Lack of exponential decay for a laminated beam with structural damping and second sound

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Abstract. In [Z. Angew. Math. Phys. 68 (2017)] Apalara considered a one-dimensional thermoelastic laminated beam under Cattaneo’s law of heat conduction and proved the exponential and polynomial decay results depending on the stability number $\chi_T$. In this short note, we continue the study of the same system and show that the solution lacks exponential decay if $\chi_T \neq 0$, which solves an open problem proposed by Apalara.

1. Introduction. In [1], Apalara considered the following laminated beam with structural damping and second sound:

$$
\begin{align*}
\rho w_{tt} + G(\psi - w_x)_x &= 0, & (x, t) \in (0, 1) \times (0, +\infty), \\
I_\rho (3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) + \delta \theta_x &= 0, & (x, t) \in (0, 1) \times (0, +\infty), \\
3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t &= 0, & (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_3 \theta_t + q_x + \delta(3s_t - \psi_t)_x &= 0, & (x, t) \in (0, 1) \times (0, +\infty), \\
\tau q_t + \alpha q + \theta_x &= 0, & (x, t) \in (0, 1) \times (0, +\infty), \\
w(x, 0) &= w_0(x), \psi(x, 0) = \psi_0(x), & x \in (0, 1), \\
s(x, 0) &= s_0(x), w_t(x, 0) = w_1(x), & x \in (0, 1), \\
\psi_t(x, 0) &= \psi_1(x), s_t(x, 0) = s_1(x), & x \in (0, 1), \\
\theta(x, 0) &= \theta_0(x), q(x, 0) = q_0(x), & x \in (0, 1), \\
w_x(0, t) &= \psi(0, t) = s(0, t) = q(0, t) = 0, & t \in [0, +\infty), \\
w(1, t) &= \psi_x(1, t) = s_x(1, t) = \theta(1, t) = 0, & t \in [0, +\infty),
\end{align*}
$$

(1.1)

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where $\rho, G, I_\rho, \rho_3, D, \gamma, \beta, \delta, \alpha, \tau$ are positive constants. Here $w(x, t)$ denotes the transverse displacement of the beam from its equilibrium position, $\psi(x, t)$ is the rotation angle, $s(x, t)$ is proportional to the amount of slip along the interface at time $t$ and for longitudinal spatial variable $x$, $3s - \psi$ denotes the effective rotation angle, the third equation of (1.1) describes the dynamics of the slip, $\theta(x, t)$ is the difference temperature, and $q(x, t)$ is the heat flux. The coefficients $\rho, G, I_\rho, D, \gamma, \beta > 0$ are the density of the beam, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. Moreover, the constants $\rho_3, \alpha, \delta > 0$ represent the physical parameters from thermoelasticity theory, and $\tau$ denotes the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature. Laminated beam structures are of considerable importance in engineering fields; the detailed physical background and derivation can be found in [5].

In [1], Apalara showed that the exponential decay and polynomial decay depend on the stability number

$$\chi_\tau = \left(1 - \frac{\rho_3 G \tau}{\rho}\right) \left(\frac{D}{I_\rho} - \frac{G}{\rho}\right) - \frac{\tau G \delta^2}{\rho I_\rho}.$$  

Moreover, he proved the well-posedness of the system.

Apalara also brought up an interesting open problem to establish the lack of exponential stability of problem (1.1), that is, to prove that the system is not exponentially stable unless $\chi_\tau = 0$. This is achieved in this note by applying the Gearhart–Herbst–Prüss–Huang theorem for dissipative systems (see Prüss [9] and Huang [6]).

To end this section, we recall some recent results on lack of exponential stability. Apalara et al. [2] studied a one-dimensional thermoelastic Timoshenko system of the form

$$\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \delta \theta_x &= 0, \quad (x, t) \in (0, L) \times (0, +\infty), \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) &= 0, \quad (x, t) \in (0, L) \times (0, +\infty), \\
\rho_3 \theta_t + q_x + \delta \varphi_{tx} &= 0, \quad (x, t) \in (0, L) \times (0, +\infty), \\
\tau \varphi_t + q + \gamma \theta_x &= 0, \quad (x, t) \in (0, L) \times (0, +\infty),
\end{aligned}$$

where the heat conduction is governed by Cattaneo’s law and the coupling is via the displacement equation. The authors proved the exponential (resp. polynomial) decay when the stability number $\chi_\tau$ is zero (resp. nonzero). Moreover, when $\chi_\tau \neq 0$, they showed that the system lacks exponential stability. In [11], Santos et al. studied a Bresse–Fourier system
They proved the system is not exponentially stable when \( k/\rho_1 \neq b/\rho_2 \) or \( k \neq k_0 \). In addition, they established that the exponential decay and polynomial decay depend on the relation between the wave speeds. For other results related to lack of exponential stability, see \([3, 4, 7, 8, 10, 12]\) and the references therein.

This note is organized as follows. In Section 2, we present some hypotheses needed for our work and state the main result. In Section 3, we show that the system is not exponentially stable if \( \chi_0 \neq 0 \). Throughout we use \( c \) to denote a generic positive constant.

### 2. Preliminaries and main results

In this section, we begin with some known results for problem \((1.1)\). First, let \( \xi = 3s - \psi \). Then \((1.1)\) can be rewritten as follows:

\[
\begin{aligned}
&\rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k\theta_x = 0, & (x, t) \in (0, L) \times (0, +\infty), \\
&\rho_2 \varphi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) - k\theta = 0, & (x, t) \in (0, L) \times (0, +\infty), \\
&\rho_1 w_{tt} - k_0 (w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) - kl\theta = 0, & (x, t) \in (0, L) \times (0, +\infty), \\
&\rho_3 \theta_t - \alpha \theta_{xx} + k(\varphi_x + \psi + lw)_t = 0, & (x, t) \in (0, L) \times (0, +\infty), \\
\end{aligned}
\]

(2.1)

Next, let

\[
\Phi = (w, u, \xi, v, s, z, \theta, q)^T, \\
\Phi_0(x) = (w_0, w_1, \xi_0, \xi_1, s_0, s_1, \theta_0, q_0)^T,
\]

where \( u = w_t, \) \( v = \xi_t \) and \( z = s_t \). Hence, problem (2.1) is equivalent to the abstract Cauchy problem

\[
\begin{aligned}
&\partial_t \Phi(x, t) = \mathcal{A} \Phi(x, t), \\
&\Phi(x, 0) = \Phi_0(x),
\end{aligned}
\]

(2.2)
where $\mathcal{A}$ is the operator defined by

$$
\mathcal{A}\Phi = \begin{pmatrix}
    u \\
    v \\
    w \\
    z \\
    -\frac{G}{\rho} (3s - \xi - w_x) \\
    -\frac{G}{I_\rho} (3s - \xi - w_x) + \frac{D}{I_\rho} \xi_{xx} - \frac{\delta}{I_\rho} \theta_x \\
    -\frac{G}{I_\rho} (3s - \xi - w_x) - \frac{4\gamma}{3I_\rho} s - \frac{4\beta}{3I_\rho} z + \frac{D}{I_\rho} s_{xx} \\
    -\frac{1}{\rho_3} q_x - \frac{\alpha}{\rho_3} v_x \\
    -\frac{\alpha}{\tau} q - \frac{1}{\tau} \theta_x
\end{pmatrix}
$$

Now, set $H^1_*(0,1) = \{ \eta \in H^1(0,1) : \eta(0) = 0 \}$, $\tilde{H}^2_*(0,1) = H^2(0,1) \cap H^1_*(0,1)$, $\tilde{H}^1_*(0,1) = \{ \eta \in H^1(0,1) : \eta(1) = 0 \}$, $\tilde{H}^2_*(0,1) = H^2(0,1) \cap \tilde{H}^1_*(0,1)$, and define the function space for $\Phi$ as follows:

$$(2.3) \quad \mathcal{H} = \tilde{H}^1_*(0,1) \times L^2(0,1) \times H^1_*(0,1) \times L^2(0,1) \times H^1_*(0,1)$$

endowed with the inner product

$$(\Phi, \tilde{\Phi})_{\mathcal{H}} = \rho \int_0^1 u\tilde{u} \, dx + I_\rho \int_0^1 v\tilde{v} \, dx + 3I_\rho \int_0^1 z\tilde{z} \, dx + \rho_3 \int_0^1 \theta\tilde{\theta} \, dx$$

$$+ \tau \int_0^1 q\tilde{q} \, dx + G \int_0^1 (3s - \xi - w_x)(3\tilde{s} - \tilde{\xi} - \tilde{w}_x) \, dx$$

$$+ D \int_0^1 \xi_x\tilde{\xi}_x \, dx + 4\gamma \int_0^1 s\tilde{s} \, dx + 3D \int_0^1 s_x\tilde{s}_x \, dx.$$

Then the domain of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \{ \Phi \in \mathcal{H} : w \in \tilde{H}^2_*(0,1), \xi, s \in H^2_*(0,1), \theta \in \tilde{H}^1_*(0,1), q \in H^1_*(0,1),$$

$$u \in \tilde{H}^1_*(0,1), v, z \in H^1_*(0,1), w_x(0) = \xi_x(1) = s_x(1) = 0 \}.$$ 

The well-posedness of problem (2.2) is ensured by

**Theorem 2.1** ([1, Theorem 3.1]). Let $\Phi_0 \in \mathcal{H}$. Then problem (2.2) has a unique weak solution $\Phi \in C(\mathbb{R}^+; \mathcal{H})$. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then

$$\Phi \in C(\mathbb{R}^+; D(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H}).$$

Our main result reads as follows:
Theorem 2.2. Let \( S(t) = e^{At} \). It is a \( C_0 \)-semigroup of contractions in the Hilbert space \( \mathcal{H} \). If \( \chi_\tau \neq 0 \), then the semigroup \( S(t) \) on \( \mathcal{H} \) is not exponentially stable.

3. Lack of exponential stability. In this section, we prove that the system (1.1) is not exponentially stable by making use of the Gearhart–Herbst–Prüss–Huang theorem for dissipative systems [6, 9, 11].

Theorem 3.1. Let \( S(t) = e^{At} \) be a \( C_0 \)-semigroup of contractions on a Hilbert space \( \mathcal{H} \). Then \( S(t) \) is exponentially stable if and only if

\[
\rho(A) \ni \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R} \quad \text{and} \quad \lim_{|\lambda| \to \infty} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty,
\]

where \( \rho(A) \) is the resolvent set of the operator \( A \).

Proof of Theorem 2.2. The main idea of the proof is to show that there exists a sequence of imaginary numbers \( \lambda_\mu, \mu \in \mathbb{N}^+ \), such that

\[
\| (\lambda_\mu I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty,
\]

which is equivalent to proving that there exist \( F_\mu \in \mathcal{H} \) with \( \| F_\mu \|_\mathcal{H} \leq 1 \) and \( V_\mu \in D(A) \) with

\[
\| (\lambda_\mu I - A)^{-1} F_\mu \|_\mathcal{H} = \| V_\mu \|_\mathcal{H} \to \infty.
\]

Hence, we obtain

(3.1) \( (\lambda_\mu - A)V_\mu = F_\mu \).

Rewriting the spectral equation (3.1) in terms of its components, we have \( \lambda_\mu = \lambda \) and

\[
\begin{aligned}
\lambda v_1 - v_2 &= f_1, \\
\lambda rv_2 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 &= \rho f_2, \\
\lambda v_3 - v_4 &= f_3, \\
\lambda I_\rho v_4 + G\partial_x v_1 + Gv_3 - D\partial_{xx} v_3 - 3Gv_5 + \delta \partial_x v_7 &= I_\rho f_4, \\
\lambda v_5 - v_6 &= f_5, \\
3\lambda I_\rho v_6 - 3G\partial_x v_1 - 3Gv_3 + (9G + 4\gamma) v_5 - 3D\partial_{xx} v_5 + 4\beta v_6 &= 3I_\rho f_6, \\
\lambda \rho v_7 + \partial_x v_8 + \partial_\tau v_7 &= \rho_3 f_7, \\
\lambda v_8 + \alpha v_8 + \partial_\tau v_7 &= \tau f_8,
\end{aligned}
\]

where \( V_\mu = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)^T \in D(A) \) and \( F_\mu = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H} \).
Next, choosing \( f_1 = f_3 = f_4 = f_5 = f_6 = f_7 = f_8 = 0 \) and \( f_2 = \frac{1}{\rho} \cos(\frac{\mu \pi}{2} x) \), and writing \( \mu' = \mu \pi / 2 \) to simplify formulas, we obtain

\[
\begin{align*}
\lambda^2 \rho v_1 - G \partial_{xx} v_1 - G \partial_x v_3 + 3G \partial_x v_5 &= \cos(\mu' x), \\
\lambda^2 I_\rho v_3 + G \partial_x v_1 + G v_3 - D \partial_{xx} v_3 - 3G v_5 + \delta \partial_x v_7 &= 0, \\
\lambda^2 I_\rho v_5 - G \partial_x v_1 - G v_3 + (3G + \frac{4\gamma}{3}) v_5 - D \partial_{xx} v_5 + \lambda \frac{4\beta}{3} v_5 &= 0, \\
\lambda \rho_3 v_7 + \partial_x v_8 + \lambda \delta \partial_x v_3 &= 0, \\
\lambda \tau v_8 + \alpha v_8 + \partial_x v_7 &= 0.
\end{align*}
\]

(3.3)

Then, taking the boundary conditions into consideration, we set

\[
\begin{align*}
v_1 &= A \cos(\mu' x), \\
v_3 &= B \sin(\mu' x), \\
v_5 &= C \sin(\mu' x), \\
v_7 &= E \cos(\mu' x), \\
v_8 &= F \sin(\mu' x)
\end{align*}
\]

for \( \mu = 2k + 1, k = 0, 1, 2, \ldots \). Consequently, (3.3) becomes

\[
\begin{align*}
(\lambda^2 \rho + G \mu'^2)A - G \mu' B + 3G \mu' C &= 1, \\
-G \mu' A + (\lambda^2 I_\rho + G + D \mu'^2) B - 3GC - \delta \mu' E &= 0, \\
G \mu' A - GB + (\lambda^2 I_\rho + 3G + \frac{4\gamma}{3} + \lambda \frac{4\beta}{3} + D \mu'^2) C &= 0, \\
\lambda \delta \mu' B + \lambda \rho_3 E + \mu' F &= 0, \\
-\mu' E + (\lambda \tau + \alpha) F &= 0.
\end{align*}
\]

(3.4)

Now, we take \( \lambda^2 \rho + G \mu'^2 = 0 \) such that \( \lambda = i \sqrt{G/\rho} \mu' \). Then, (3.4) can be rewritten as

\[
\begin{align*}
-G \mu' B + 3G \mu' C &= 1, \\
-G \mu' A + (-\frac{L G}{\rho} \mu'^2 + G + D \mu'^2) B - 3G C - \delta \mu' E &= 0, \\
G \mu' A - GB + (-\frac{L G}{\rho} \mu'^2 + 3G + \frac{4\gamma}{3} + i \frac{4\beta}{3} \sqrt{G/\rho} \mu' + D \mu'^2) C &= 0, \\
i \delta \sqrt{\frac{G}{\rho}} \mu'^2 B + i \rho_3 \sqrt{\frac{G}{\rho}} \mu' E + \mu' F &= 0, \\
-\mu' E + (i \tau \sqrt{\frac{G}{\rho}} \mu' + \alpha) F &= 0.
\end{align*}
\]

(3.5)

From (3.5), we have

\[
F = \frac{\mu'}{i \tau \sqrt{G/\rho} \mu' + \alpha} E.
\]

(3.6)

Substituting (3.6) into (3.5) yields

\[
E = \frac{(G \delta \tau / \rho) \mu'^3 - i \delta \alpha \sqrt{G/\rho} \mu'^2}{(1 - G \tau \rho_3 / \rho) \mu'^2 + i \rho_3 \alpha \sqrt{G/\rho} \mu'} B.
\]

(3.7)
Combining \( (3.5)_2 \) and \( (3.5)_3 \), we obtain
\[
(3.8) \quad \left( -\frac{I_\rho G}{\rho} \mu'^2 + D\mu'^2 \right) B + \left( -\frac{I_\rho G}{\rho} \mu'^2 + \frac{4\gamma}{3} + i\frac{4\beta}{3} \sqrt{\frac{G}{\rho} \mu'} + D\mu'^2 \right) C - \delta \mu' E = 0.
\]
Then, replacing \( (3.7) \) in \( (3.8) \), we get
\[
(3.9) \quad C = -\frac{(-\frac{I_\rho G}{\rho} \mu'^2 + D\mu'^2) + (\frac{-G\delta^2 \tau/\rho}{1-G\tau_3/\rho} e^{4i\alpha \sqrt{\frac{G}{\rho} \mu'}} B + \frac{3GA}{C\mu} B.}
\]
Gathering \( (3.5)_3 \) and \( (3.9) \), we obtain
\[
(3.10) \quad A = \frac{G + A_\mu + 3GA_\mu/C\mu}{G\mu'} B.
\]
Next, substituting \( (3.9) \) into \( (3.5)_1 \), we arrive at
\[
B = -\frac{1}{G\mu' + (3GA_\mu/C\mu) \mu'}.
\]
Let \( \chi = D/G - I_\rho/\rho \) and \( \chi_1 = 1 - \tau \rho_3 G/\rho \). In case \( \chi_1 = 0 \), we get
\[
B \rightarrow \frac{i\rho_3 \alpha \chi}{3\tau \delta^2 \sqrt{G/\rho} \mu'^2} \text{ for } \mu \text{ large.}
\]
Then
\[
A \rightarrow -\frac{\chi}{3G}, \quad B \rightarrow 0, \quad C \rightarrow 0, \quad E \rightarrow \frac{\chi}{3\delta}, \quad F \rightarrow \frac{\chi}{3i\delta \sqrt{G/\rho}}, \quad \mu \rightarrow \infty.
\]
Thus, we obtain
\[
\|V_\mu\|_H^2 \geq G \int_0^1 (3v_5 - v_3 - v_{1x})^2 \, dx
\]
\[
= G \int_0^1 (3C \sin(\mu'x) - B \sin(\mu'x) + A\mu' \sin(\mu'x))^2 \, dx
\]
\[
= G (3C - B + A\mu')^2 \int_0^1 (\sin(\mu'x))^2 \, dx
\]
\[
\approx \frac{\chi^2}{18G} \mu'^2 \rightarrow \infty \quad \text{as } \mu \rightarrow \infty.
\]
In case \( \chi_1 \neq 0 \), we have
\[
B \rightarrow -\frac{\chi \chi_1}{(3I_\rho \chi \tau + \chi \chi_1 G) \mu'} \text{ for } \mu \text{ large.}
\]
Therefore
\[ A \rightarrow -\frac{I_\rho \chi \tau}{3GI_\rho \chi + \chi \chi_1 G^2}, \quad B \rightarrow 0, \quad C \rightarrow 0, \quad E \rightarrow -\frac{\tau \delta G \chi}{3\rho I_\rho \chi + \rho G \chi \chi_1}, \]
\[ F \rightarrow \frac{iG \delta \chi}{(3\rho I_\rho \chi + \rho G \chi_1 G)\sqrt{G/\rho}}, \quad \mu \rightarrow \infty. \]

Consequently,
\[
\|V_\mu\|_{_H}^2 \geq G \int_0^1 \left( 3v_5 - v_3 - v_{1x} \right)^2 \, dx
= G \int_0^1 \left( 3C \sin(\mu' x) - B \sin(\mu' x) + A \mu' \sin(\mu' x) \right)^2 \, dx
= G(3C - B + A \mu')^2 \int_0^1 (\sin(\mu' x))^2 \, dx
\approx \frac{GI_\rho^2 \chi^2 \chi_1^2}{2(3GI_\rho \chi + \chi \chi_1 G^2)^2} \mu^2 \rightarrow \infty \quad \text{as } \mu \rightarrow \infty.
\]

From the above, it is easy to see that
\[ \|V_\mu\|_{_H} \to \infty \quad \text{as } \mu \to \infty, \]
which is the desired conclusion.

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