S = \{s\}$, we write $F(s)$ instead of $F(\{s\})$. Following the notation of [FK82], for any prime $p$, any field $F$, and any set $S$ of algebraically independent elements over $F$, we denote by $F(S)(S,p)$ the smallest field extension of $F(S)$ containing $\{s(n) \mid s \in S, n < \omega\}$, where

- $s(0) = s$,
- $s(n + 1)$ is such that $(s(n + 1))^p = s(n)$.

Notice that this uniquely determines $F(S)(S,p)$ up to isomorphism. We use the convention $F(s)(s,p) = F(\{s\})(\{s\},p)$.

We now recall a construction of Fried and Kollár [FK82] that, given a combinatorial tree $T$ of infinite cardinality, produces a field $K_T$, and furthermore this construction respects embedding. For clarity we denote by $V = \{v_0, v_1, \ldots\}$ the set of vertices of the graphs in $X_{\text{CT}}$.

**Definition 5.1 ([FK82, Section 3])**. Fix a characteristic $p$ equal to 0 or an odd prime number, fix $F$ a countable field of characteristic $p$, and fix an increasing sequence of odd prime numbers $\{p_n \mid n \in \mathbb{N}\}$ not containing $p$. For any $T$ in $X_{\text{CT}}$, we define $K_T$ as the union of an increasing chain of fields $K_n(T)$. These fields $K_n(T)$ are defined recursively. First define

$$K_0(T) := F(V)(V,p_0) \quad \text{and} \quad H_0(T) := \{u + v \mid (u,v) \in T\}.$$ 

Next suppose $K_n(T)$ and $H_n(T)$ have already been defined. Fix a transcendental element $t_n$ over $K_n(T)$, and let $L_n$ be the field $K_n(T)(t_n)(\{t_n\},p_{n+1})$. Now we define $K_{n+1}(T)$ as the splitting field over $L_n$ of the set of polynomials

$$P_n = \{x^2 - (t_n - a) \mid a \in H_n(T)\}.$$ 

Further, we define $H_{n+1}(T)$ to be a set containing exactly one root of each of the polynomials in $P_n$.

The next two lemmas summarize the essential properties of the map sending any $T$ of $X_{\text{CT}}$ to $K_T$. They were implicitly obtained in the paper of Fried and Kollár [FK82].

**Lemma 5.2.** If there is a graph embedding from $S$ to $T$, then $K_S$ is a subfield of $K_T$. 

In fact, Fried and Kollár \cite{FK82} proved inductively that if there is a graph embedding from $S$ to $T$, then each $K_n(S)$ is a subfield of $K_n(T)$.

**Lemma 5.3.** Let $\phi: K_S \to K_T$ be a homomorphism.

(a) For every $n$ in $\mathbb{N}$, $\phi$ maps $H_n(S)$ into $H_n(T)$. In particular, we have $\phi[H_0(S)] \subseteq H_0(T)$.

(b) Suppose that $u$ is a vertex of $S$. If $u$ is not isolated and $(u, v)$ is an edge in $S$, then $\phi(u)$ is in $V$ and $(\phi(u), \phi(v))$ is an edge in $T$.

The next theorem is a consequence of the previous two lemmas. The structure of the proof is as for \cite[Theorem 2.1]{FK82}, but since we are concerned only with the embeddability relation rather than all embeddings, we are able to include the odd characteristic case, unlike that theorem.

**Theorem 5.4.** For every $p$ equal to 0 or an odd prime number, the quasi-order $\sqsubseteq_{CT}$ Borel reduces to $\sqsubseteq_{\text{Fld},p}$. Thus $\sqsubseteq_{\text{Fld},p}$ is a complete $\Sigma^1_1$ quasi-order.

**Proof.** The map taking each $T$ in $X_{CT}$ to $K_T$ can be realized as a Borel map from $X_{CT}$ to $X_{\text{Fld},p}$. If $S$ is embeddable into $T$, then $K_T$ is a field extension of $K_S$ by Lemma 5.2. Now suppose that $\rho: K_S \to K_T$ is a homomorphism. We claim that $f$ defined as the restriction map $\rho|V$ is a graph embedding from $S$ to $T$. Since $S$ is a combinatorial tree, it has no isolated vertices and therefore Lemma 5.3(b) ensures that every edge $(u, v)$ in $S$ is preserved by $f$. For the converse, when $u$ and $v$ are not adjacent in $S$, we have a sequence of vertices $u = v_0, \ldots, v_n = v$ which is a path in $S$, namely, such that $(v_i, v_{i+1})$ is in $S$ for every $i < n$. Since $f$ preserves edges and is one-to-one, the vertices $f(v_0), \ldots, f(v_n)$ are all distinct and $(f(v_i), f(v_{i+1}))$ is an edge in $T$, for every $i < n$. As a result, $f(u)$ and $f(v)$ are not adjacent in $T$ by acyclicity.

The arguments of Fried and Kollár show that for any $T$, the automorphisms of $K_T$ are uniquely determined by their action on $V$, so we have the following.

**Corollary 5.5.** The groups $\text{Aut}(K_T)$ and $\text{Aut}(T)$ are isomorphic via the map sending any automorphism $\phi$ of $K_T$ to the restriction of $\phi$ to $V$.

Now we use Theorem 5.4 and Corollary 5.5 to prove that $\sqsubseteq_{\text{Fld},p}$ is invariantly universal.

**Theorem 5.6.** For $p$ not equal to 2, the quasi-order $\sqsubseteq_{\text{Fld},p}$ is invariantly universal.

**Proof.** It suffices to check that $\sqsubseteq_{\text{Fld},p}$ and $\sqsubseteq_{\text{Fld}}$ satisfy conditions (i)–(iii) of Theorem 3.2. Let $f: X \to X_{\text{Fld},p}$ be the map sending $T$ to $K_T$. Theorem 5.4 gives (i) To see (ii) notice that if $\phi: K_S \to K_T$ is an isomorphism then $\phi|V$ is an isomorphism from $S$ to $T$ as $(\phi|V)^{-1} = \phi^{-1}|V$. Moreover, condition (iii)
is immediate as the map $T \mapsto \text{Aut}(K_T)$ is the constant map $T \mapsto \{\text{id}\}$ by Corollary 5.5.

**Corollary 5.7.** For every $\Sigma^1_1$ quasi-order $P$ there is an $\mathcal{L}_{\omega_1\omega}$-elementary class of countable fields of characteristic $p$ such that the embeddability relation on it is Borel bi-reducible with $P$.

**Question 5.8.** Is the embeddability relation $\sqsubseteq_{\text{Fld},2}$ between countable fields of characteristic 2 an invariantly universal quasi-order?

**Acknowledgements.** The second author would like to thank Simon Thomas for interesting discussions and pointing out [FK82]. We thank the anonymous referee for their careful reading and suggestions.

The first author was supported during this research by EPSRC Early Career Fellowship EP/K035703/2, “Bringing set theory and algebraic topology together”. This work was carried out while the second author was visiting Rutgers University supported by the “National Group for the Algebraic and Geometric Structures and their Applications” (GNSAGA–INDAM).

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