NEW UPPER BOUNDS FOR THE NUMBER OF DIVISORS FUNCTION

BY

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Abstract. Let \( \tau(n) \) stand for the number of divisors of the positive integer \( n \). We obtain upper bounds for \( \tau(n) \) in terms of \( \log n \) and the number of distinct prime factors of \( n \).

1. Introduction and notation. Let \( \tau(n) \) denote the number of divisors of the positive integer \( n \) and \( \omega(n) \) the number of prime factors of \( n \). We shall also be using the functions

\[
\gamma(n) := \prod_{p|n} p, \quad \beta(n) := \prod_{p|n} \frac{1}{\log p}.
\]

In 1915, Ramanujan [8, (3)] obtained the inequality

(1.1) \[
\tau(n) \leq \left( \frac{\log(n \gamma(n))}{\omega(n)} \right)^{\omega(n)} \beta(n) \quad (n \geq 2).
\]

In this paper, we explicitly compute some interesting limit cases of (1.1) and show that for \( k = \omega(n) \geq 74 \),

\[
\tau(n) < \left( 1 + \frac{\log n}{k \log k} \right)^k.
\]

We also provide another proof of (1.1) in Corollary 4.5.

From here on, for each integer \( k \geq 0 \), we let

\[
n_k := p_1 p_2 \cdots p_k, \quad \text{the product of the first } k \text{ primes (with } n_0 = 1).\]

Also, when we write \( \log_+ x \), we mean \( \log \max(2, x) \).

Finally, given the factorization of an integer \( n = q_1^{\alpha_1} \cdots q_k^{\alpha_k} \) with \( q_1 < \cdots < q_k \), we call the vector \( (\alpha_1, \ldots, \alpha_k) \) the exponent vector of \( n \).
2. Background results. It is well known that
\[
2^{\omega(n)} \leq \tau(n) \leq \left(1 + \frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)} \quad (n \geq 2),
\]
where \(\Omega(n)\) stands for the number of prime divisors of \(n\) counting multiplicities. Here, the lower bound is best possible in general and the upper bound, which follows from the inequality between arithmetic and geometric means, is of great interest. For instance, it is known that the quotient \(\Omega(n)/\omega(n)\) is near 1 for almost all integers \(n\), as was shown for instance by the first author in [2]. In fact, one can use (2.1) and the estimate
\[
|\{n \leq x : \Omega(n) \geq \kappa \omega(n)\}| \ll x(\log \log x)(\log x)^{2\kappa - 1}
\]
valid for all \(\kappa \geq 1\) and \(x \geq 3\) (see Tenenbaum [11, Corollaire 3.6, p. 436] or for an even sharper estimate, Balazard [1, Théorème 3]) to show that for every fixed \(\varepsilon > 0\),
\[
\tau(n) \leq (2 + \varepsilon)^{\omega(n)} \quad \text{for almost all } n.
\]

We are motivated by the fact that (see Wigert [12])
\[
\log \tau(n) \leq \frac{(\log 2)(\log n)}{\log \log n} + O\left(\frac{\log n}{(\log \log n)^2}\right),
\]
and by the fact proved by Nicolas and Robin [6] that the maximum value of the function
\[
n \mapsto \frac{\log(\tau(n)) \log \log n}{(\log 2)(\log n)} \quad (n \geq 3)
\]
is attained at \(n = 6,983,776,800 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = 720n_8\) and its value is approximately 1.5379. Much more is known on the ratio (2.2), as explained in [5]. But, meanwhile, those large values are almost never attained since it has been proved by Erdős and Nicolas [4] that, given any real \(\vartheta \in (0, 1)\), the cardinality of the set of those \(n \leq x\) for which
\[
\omega(n) \geq \vartheta \frac{\log x}{\log \log x}
\]
is \(\ll x^{1-\vartheta+o(1)}\) as \(x \to \infty\). Furthermore, this set corresponds exactly to the set of values where \(\tau(n)\) is large. This can be deduced from (1.1), that is, if we define \(\vartheta\) by \(\omega(n) = \vartheta \frac{\log n}{\log \log n}\) for \(n \geq 17\), then we find that
\[
\tau(n) \leq \exp\left(\vartheta \log \left(1 + \frac{1}{\vartheta}\right) \frac{\log n}{\log \log n} \left(1 + O\left(\frac{\log \log \log n}{\log \log n}\right)\right)\right).
\]
Observe that the function \(\vartheta \mapsto \vartheta \log(1 + 1/\vartheta)\) is strictly increasing from 0 to \(\log 2\) as \(\vartheta\) goes from 0 to 1.
Robin [9] also designed an algorithm that allows one to easily obtain the list of all highly composite numbers with less than $k$ prime factors, which yields the absolute best estimate for $\tau(n)$ for every $n \leq x$ with any given $x$.

Before stating our main results, we introduce the function $\lambda(n)$ defined implicitly by

$$\tau(n) = \left(1 + \frac{\lambda(n) \log n}{k \log k}\right)^k,$$

where $k = \omega(n) \geq 2$. Therefore, for each integer $n \geq 2$ with $\omega(n) = k \geq 2$, we set

$$\lambda(n) := \frac{(\tau(n)^{1/k} - 1)k \log k}{\log n}. \quad (2.3)$$

3. Main results

**Theorem 3.1.** For every integer $n \geq 2$,

$$\tau(n) \leq \left(\frac{\eta_2 \log n}{\omega(n) \log \omega(n)}\right)^{\omega(n)}, \quad (3.1)$$

where

$$\eta_2 := \exp\left(\frac{1}{6} \log 96 - \log \left(\frac{\log 60060}{6 \log 6}\right)\right) = 2.0907132 \ldots$$

**Theorem 3.2.** For every integer $n > 24n_{16} = 782139803452561073520$,

$$\tau(n) \leq \left(\frac{2 \log n}{\omega(n) \log \omega(n)}\right)^{\omega(n)}. \quad (3.2)$$

Moreover, the inequality remains true for all $n \geq 2$ with $\omega(n) \leq 3$.

**Theorem 3.3.** For every integer $n \geq 2$,

$$\tau(n) \leq \left(1 + \frac{\log n}{\omega(n) \log \omega(n)}\right)^{\omega(n)} \quad (3.3)$$

where

$$\eta_3 := \lambda(720n_7) = \frac{(1152^{1/7} - 1)7 \log 7}{\log 367567200} = 1.1999953 \ldots$$

**Theorem 3.4.** For every positive integer $n$ with $k = \omega(n) \geq 74$,

$$\tau(n) < \left(1 + \frac{\log n}{k \log k}\right)^k. \quad (3.4)$$

**Remark 3.5.** The number

$$n' = 2^{13} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^3 \cdot 19^2 \cdots 53^2 \cdot 59 \cdots 367,$$

whose prime factors are the first 73 prime numbers, shows that Theorem 3.4 is the best possible since $\lambda(n') = 1.0008832 \ldots$. In fact, one can find similar
examples of $n$ (that is, with $\lambda(n) > 1$) for each $\omega(n) = k \in [3, 73]$. Also, the methods used in the proof of Theorem 3.4 allow one to show that the largest value of $\lambda(n)$, with $\omega(n) = 74$, is attained only by the number

$$n'' = 2^{13} \cdot 3^8 \cdot 5^7 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17^3 \cdot 19^2 \cdot 53^2 \cdot 59 \cdots 373,$$

for which $\lambda(n'') = 0.99991077\ldots$. (Observe that $n'$ realizes the unique maximum of the function $\lambda$ among the integers $n$ with exactly 73 distinct prime factors.)

By comparing the lower bound in (2.1) with (3.4) and after some computation, one can show that the inequality

$$n \geq \omega(n) \quad (n \geq 2)$$

holds for each $n$ satisfying $\omega(n) \notin [4, 12]$ or $n > 43n_1$. This helps us understand why Theorem 3.4 is more powerful than Theorem 3.2.

**Theorem 3.6.** The largest integer $n$ with $k = \omega(n) \geq 44$ for which $\lambda(n) \geq 1$ is the integer made up of the first 44 primes that has the exponent vector

$$\omega = (354, 223, 152, 125, 102, 95, 86, 83, 77, 72, 71, 67, 65, 64, 63, 61, 59, 59, 57, 57, 56, 55, 55, 54, 53, 52, 52, 52, 51, 51, 50, 49, 49, 49, 48, 48, 47, 47, 47, 46, 46, 46, 46).$$

There are infinitely many integers $n$ for which $\lambda(n) > 1$. Most of them satisfy $\omega(n) = 43$; see the final remarks of this paper for more information.

### 4. Preliminary lemmas

**Definition 4.1.** Let $x_i$, with $i \in \{1, \ldots, k\}$, be fixed real numbers that satisfy $0 < x_1 \leq \cdots \leq x_k$. Let

$$\mu := \frac{x_1 + \cdots + x_k}{k} \quad \text{and} \quad \varpi := \sum_{i=1}^{k} |x_i - \mu|.$$  

Assume also that $x_1 \leq \cdots \leq x_m \leq \mu \leq x_{m+1} \leq \cdots \leq x_k$ for a fixed $m \in \{1, \ldots, k-1\}$ where $k \geq 2$. Further, set

$$\mu_1 := \frac{x_1 + \cdots + x_m}{m} = \mu - \frac{\varpi}{2m},$$

$$\mu_2 := \frac{x_{m+1} + \cdots + x_k}{k-m} = \mu + \frac{\varpi}{2(k-m)}$$

and also

$$\varpi_1 := \sum_{i=1}^{m} |x_i - \mu_1| \quad \text{and} \quad \varpi_2 := \sum_{i=m+1}^{k} |x_i - \mu_2|.$$  

**Example 4.2.** Here it is how this notation is used throughout the proof of Theorem 3.6. Fix an integer $n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$. For each $i \in \{1, \ldots, k\}$, we
define $\theta_i$ implicitly by $n^{\theta_i} = q_i^{\alpha_i}$, so that $\theta_1 + \cdots + \theta_k = 1$. We write

\begin{equation}
(4.1) \quad x_i := \frac{(\alpha_i + 1) \log q_i}{\log n}
\end{equation}

and assume that the primes $q_i$ are ordered in such a way that (4.7) below holds. In this case we have

\begin{align*}
\mu &= \frac{1}{k} \left( 1 + \frac{\log \gamma(n)}{\log n} \right), \\
\varpi &= \frac{1}{k} \sum_{i=1}^{k} \left| \frac{(\alpha_i + 1) k \log q_i - \log \gamma(n)}{\log n} - 1 \right| =: \frac{\varpi'}{k}, \\
\mu_1 &= \frac{1}{k} \left( 1 + \frac{\log \gamma(n)}{\log n} \right) - \frac{\varpi'}{2km}, \\
\mu_2 &= \frac{1}{k} \left( 1 + \frac{\log \gamma(n)}{\log n} \right) + \frac{\varpi'}{2(k(k - m))},
\end{align*}

and

\begin{align*}
\varpi_1 &= \frac{1}{k} \sum_{i=1}^{m} \left| \frac{(\alpha_i + 1) k \log q_i - \log \gamma(n)}{\log n} - 1 + \frac{\varpi'}{2m} \right| =: \frac{\varpi_1'}{k}, \\
\varpi_2 &= \frac{1}{k} \sum_{i=m+1}^{k} \left| \frac{(\alpha_i + 1) k \log q_i - \log \gamma(n)}{\log n} - 1 - \frac{\varpi'}{2(k - m)} \right| =: \frac{\varpi_2'}{k}. 
\end{align*}

**Lemma 4.3.**

(i) For $k \geq 1$ we have

\begin{equation}
(4.5) \quad x_1 \cdots x_k \leq \mu^k.
\end{equation}

(ii) For $k \geq 2$ we have

\begin{equation}
(4.6) \quad x_1 \cdots x_k \leq \mu_1^m \mu_2^{k-m}.
\end{equation}

(iii) For $k \geq 4$, let $m \in \{2, \ldots, k-2\}$, $m_1 \in \{1, \ldots, m-1\}$, $m_2 \in \{1, \ldots, k-m-1\}$ and assume that

\begin{equation}
(4.7) \quad 0 < x_1 \leq \cdots \leq x_{m_1} \leq \mu_1 \leq x_{m_1+1} \leq \cdots \leq x_m \leq \mu \leq x_{m+1} \leq \cdots \leq x_{m+m_2} \leq \mu_2 \leq x_{m+m_2+1} \leq \cdots \leq x_k.
\end{equation}

Then

\begin{equation}
(4.8) \quad x_1 \cdots x_k \leq \left( \mu_1 - \frac{\varpi_1}{2m_1} \right)^{m_1} \left( \mu_1 + \frac{\varpi_1}{2(m-m_1)} \right)^{m-m_1} \times \left( \mu_2 - \frac{\varpi_2}{2m_2} \right)^{m_2} \left( \mu_2 + \frac{\varpi_2}{2(k-m-m_2)} \right)^{k-m-m_2}.
\end{equation}

**Proof.** In each case, we simply use the arithmetic-geometric inequality for the corresponding subproduct of variables for which we know the average. \hfill \blacksquare
Lemma 4.4. Let \( k \geq 1 \) be an integer, \( z_i > 0 \) and \( \theta_i \geq -1/z_i \) be real numbers for \( i = 1, \ldots, k \), and assume that
\[
\theta_1 + \cdots + \theta_k = 1.
\]
Then
\[
(4.9) \quad \prod_{i=1}^{k} (1 + \theta_i z_i) \leq \prod_{i=1}^{k} \left( \frac{z_i}{k} \right) \left( 1 + \sum_{j=1}^{k} \frac{1}{z_j} \right)^k,
\]
with equality if and only if
\[
\theta_i = \frac{1}{k} \left( 1 + \sum_{j=1}^{k} \frac{1}{z_j} \right) - \frac{1}{z_i} \quad (i = 1, \ldots, k).
\]

Proof. Using the arithmetic-geometric mean inequality, the hypothesis \( z_i > 0 \) and the fact that for each \( i \) we have \( 1 + \theta_i z_i \geq 0 \), we can write
\[
\prod_{i=1}^{k} (1 + \theta_i z_i) = \left( \prod_{i=1}^{k} z_i \right) \left( \prod_{j=1}^{k} \left( \theta_j + \frac{1}{z_j} \right) \right) \leq \prod_{i=1}^{k} \left( \frac{z_i}{k} \right) \left( \sum_{j=1}^{k} \left( \theta_j + \frac{1}{z_j} \right) \right)^k = \prod_{i=1}^{k} \left( \frac{z_i}{k} \right) \left( 1 + \sum_{j=1}^{k} \frac{1}{z_j} \right)^k.
\]
We have equality if and only if
\[
\theta_i + \frac{1}{z_i} = \frac{1}{k} \left( 1 + \sum_{j=1}^{k} \frac{1}{z_j} \right) \quad (i = 1, \ldots, k). \]

Corollary 4.5. Assume the above notation. Then, for every integer \( n \geq 2 \),
\[
(4.10) \quad \tau(n) \leq \left( \frac{\log n}{\omega(n)} \right)^{\omega(n)} \left( 1 + \frac{\log \gamma(n)}{\log n} \right)^{\omega(n)} \beta(n)
\]
and
\[
(4.11) \quad \tau(n) \leq \left( \frac{2 \log n}{\omega(n)} \right)^{\omega(n)} \beta(n).
\]

Proof. We write \( n = q_1^{\alpha_1} \cdots q_k^{\alpha_k} \) and \( n^{\theta_i} = q_i^{\alpha_i} \) as in Example 4.2. Using (4.9) with \( z_i = \frac{\log n}{\log q_i} \) we have
\[
\tau(n) = \prod_{i=1}^{k} (1 + \alpha_i) = \prod_{i=1}^{k} \left( 1 + \theta_i \log n / \log q_i \right) \leq \left( \frac{\log n}{k} \right)^k \left( 1 + \frac{\log \gamma(n)}{\log n} \right)^k \prod_{p|n} \frac{1}{\log p},
\]
which proves (4.10). Since \( \log \gamma(n) \leq \log n \), (4.11) follows immediately from (4.10). \( \blacksquare \)
In any event, it follows from Corollary 4.5 that
\begin{equation}
\lambda(n) \leq \left( \prod_{p|n} \log k \right)^{1/k} + \frac{\log \gamma(n)}{\log n} \left( \prod_{p|n} \log k \right)^{1/k} - k \frac{\log k}{\log n}.
\end{equation}

**Lemma 4.6.**

(i) Assume that \( \mu > 0, \ m \geq 1 \) and \( k - m \geq 1 \). Then the function
\begin{equation}
\varpi \mapsto (\mu - \frac{\varpi}{2m})^m \left( \mu + \frac{\varpi}{2(k-m)} \right)^{k-m}
\end{equation}
decreases when \( \varpi \) increases from 0 to 2m\( \mu \).

(ii) Assume that \( \mu > 0, \ \varpi_1, \varpi_2 \geq 0, \ m \geq 1, \ k - m \geq 1, \ m_1, m_2 \geq 1, \ m - m_1 \geq 1, \ k - m - m_2 \geq 1, \ \mu - \frac{\varpi_1}{2m} - \frac{\varpi_2}{2m_1} > 0 \) and \( \mu + \frac{\varpi}{2(k-m)} - \frac{\varpi_2}{2m_2} > 0 \). Then the function
\begin{equation}
f(\varpi) := (\mu - \frac{\varpi}{2m} - \frac{\varpi_1}{2m_1})^{m_1} \left( \mu - \frac{\varpi}{2m} + \frac{\varpi_1}{2(m - m_1)} \right)^{m - m_1} \times \left( \mu + \frac{\varpi}{2(k-m)} - \frac{\varpi_2}{2m_2} \right)^{m_2} \left( \mu + \frac{\varpi}{2(k-m)} + \frac{\varpi_2}{2(k - m - m_2)} \right)^{k - m - m_2}
\end{equation}
has the property that if \( f'(\varpi_0) < 0 \) for some \( \varpi_0 > 0 \), then \( f(\varpi) < f(\varpi_0) \) for each \( \varpi > \varpi_0 \).

(iii) Assume that \( A, B, C > 0, \ \gamma_1, \gamma_2 \geq 0, \ \varrho_1, \varrho_2 \geq 0, \ \varrho_1 + \varrho_2 = 1 \) and \( C < AB \). Then the function
\begin{equation}
z \mapsto B \left( \gamma_1 + \frac{A}{z} \right)^{\varrho_1} \left( \gamma_2 + \frac{A}{z} \right)^{\varrho_2} - \frac{C}{z}
\end{equation}
decreases when \( z \) increases for \( z > 0 \).

(iv) Assume that \( A, B, C > 0, \ z > 0, \ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0, \ \varrho_1, \varrho_2, \varrho_3, \varrho_4 \geq 0, \ \varrho_1 + \varrho_2 + \varrho_3 + \varrho_4 = 1 \) and \( C < AB \). Then the expression
\begin{equation}
B \left( \gamma_1 + \frac{A}{z} \right)^{\varrho_1} \left( \gamma_2 + \frac{A}{z} \right)^{\varrho_2} \left( \gamma_3 + \frac{A}{z} \right)^{\varrho_3} \left( \gamma_4 + \frac{A}{z} \right)^{\varrho_4} - \frac{C}{z}
\end{equation}
decreases when \( z \) increases.

**Proof.** (i) Since the function (4.13) is assumed to be positive, its derivative with respect to \( \varpi \) has the same sign as its logarithmic derivative with respect to \( \varpi \). Since the logarithmic derivative is
\[
\frac{-1}{2\mu - \frac{\varpi}{m}} + \frac{1}{2\mu + \frac{\varpi}{k-m}},
\]
it is clearly strictly negative when \( 0 < \varpi < 2m\mu \).

(ii) Again, \( f \) is assumed to be positive, so its derivative with respect to \( \varpi \) has the same sign as its logarithmic derivative with respect to \( \varpi \). Also,
\[
\left( \frac{f'(\varpi)}{f(\varpi)} \right)' = \frac{-m_1}{m^2(2\mu - \frac{\varpi}{m} - \frac{\varpi_1}{m_1})^2} - \frac{m - m_1}{m^2(2\mu - \frac{\varpi}{m} + \frac{\varpi_1}{m_1})^2} - \frac{m_2}{(k - m)^2(2\mu - \frac{\varpi}{k} - \frac{\varpi_2}{m_2})^2} - \frac{k - m - m_2}{(k - m)^2(2\mu + \frac{\varpi}{k} - \frac{\varpi_2}{m_2})^2},
\]

which is clearly negative. We deduce that if \( f'(\varpi_0) < 0 \) for some \( \varpi_0 > 0 \), then \( f'(\varpi) < 0 \) for each \( \varpi > \varpi_0 \), which in turn implies
\[
f(\varpi) - f(\varpi_0) = \int_{\varpi_0}^{\varpi} f'(t) \, dt < 0
\]
for each \( \varpi > \varpi_0 \), thus establishing our claim.

(iii) We take the derivative of (4.15) with respect to \( z \) and multiply by \( z^2 \). We then see that the desired property is equivalent to
\[
C < AB \left( \gamma_1 + \frac{A}{z} \right) \left( \gamma_2 + \frac{A}{z} \right) \left( \frac{\varphi_1}{\gamma_1 + \frac{A}{z}} + \frac{\varphi_2}{\gamma_2 + \frac{A}{z}} \right).
\]

Now, from Jensen’s inequality for the exponential function, we have
\[
\frac{1}{z_1\varphi_1} \leq \frac{\varphi_1}{z_1} + \frac{\varphi_2}{z_2} \quad (z_1, z_2 > 0).
\]
We deduce that the hypothesis \( C < AB \) implies (4.17).

(iv) is proved in the same manner. ■

**Lemma 4.7.** Let \( A \) and \( B \) be fixed positive real constants. Consider the function \( \psi := \mathbb{Z} \times \mathbb{R}^* \times \mathbb{R} \to \mathbb{R}_0^+ \) defined by
\[
\psi(\alpha, x, \varphi) = \left| \frac{(\alpha + 1)B - A}{x} - \varphi \right|.
\]

(i) Assume \( x_1, \varphi_1 > 0 \). The minimum of \( \psi(\alpha, x, \varphi) \) for \( \alpha \in \mathbb{Z} \), \( x \in [x_1, x_2] \) and \( \varphi \in [\varphi_1, \varphi_2] \) is either 0 or the minimum over the eight possibilities provided by
\[
\alpha \in \left\{ \left\lfloor \frac{x_2\varphi_2 + A}{B} \right\rfloor - 1, \left\lfloor \frac{x_1\varphi_1 + A}{B} \right\rfloor - 1 \right\}, \quad x \in \{x_1, x_2\} \text{ and } \varphi \in \{\varphi_1, \varphi_2\}.
\]
The minimum is 0 if and only if
\[
\left\lfloor \frac{x_1\varphi_1 + A}{B} \right\rfloor \leq \left\lfloor \frac{x_2\varphi_2 + A}{B} \right\rfloor.
\]

(ii) Fix \( \delta > 0 \) and assume that \( x_1, \varphi_1 > 0 \). The minimum of \( \psi(\alpha, x, 1) \) for \( x \in [x_1, x_2] \) and \( \alpha \in \mathbb{Z} \}\left\{ \left\lfloor \frac{(1-\delta)x_1 + A}{B} \right\rfloor - 1 \text{,} \ldots, \left\lfloor \frac{(1+\delta)x_2 + A}{B} \right\rfloor - 1 \right\}\] is the minimum over the four possibilities provided by
\[
\alpha \in \left\{ \left\lfloor \frac{(1-\delta)x_1 + A}{B} \right\rfloor - 2, \left\lfloor \frac{(1+\delta)x_2 + A}{B} \right\rfloor \right\} \text{ and } x \in \{x_1, x_2\}.
Proof. (i) First, assume that the minimum is 0. Choose $(\alpha, x, \varphi)$ that realizes 0. We deduce that $\alpha + 1 = \frac{x_1 \varphi + A}{B}$ and this is equivalent to having (4.19). Now, assume that the minimum is not 0. In this case, $\left\lfloor \frac{x_2 \varphi + A}{B} \right\rfloor < \left\lceil \frac{x_1 \varphi + A}{B} \right\rceil + 1$. Also, if $(\alpha, x, \varphi)$ realizes the minimum then there are two cases: either $(\alpha + 1)B - A - x \varphi \geq 0$, in which case $\alpha + 1 \geq \left\lceil \frac{x_1 \varphi + A}{B} \right\rceil$, or $(\alpha + 1)B - A - x \varphi \leq 0$, in which case $\alpha + 1 \leq \left\lfloor \frac{x_2 \varphi + A}{B} \right\rfloor$. It is then clear that the minimum is attained for $\alpha \in \{\left\lfloor \frac{x_2 \varphi + A}{B} \right\rfloor - 1, \left\lceil \frac{x_1 \varphi + A}{B} \right\rceil - 1\}$. To conclude, once $\alpha$ is fixed, $(\alpha + 1)B - A - x \varphi$ attains its extremum at the extremities of the intervals since it is a sum of independent monotone functions.

(ii) The choice for $\alpha$ is clear. Also, if we assume that the minimum is not 0 then the choice for $x$ is also clear. Now, assume that the minimum is 0 and that it is attained at $(\alpha, x)$ with $\alpha = \left\lfloor \frac{(1 - \delta) x_1 + A}{B} \right\rfloor - 2 = \frac{(1 - \delta) x_1 + A}{B} - 2 + \xi$ for some $\xi \in [0, 1]$. In this case,

$$\frac{(\alpha + 1)B - A}{x} = \frac{(1 - \delta) x_1 + A - 2 + \xi + 1)B - A}{x} = \frac{(1 - \delta) x_1 + (\xi - 1)B}{x} \leq 1 - \delta < 1,$$

and similarly for the other choice of $\alpha$. This shows that the minimum is not 0 and the proof is complete. 

Lemma 4.8. We have

\begin{align*}
(4.20) & \sum_{i=1}^{k} \log p_i \leq k(\log k + \log \log k - 3/4) \quad \text{for } k \geq 8, \\
(4.21) & \sum_{i=1}^{k} \log \log p_i \geq k \left( \log \log k + \frac{\log \log k - 5/4}{\log k} \right) \quad \text{for } k \geq 319
\end{align*}

and

\begin{equation}
(4.22) \quad \beta(n_k) \leq (\log k)^{-k} \quad \text{for } k \geq 44.
\end{equation}

Proof. We first prove (4.21) by induction. We verify using a computer that it holds for each $k \in [319, 900000]$. Then, we assume that (4.21) holds for some $k \geq 900000$ and we want to establish it for $k + 1$. Define

$$W(x) := x \left( \log \log x + \frac{\log \log x - 5/4}{\log x} \right) \quad (x \geq e).$$

It will be enough to show that

$$\log \log p_{k+1} \geq W(k + 1) - W(k) \quad (k \geq 900000).$$

On the one hand, it is known that $p_j \geq j \log j$ (see for instance Rosser and
Schoenfeld [10] for each \( j \geq 1 \), so
\[
\log \log p_{k+1} \geq \log \log (k+1) + \frac{\log \log (k+1)}{\log (k+1)} - \frac{1}{2} \left( \frac{\log \log (k+1)}{\log (k+1)} \right)^2.
\]
Also, by the mean value theorem there is \( \xi \in (k, k+1) \) for which \( W(k+1) - W(k) = W'(\xi) \). Since \( \exp(\exp(9/4)) < 90000 \), we deduce that (4.21) holds at \( k+1 \) if
\[
\frac{1}{4} \geq \frac{1}{2} \left( \frac{\log \log (k+1)}{\log (k+1)} \right)^2,
\]
which is the case for \( k \geq 900000 \). The proof is complete.

Inequality (4.20) can be proved using a similar method or one can also use the sharper result of Massias and Robin [7, (1.14)], along with a computer verification for \( k = 8, \ldots, 13 \). Finally, (4.22) follows from (4.21) and an easy computer verification.

Let us further introduce the function
(4.23) \[ t(n) := \frac{\tau(n)^{1/k}}{\log n} \quad (n \geq 2). \]

**Lemma 4.9.** Let \( n \geq 2 \) be an integer, \( 2 \leq k = \omega(n) \) and \( p \) be a prime number. If \( p^\alpha \| n \) with \( \alpha \geq 2 \), then
\begin{align*}
\frac{\lambda(n)}{\lambda(n/p)} &\leq \left( 1 + \frac{2}{k\alpha} \right) \left( 1 - \frac{\log p}{\log n} \right), \\
\frac{t(n)}{t(n/p)} &\leq \left( 1 + \frac{1}{k\alpha} \right) \left( 1 - \frac{\log p}{\log n} \right).
\end{align*}

Also, for \( \ell \in \{1, 2\} \), we have
\begin{align*}
1 + \frac{\ell}{k\alpha} &\left( 1 - \frac{\log p}{\log n} \right) < 1 \iff p > n^{\frac{\ell}{\alpha k + \ell}}
\end{align*}
and
\begin{align*}
\alpha &= \max \left( 2, \left\lfloor \frac{\ell}{k} \left( \frac{\log n}{\log p} - 1 \right) \right\rfloor \right) \implies \left( 1 + \frac{\ell}{k\alpha} \right) \left( 1 - \frac{\log p}{\log n} \right) < 1.
\end{align*}

**Proof.** We write \( n = p^\alpha m \), so that \( (p, m) = 1 \) and therefore
\begin{align*}
\frac{\lambda(n)}{\lambda(n/p)} &= \frac{\tau(n)^{1/k} - 1}{\tau(n/p)^{1/k} - 1} \frac{\log n/p}{\log n} \\
&= \left( 1 + \frac{\tau(n)^{1/k} - \tau(n/p)^{1/k}}{\tau(n/p)^{1/k} - 1} \right) \left( 1 - \frac{\log p}{\log n} \right) \\
&= \left( 1 + \frac{\tau(m)^{1/k}}{\tau(n/p)^{1/k} - 1} \left( (\alpha + 1)^{1/k} - \alpha^{1/k} \right) \right) \left( 1 - \frac{\log p}{\log n} \right) \\
&\leq \left( 1 + \frac{\tau(n/p)^{1/k}}{\tau(n/p)^{1/k} - 1} \frac{1}{k\alpha} \right) \left( 1 - \frac{\log p}{\log n} \right),
\end{align*}
where the last inequality follows from the fact that
\[(\alpha + 1)^{1/k} - \alpha^{1/k} \leq \sup_{\xi \in [\alpha, \alpha+1]} \frac{\xi^{1/k}}{k \xi} = \frac{\alpha^{1/k}}{k \alpha}.
\]
Since the function \( z \mapsto \frac{z}{z-1} \) is strictly decreasing for \( z > 1 \), (4.24) follows from \( \tau(n/p)^{1/k} \geq 2 \). The proof of (4.25) is similar, and (4.26) and (4.27) follow from an easy computation.

**Lemma 4.10.** For any real \( z > 1 \) and integer \( n = q_1^{\alpha_1} \cdots q_k^{\alpha_k} \geq 2 \), let
\[(4.28) \quad v(n, z) := \log k \left( 1 + \frac{\log \gamma(n)}{\log z} \right) \beta(n)^{1/k} - \frac{k \log k}{\log z}.
\]
Then
\[(4.29) \quad \frac{d}{dz} v(n, z) \leq 0 \quad (n \geq 2)
\]
with strict inequality if \( \omega(n) \geq 2 \). Also,
\[(4.30) \quad v(n_k, n_k) < 1 \quad (k \geq 95).
\]

**Proof.** To prove (4.29), we first observe that
\[
\frac{d}{dz} v(n, z) = - \log k \frac{\log \gamma(n)}{z \log^2 z} \beta(n)^{1/k} + \frac{k \log k}{z \log^2 z},
\]
which implies that (4.29) is equivalent to
\[k \leq \beta(n)^{1/k} \log \gamma(n),
\]
which itself is a consequence of the arithmetic-geometric mean inequality applied to \( \log p \) for \( p | n \).

To prove (4.30), by observing that \( \gamma(n_k) = n_k \), we must show that
\[(4.31) \quad 2 < \left( \frac{k}{\log n_k} + \frac{1}{\log k} \right) \beta(n_k)^{-1/k}.
\]
Using (4.21) and (4.20) we see that the right hand side of (4.31) is
\[
\geq \left( \exp \left( \log \log k + \frac{\log \log k - 5/4}{\log k} \right) \right) \cdot \left( \frac{1}{\log k} + \frac{1}{\log k + \log \log k - 3/4} \right)
\]
\[
= \left( \exp \left( \frac{\log \log k - 5/4}{\log k} \right) \right) \cdot \left( 1 + \frac{\log k}{\log k + \log \log k - 3/4} \right)
\]
\[
> \left( 1 + \frac{\log \log k - 5/4}{\log k} \right) \cdot \left( 1 + \frac{\log k}{\log k + \log \log k - 3/4} \right)
\]
\[
= 2 + \frac{\log \log k - 5/4}{\log k} - \frac{1/2}{\log k + \log \log k - 3/4} > 2
\]
for each \( k \geq 319 \). On the other hand, an easy computer check shows that (4.30) holds for each integer \( k \in [95, 318] \), thus completing the proof.
Lemma 4.11. Let \( \alpha \in (0, 1) \), \( c_1, c_2 \in \mathbb{R}_{> 0} \) and \( I := (c_1^{-1/\alpha} + c_2, \infty) \). Consider the function \( g : I \to \mathbb{R} \) defined by

\[
g(z) := \frac{c_1(z - c_2)^\alpha - 1}{z}.
\]

Then \( g \) attains its unique maximum at some point \( z_0 > c_1^{-1/\alpha} + c_2 \).

Proof. Consider the function \( h : I \to \mathbb{R} \) given by

\[
h(z) := z^2(z - c_2)^{1-\alpha}g'(z) = c_1\alpha z - c_1(z - c_2) + (z - c_2)^{1-\alpha}.
\]

It follows that \( h \) and \( g' \) have the same sign and the same zeros in \( I \). Moreover, \( h(c_1^{-1/\alpha} + c_2) \) is positive and \( h(\infty) = -\infty \). On the other hand, \( h'(z) = c_1(\alpha - 1) + \frac{1-\alpha}{(z - c_2)^\alpha} \), in which case

\[h'(z) = 0 \iff 1 = c_1(z - c_2)^\alpha,
\]

which is impossible for \( z \in I \). Now, because \( h'(\infty) < 0 \), this means that \( h'(z) < 0 \) for \( z \in I \). The result follows. \( \Box \)

5. Proof of Theorem 3.1. It is easy to verify that (3.1) holds when \( \omega(n) = 1 \). For any \( n \) with \( \omega(n) \geq 2 \), we introduce the function \( r(n) \) defined implicitly by

\[
\tau(n) = \left( \frac{e^{r(n)} \log n}{\omega(n) \log \omega(n)} \right)^{\omega(n)}.
\]

Hence, for any \( n \) with \( \omega(n) \geq 2 \),

\[
r(n) = \frac{1}{\omega(n)} \left( \log \tau(n) - \omega(n) \log \left( \frac{\log n}{\omega(n) \log \omega(n)} \right) \right).
\]

Observe that for \( n_* := 60060 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \) we have \( r(n_*) = 0.737505 \ldots = \log \eta_2 \). We claim that \( n_* \) is the only integer \( n \) with \( \omega(n) \geq 2 \) that maximizes \( r \) (this function is bounded, as will become clear below). To prove it, assume for contradiction that, for some \( k \geq 2 \), there exists an integer \( n' \neq n_* \) with \( \omega(n') = k \) for which (3.1) is false and moreover that \( r(n') \) is maximal. It is clear that the factorization of \( n' \) takes the form

\[
n' = \prod_{i=1}^{k} p_i^{\alpha_i}
\]

with \( \alpha_1 \geq \cdots \geq \alpha_k \),

where the \( p_i \)'s are the primes in ascending order.

Using (4.11) (from Corollary 4.5) and (4.22) (from Lemma 4.8), one can easily see that \( r(n') < \log 2 = 0.693 \ldots \) if \( k \geq 44 \), which is nonsense since
Thus \( k \leq 43 \). Now, it follows from (4.11) that

\[
\tau(n') \leq \left( \frac{2 \log n'}{k} \right)^k \beta(n') \leq \left( \frac{2 \log n'}{k} \right)^k \beta(n_k).
\]

Inserting (5.3) into (5.1), we then get

\[
r(n') \leq \frac{1}{k} \left( \log \beta(n_k) + k \log \left( \frac{2 \log n'}{k} \right) - k \log \left( \frac{\log n'}{k \log k} \right) \right)
= \log 2 + \log \log k + \frac{\log \beta(n_k)}{k},
\]

a quantity which depends only on \( k \). On the other hand, a computer check reveals that \( r(n') < \log \eta_2 \) for each \( k \in \{2, 3 \} \cup \{25, 43 \} \). This contradicts the choice of \( n' \). Therefore we only need to consider the cases when \( k \in \{4, \ldots, 24 \} \).

Now, inserting (4.10) in (5.1), we have

\[
r(n') \leq \frac{1}{k} \left( \log \beta(n_k) + k \log \left( \frac{\log n'}{k} \right) + k \log \left( 1 + \frac{\log n_k}{\log n'} \right) - k \log \left( \frac{\log n'}{k \log k} \right) \right)
= \frac{\log \beta(n_k)}{k} + \log \log k + \log \left( 1 + \frac{\log n_k}{\log n'} \right) = r_1(n', k),
\]

where

\[
(5.4) r_1(z, k) := \frac{\log \beta(n_k)}{k} + \log \log k + \log \left( 1 + \frac{\log n_k}{\log z} \right).
\]

We observe that the function \( r_1(z, k) \) decreases when \( z \) increases. Thus, defining \( z_k \) as the unique solution in \( z \) of \( r_1(z, k) = \log \eta_2 \), we find that \( n' \leq z_k \) given that \( \omega(n') = k \).

We now consider the function

\[
(5.5) u(x) := \max_{\ell \geq 0} \{ \ell : n_\ell \leq x \}.
\]

Observe that, since \( n' \) is of the form (5.2), \( u(z_k/n_k) \) is an upper bound for the rank \( j \) of the largest prime \( p_j \) such that \( p_j^2 \mid n' \). One may verify that for each \( k \in \{4, \ldots, 24 \} \) we have \( u(z_k/n_k) \leq 3 \) implying that \( j \leq 3 \). Now, recalling the definition of \( t(n) \) given in (4.23), we may write

\[
r(n) = \log t(n) + \log(\omega(n) \log \omega(n)).
\]

Hence, for a fixed value of \( k = \omega(n) \), it follows that \( r(n) \) increases or decreases along with \( t(n) \). Therefore, our hypothesis implies that \( t(n') \) is maximal. Thus for each \( j \in \{1, 2, 3 \} \), using (4.25) and the maximality of \( t(n') \), we can write

\[
1 \leq \frac{t(n')}{t(n'/p_j)} \leq \left( 1 + \frac{1}{k \alpha} \right) \left( 1 - \frac{\log p_j}{\log n'} \right) \leq \left( 1 + \frac{1}{k \alpha} \right) \left( 1 - \frac{\log p_j}{\log z_k} \right),
\]
and we obtain the desired contradiction if this last expression is less than 1, which will happen if the integer $\alpha \geq 2$ satisfying $p_j^\alpha \parallel n'$ is large enough. Using (4.27) we get an upper bound for each of the first three components in the exponent vector of $n'$. In fact, one may verify that, for each $k \in \{4, \ldots, 24\}$,

$$(4, 2, 2, 1, \ldots, 1)$$

is an upper bound (coordinatewise) for the exponent vector of $n'$, implying that there are just a small number of cases to verify. After all the computations are done, we obtain a finite set of pairs $(n, r(n))$ including $(n_*, r(n_*))$ and find that all the other pairs in this set satisfy $r(n) < r(n_*)$. This contradicts the existence of $n'$ and completes the proof of Theorem 3.1.

6. Proof of Theorem 3.2. We first verify that (3.2) does not hold for the integer

$$n_* := 782139803452561073520 = 24n_{16}.$$ 

If $n$ is any integer such that $\omega(n) \geq 44$, then it follows from Corollary 4.5 and Lemma 4.8 that (3.2) is satisfied (see the proof of Theorem 3.1). Since it is clear that (3.2) holds when $\omega(n) = 1$, it remains only to consider the set of integers $n$ such that $2 \leq \omega(n) \leq 43$. For any such $k$, let $z_k$ be the unique solution in $z$ to

$$r_1(z, k) = \log 2,$$

where $r_1(z, k)$ is defined in (5.4).

To reach a contradiction, assume that there exists an integer $n'$ such that $\omega(n') \in \{17, \ldots, 43\}$ and for which (3.2) is false. We may also assume that $n'$ realizes the maximum of $r$ and that $n'$ is of the form (5.2). As in Theorem 3.1 we have $n' \leq z_k$ and one can verify that $u(z_k/n_k) \leq 5$. Thus, the same method that we used in the proof of Theorem 3.1 leads to an upper bound for the exponent vector of $n'$ given by

$$(5, 3, 2, 2, 1, \ldots, 1).$$

One can then verify, using a computer, that none of these finite numbers of possibilities leads to a number $n$ that does not satisfy (3.2), thus contradicting the existence of $n'$.

We can therefore assume that $2 \leq k \leq 16$. Since $z_2 = 3.25 \ldots$, $z_3 = 36.12 \ldots$ and $r(30) < \log 2$, we deduce that in the particular cases $k = 2$ and $k = 3$, there is no counterexample to inequality (3.2) in integers $n$ of the form (5.2). Thus, there is no counterexample in $n \geq 2$ with $\omega(n) \leq 3$. For $4 \leq k \leq 16$ there are counterexamples to (3.2) and thus we need to focus on getting a good upper bound for every such integer in terms of $k$. 
only. In order to do this, we first exhibit the values of \( u_k := u(z_k/n_k) \) (easily obtained using a computer) in Table 1.

<table>
<thead>
<tr>
<th>( k )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_k )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

We can use this information to obtain an upper bound for \( \tau(n) \) for any such counterexample \( n \) of (3.2). Indeed, by using the multiplicativity of the function \( \tau \) and inequality (4.10), we find that for any such \( n \) with \( \omega(n) = k \),

\[
\tau(n) \leq d_k := 2^{k-u_k} \beta(n_{u_k}) \left( \log \frac{z_k n_{u_k}^2}{n_k} \right)^{u_k}.
\]

A priori this inequality is valid only for integers \( n \) of the form (5.2), but it is then clearly also true for any counterexample to (3.2) since any general counterexample to (3.2) has an associated counterexample of the type (5.2) with the same exponent vector once the prime factors are properly ordered. We use this inequality in (5.1) and introduce the function

\[
r_2(z, k) := \frac{1}{k} \left( \log d_k - k \log \left( \frac{\log z}{k \log k} \right) \right).
\]

Now, let \( z_k' \) be the unique solution in \( z \) to

\[
r_2(z, k) = \log 2.
\]

Since \( \frac{dz}{dz} r_2(z, k) < 0 \), we deduce that \( z_k' \) is an upper bound for the largest possible counterexample \( n \) to (3.2) with a hypothetical value of \( \tau(n) \) equal to \( d_k \); clearly this is the largest among those we find with any smaller value of \( \tau(n) \). We then find, using a computer, that \( z_k' \) is smaller than \( 24 n_{16} \) for each \( k \in \{4, \ldots, 15\} \).

For \( k = 16 \), the situation is somewhat different. Instead, we verify by using \( z_{16}' \) that there are only three possible exponent vectors, namely

\[
(3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),
\]

(3,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1),

(4,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1),

(6.1)

that yield a counterexample to (3.2) in integers \( n \) of the type (5.2). For each of these, the smallest number strictly larger than the basic form is obtained by replacing the largest prime factor \( p_{16} = 53 \) by 59. We then obtain numbers \( n \) which give \( r(n) < \log 2 \). We deduce that \( 24 n_{16} \) (which corresponds to the last exponent vector in (6.1)) is the largest of these. The proof of Theorem 3.2 is complete.
7. Proof of Theorem 3.3. We first verify that for \( n^\ast := 720n_7 \) we have \( \lambda(n^\ast) = 1.1999953 \ldots =: \eta_3 \). We will show that \( n^\ast \) is the only integer that maximizes \( \lambda \). To reach a contradiction, assume that there exists \( n' \neq n^\ast \) for which \( \lambda(n') \geq \lambda(n^\ast) \). Again, it is clear that the maximal value of \( \lambda \) exists and is attained by an integer of the form (5.2). Therefore we assume that \( n' \) is of this form with \( \omega(n') = k \). From (4.12) and (4.22), it follows that

\[
\lambda(n') \leq 1 + \sum_{i=1}^{k} \frac{\log p_i - k \log k}{\log n'}
\]

for each \( k \geq 44 \). On the other hand, we cannot have

\[
(7.1) \quad \sum_{i=1}^{k} \frac{\log p_i - k \log k}{\log n'} > \eta_3 - 1
\]

if \( k \geq 44 \), the reason being that since \( n' \) has \( k \) prime factors, it must satisfy \( \log n' \geq \log n_k = \sum_{i=1}^{k} \log p_i \), in which case (7.1) would imply

\[
(7.2) \quad (2 - \eta_3) \sum_{i=1}^{k} \log p_i > k \log k;
\]

but, using (4.20), it is easy to verify that (7.2) is impossible when \( k \geq 44 \). This proves that we must have \( k \leq 43 \). Considering (4.12), we let \( z_k \) be the unique solution in \( z \) of

\[
v(n_k, z) = \left( \prod_{p|n_k} \frac{\log k}{\log p} \right)^{1/k} + \frac{\log n_k}{\log z} \left( \prod_{p|n_k} \frac{\log k}{\log p} \right)^{1/k} - \frac{k \log k}{\log z} = \eta_3,
\]

where \( v(n, z) \) is the function defined in (4.28). Since \( \frac{d}{dz}v(n_k, z) < 0 \) by (4.29), we deduce that \( n' \leq z_k \). We find that the only possibilities for \( n' \) are those with \( k \in \{5, \ldots, 13\} \), since otherwise we would have \( n' \leq z_k < n_k \), which is impossible since by hypothesis we have \( n_k | n' \).

Now, for \( 5 \leq k \leq 13 \), from the fact that \( n' \) is of the form (5.2) with \( \omega(n') = k \), we deduce that \( n' = sn_k \leq z_k \) for some integer \( s \) which satisfies \( n_j | s \) with \( j \leq k \). One can calculate that the largest ratio \( z_k/n_k \) (for \( 5 \leq k \leq 13 \)) is less than 264,507. This forces \( j \leq 6 \). Now, consider the set

\[
U := \{ s \leq 264,507 : P(s) \leq 13 \},
\]

where \( P(s) \) stands for the largest prime factor of \( s \), and the set \( V := \{ (sn_k, \lambda(sn_k)) : s \in U \text{ and } 5 \leq k \leq 13 \} \). By computation, we observe that \( V \) contains the element \( (n^\ast, r(n^\ast)) \) and that for any other \( n \) we have \( r(n) < r(n^\ast) \). This contradicts the existence of \( n' \) and the proof of Theorem 3.3 is complete.

8. Proof of Theorem 3.4. To reach a contradiction, assume that there exists an integer \( n' \) with \( \omega(n') = k \), for some \( k \geq 74 \), for which (3.4) does
not hold. For fixed values of $\omega(n)$ and $\tau(n)$, we see by definition (2.3) that the function $\lambda(n)$ decreases as $n$ increases. For this reason, we will assume that $n'$ is of the form (5.2). We will also assume that $\lambda(n')$ is maximal.

For $k \geq 95$, we deduce from (4.12) and (4.28)–(4.30) that

$$
\lambda(n') \leq v(n', n') \leq v(n_k, n_k) < 1.
$$

This means that (3.4) holds for $k \geq 95$.

For $k \in \{74, \ldots, 94\}$, we cannot conclude the proof since $v(n_k, n_k) > 1$. However, since by Lemma 4.10 we have $\frac{d}{dz}v(n_k, z) \leq 0$ and $v(n_k, \infty) < 1$, we can define $z_k$ implicitly by $v(n_k, z_k) = 1$, in which case $n' \leq z_k$. Also, by computation,

$$
\log z_k / \log n_k < 2
$$

for each $k$. This last inequality implies that the largest prime factor of $n'$ has exponent 1.

As we have already seen, $u_k := u(z_k/n_k)$ provides an upper bound for the rank $j$ of the largest prime $p_j$ such that $p_j^2 | n'$ since $n'$ is of the form (5.2). Our goal from now on is to check all the remaining possibilities. To do so, we proceed in four steps. In the first step, we introduce a variable $j_1$ that will take the values $0, 1, \ldots, u_k$ and a variable $j_2$ that will take the values $0, 1, \ldots, \min(j_1, u_k)$. Then, we assume that

$$
n' = p_1^{\alpha_1} \cdots p_{j_2}^{\alpha_{j_2}} p_{j_2+1}^2 \cdots p_{j_1}^2 p_{j_1+1} \cdots p_k
$$

for some integers $\alpha_i \geq 3$ and that $n'$ is of the form (5.2). Now, if $0 < j_2 \leq j_1$, by using the multiplicativity of $\tau$ together with (4.12), recalling the definition of $\lambda$ in (2.3), we are led to consider the function

$$
f_1(j_2, j_1, k, z) := (c_1(j_2, j_1, k)(\log z - c_2(j_2, j_1, k))^{j_2/k} - 1)k \log k
$$

where

$$
c_1 = c_1(j_2, j_1, k) := 2^{(k-j_1)/k} 3^{(j_1-j_2)/k} \frac{1}{j_2^{j_2/k}} \beta(n_{j_2})^{1/k},
$$

$$
c_2 = c_2(j_2, j_1, k) := \log(n_k n_j) / (n_{j_2})^3).
$$

Assume for now that each constant $c_2$ that will be considered through this proof satisfies

$$
c_2 > 0.
$$

Then, using Lemma 4.11, we have

$$
\lambda(n') \leq f_1(j_2, j_1, k, n') \leq \max_{z > c_2(j_2, j_1, k)} f_1(j_2, j_1, k, z).
$$

Therefore, we will get the desired contradiction if $n'$ is of the type (8.1) and the unique maximum of $f_1(j_2, j_1, k, z)$ is proven to be less than 1 (see
Remark 8.1 for more details). The cases with \( j_2 = 0 \) or \( j_1 = 0 \) must be verified directly.

The values of \( u_k \) are recorded in Table 2.

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<tr>
<th>( k )</th>
<th>( 74 )</th>
<th>( 75 )</th>
<th>( 76 )</th>
<th>( 77 )</th>
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<th>( 79 )</th>
<th>( 80 )</th>
<th>( 81 )</th>
<th>( 82 )</th>
<th>( 83 )</th>
<th>( 84 )</th>
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<tr>
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<td>29</td>
<td>26</td>
<td>25</td>
</tr>
<tr>
<td>( k )</td>
<td>( 85 )</td>
<td>( 86 )</td>
<td>( 87 )</td>
<td>( 88 )</td>
<td>( 89 )</td>
<td>( 90 )</td>
<td>( 91 )</td>
<td>( 92 )</td>
<td>( 93 )</td>
<td>( 94 )</td>
<td></td>
</tr>
<tr>
<td>( u_k )</td>
<td>23</td>
<td>21</td>
<td>19</td>
<td>17</td>
<td>15</td>
<td>13</td>
<td>11</td>
<td>8</td>
<td>6</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

All the computations being done, one is left with a reduced set of possibilities for the form of \( n' \). In fact, we now have \( k \in \{74, 75, 76, 77\} \) and also the number of values that \( j_1 \) can take is significantly reduced. The final result is given in Table 3.

| \( k \) | \( 74 \) | \( 75 \) | \( 76 \) | \( 77 \) |
|---|---|---|---|
| \( j_1 \in \{14, \ldots, 28\} \) | \( 16, \ldots, 26 \) | \( 18, \ldots, 25 \) | \( 20, \ldots, 23 \) |

What we mean here is that a fixed pair \((k, j_1)\) is not in Table 3 if for all \( j_2 \leq \min(j_1, u_k) \) we have \( \max_{z > c_2(j_2, j_1, k)} f_1(j_2, j_1, k, z) < 1 \).

It is here that the second step of verifications starts. We now assume that

\[
n' = p_1^{a_1} \cdots p_{j_3}^{a_{j_3}} \cdot p_{j_3+1}^{3} \cdots p_{j_2+1}^{2} \cdots p_{j_1+1} \cdots p_k
\]

for some integers \( a_i \geq 4 \) and we use the same argument as before to define

\[
f_2(j_3, j_2, j_1, k, z) := \frac{(c_1(j_3, j_2, j_1, k)(\log z - c_2(j_3, j_2, j_1, k))^{j_3/k} - 1)k \log k}{\log z},
\]

where

\[
c_1(j_3, j_2, j_1, k) := 2^{(k-j_1)/k} 3^{(j_1-j_2)/k} 4^{(j_2-j_3)/k} \frac{1}{j_3^{j_3/k}} \beta(n_{j_3})^{1/k},
\]

\[
c_2(j_3, j_2, j_1, k) := \log(n_k n_{j_1} n_{j_2}/(n_{j_3})^4).
\]

We still have the inequalities

\[
\lambda(n') \leq f_2(j_3, j_2, j_1, k, n') \leq \max_{z > c_2(j_3, j_2, j_1, k)} f_2(j_3, j_2, j_1, k, z).
\]

This time, we run this over the remaining values of \( j_1 \), and for

\[
\begin{align*}
  j_2 & \in \{1, \ldots, \min(j_1, u(z_k/(n_{j_1} n_k)))\}, \\
  j_3 & \in \{1, \ldots, \min(j_2, u(z_k/(n_{j_2} n_{j_1} n_k)))\}.
\end{align*}
\]
The cases with $j_3 = 0$ must be treated separately. Once again, these computations lead to further progress. We record in Table 4 the remaining values which need to be examined.

<table>
<thead>
<tr>
<th>$k$</th>
<th>74</th>
<th>75</th>
<th>76</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j_1$ $\in{14, \ldots, 23}$</td>
<td>${16, \ldots, 21}$</td>
<td>${18, 19}$</td>
<td></td>
</tr>
</tbody>
</table>

We are now ready to begin the third step of verifications. We assume that

$$n' = p_1^{\alpha_1} \cdots p_j^{\alpha_j} \cdot p_{j+1}^4 \cdots p_{j+1}^4 \cdot p_{j+1}^3 \cdots p_{j+1}^3 \cdot p_{j+1}^2 \cdots p_{j+1}^2 \cdot p_{j+1} \cdots p_k$$

for some integers $\alpha_i \geq 5$ and define $f_3(j_4, j_3, j_2, j_1, k, z)$ in a similar manner by using the same ideas. However, we do introduce a new idea in the way of reducing the number of values that the variables $j_s$ $(s \geq 3)$ can take. We first assume that $p^\alpha \mid n'$ for a fixed $\alpha \geq 2$, and then we use (4.24), the fact that $n' \leq z_k$ and the maximality of $\lambda(n')$ to write

$$1 \leq \frac{\lambda(n')}{\lambda(n'/p)} \leq \left(1 + \frac{2}{k\alpha}\right)\left(1 - \frac{\log p}{\log n'}\right) \leq \left(1 + \frac{2}{k\alpha}\right)\left(1 - \frac{\log p}{\log z_k}\right).$$

We find a contradiction if $p$ is large enough to force the last expression to be less than 1. In particular, we get an upper bound for the rank $j$ of such a prime $p_j$. Since this upper bound decreases when $\alpha$ increases, we obtain an upper bound for the rank $j$ of any prime $p_j$ for which $p_j^\alpha \mid n'$. Thus, by using (4.26), we obtain Table 5.

<table>
<thead>
<tr>
<th>$\alpha \backslash k$</th>
<th>74</th>
<th>75</th>
<th>76</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>11</td>
<td>11</td>
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<td>6</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Using this, we let $j_1$ take the values in Table 4, and we let

$$j_2 \in \{1, \ldots, \min(j_1, u(z_k/(n_j n_k)))\},$$

$$j_3 \in \{1, \ldots, \min(j_2, u(z_k/(n_{j_2} n_{j_1} n_k)), 11 \text{ or } 10)\},$$

$$j_4 \in \{1, \ldots, \min(j_3, u(z_k/(n_{j_3} n_{j_2} n_{j_1} n_k)), 7 \text{ or } 6)\}.$$ Again, we treat the cases with $j_4 = 0$ independently. Computations show that we must have $k = 74$ and $j_1 \in \{16, 17, 18\}$. We rule out these cases by defining $f_4(j_5, j_4, j_3, j_2, j_1, k, z)$ and by using Table 5 to limit the range of the variables $j_3$, $j_4$ and $j_5$. This completes the verifications.
It remains to prove (8.2). To do so, we use the fact that \( j_{s+1} \leq j_s \) and
\[
n_{j_s} \leq \frac{z_k}{n_k n_{j_1} \cdots n_{j_{s-1}}} \quad \text{and} \quad -c_2(j_s, \ldots, j_1, k) = \log n_{j_s} + \log \frac{n_{j_s}^s}{n_k n_{j_1} \cdots n_{j_{s-1}}} ,
\]
from which we deduce that
\[
-c_2(j_s, \ldots, j_1, k) \leq \log \frac{z_k}{n_k n_{j_1} \cdots n_{j_{s-1}}} + \log \frac{n_{j_s}^s}{n_k n_{j_1} \cdots n_{j_{s-1}}}
\leq \log \frac{z_k}{n_k} + \log \frac{n_{j_s}}{n_k} \leq \log \frac{z_k}{n_k^s} < -159.6
\]
by direct computation, which proves (8.2).

We also observe that
\[
-c_2(j_s, \ldots, j_1, k) = \log \frac{n_{j_s}^{s+1}}{n_k n_{j_1} \cdots n_{j_{s-1}}} \geq \log \frac{1}{n_k n_{j_1} \cdots n_{j_{s-1}}}
\geq \log \frac{1}{n_k} > -5 \log n_{94} > -2342.
\]
This completes the proof of Theorem 3.4.

**Remark 8.1.** We now provide some key details concerning the computations used in the proof of Theorem 3.4. The information provided by the previous proof may differ from other information obtained with another strategy. We used 50 decimal places for all computations. We used the criterion \( f_s(\ldots) < 0.999999 \) (for \( s = 1, 2, 3 \) or 4) for each comparison in the four steps of the computation and we kept a pair \((k, j_1)\) if we found \( f_s(\ldots) \geq 0.999999 \) somewhere in the process. By considering the function \( h \) defined in (4.32), we approximated \( c_1 \) and \( c_2 \) with 50 decimals and we called these approximations \( c'_1 \) and \( c'_2 \). Then we solved for \( z_1 \) in \( h(z_1) = 0 \). From
\[
0 = c'_1 \alpha z_1 - c'_1 (z_1 - c'_2) + (z_1 - c'_2)^{1-\alpha} > c'_1 \alpha z_1 - c'_1 (z_1 - c'_2) ,
\]
\( \alpha \geq 1/94 \) and \( c'_2 > 159 \), we deduced that \( z_1 - c'_2 > 1.61 \). The same is true also for the solution \( z \) of
\[
0 = c_1 \alpha z - c_1 (z - c_2) + (z - c_2)^{1-\alpha}.
\]
It is easy to see that we always have \( c_1 < 6! / \log 2 < 1039 \). With this information at hand and using the mean value theorem, one finds that
\[
\left| \frac{c'_1(z_1 - c'_2)^{\alpha} - 1}{z_1} - \frac{c_1(z_1 - c_2)^{\alpha} - 1}{z_1} \right| < 10^{-43} ,
\]
thus concluding that the two functions are of about the same size for all values of \( z \) or \( z_1 \) such that \( z_1 - c'_2 > 1.6 \) or \( z - c_2 > 1.6 \). From the fact that \( 94 \log 94 < 428 \) and that an error in \( z_1 \) of about \( 10^{-50} \) costs less than \( 10^{-45} \) in the evaluation of \( \frac{c'_1(z_1 - c'_2)^{\alpha} - 1}{z_1} \), we end up with an error of at most \( 10^{-40} \). This is small enough for the criterion we used.
9. Proof of Theorem 3.6. First, we verify that the integer $n_*$ defined in the statement of the theorem satisfies $\lambda(n_*) > 1$ and is of size $\exp(10640.8428 \ldots)$. Then, we claim that $n_*$ is the largest integer $n$ with $\omega(n) \geq 44$ and $\lambda(n) \geq 1$. For a contradiction, assume that there exists an integer $n'$ such that $n' > n_*$ with $\lambda(n') \geq 1$ and $\omega(n') \geq 44$. The argument is in several steps.

9.1. Preliminary steps. The first step consists in showing that $\omega(n') = 44$. For this, we use (4.12), (4.28) and (4.29) to deduce that if we define $z_k$ by $\nu(n_k, z_k) = 1$ then $n' \leq z_k$. We verify that $z_k \leq \exp(4569.68) < n_*$ for each $k \in \{45, \ldots, 73\}$ and then we conclude using Theorem 3.4.

We then want to show that $\gamma(n') = n_{44}$. This is done in two steps. We first assume that $n'$ has its prime factors in a set $S$ of 44 distinct primes in $\{p_1, \ldots, p_{45}\}$ and that $S$ is not $S' := \{p_1, \ldots, p_{44}\}$. There are 44 possibilities and if we write $n_S := \prod_{p \in S} p$, then using again the same argument, we define $z_S$ by $\nu(n_S, z_S) = 1$ and verify that $z_S \leq \exp(9927.67) < n_*$ for each $S$.

Now, we assume that $n'$ has a general set of prime factors which has not been previously considered (and is not $S'$), implying that there exists an integer $n'' < n'$ such that $\tau(n'') = \tau(n')$ and the set of prime divisors of $n''$ is $S$ for some $S \neq S'$. We then have

$$\lambda(n') < \lambda(n'') < 1$$

if $n'' > z_S$. This proves that the set of prime factors of $n'$ must be $S'$.

We solve the equation $\nu(n_{44}, z) = 1$ for $z$ to find that $n' < \exp(10758.21)$. We have thus proved that

$$10640.8 < \log n' < 10758.8.$$ 

Consider the intervals $I_j := [10639.8 + j, 10640.8 + j]$ for each $j = 1, \ldots, 118$. From now on, we want to show that $\log n'$ cannot be in any of these $I_j$.

9.2. A first argument. Recall the notation in (4.1) and (4.2), that is, $x_i (i = 1, \ldots, 44)$, $\mu$, $\mu_1$, $\mu_2$, $\varpi$ and $\varpi'$. The first argument that we use to eliminate some intervals $I_j$ relies on the inequality (4.6) and on the proof of Corollary 4.5. For $m \in \{1, \ldots, 43\}$, we have

$$\tau(n') \leq \log^{44} n' \beta(n_{44}) \mu_1^m \mu_2^{44-m}$$

$$= \beta(n_{44}) \left( \frac{\log n'}{44} \right)^{44} \left( 1 + \frac{\log n_{44}}{\log n'} - \frac{\varpi'}{2m} \right)^m$$

$$\times \left( 1 + \frac{\log n_{44}}{\log n'} + \frac{\varpi'}{2(44 - m)} \right)^{44-m}$$
so that if we write

\[
\nu_m(z, w) := \beta(n_{44})^{1/44} \log 44 \left( \frac{1 + \log n_{44}}{z} - \frac{w}{2m} \right)^{m/44} \\
\times \left( 1 + \frac{\log n_{44}}{z} + \frac{w}{2(44 - m)} \right)^{1-m/44} - \frac{44 \log 44}{z}
\]

then

\[
\lambda(n') \leq \max_{m \in \{1, \ldots, 43\}} \nu_m(\log n', \varpi').
\]

Thus, we define \( z_{m, \varpi} \) by \( \nu_m(z_{m, \varpi}, \varpi) = 1 \). We have seen that \( z_{m, 0} = 10758.2 \ldots \) From Lemma \ref{lem:4.6}(i, iii), we know that \( z_{m, w} \) decreases when \( w \) increases. We record in Table 6 the values of \( w := w(j) \) such that

\[
\max_{m \in \{1, \ldots, 43\}} z_{m, w(j)} < 10639.8 + j
\]

for the first 39 values of \( j \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>0.2137</td>
<td>0.2128</td>
<td>0.2119</td>
<td>0.2109</td>
<td>0.2100</td>
<td>0.2091</td>
<td>0.2081</td>
<td>0.2072</td>
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<table>
<thead>
<tr>
<th>( j )</th>
<th>9</th>
<th>10</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>0.2062</td>
<td>0.2053</td>
<td>0.2043</td>
<td>0.2034</td>
<td>0.2024</td>
<td>0.2014</td>
<td>0.2004</td>
<td>0.1995</td>
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</table>

<table>
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<tr>
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<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>0.1985</td>
<td>0.1975</td>
<td>0.1965</td>
<td>0.1955</td>
<td>0.1945</td>
<td>0.1935</td>
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<table>
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<tr>
<th>( j )</th>
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<th>29</th>
<th>30</th>
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<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>0.1904</td>
<td>0.1894</td>
<td>0.1884</td>
<td>0.1873</td>
<td>0.1863</td>
<td>0.1852</td>
<td>0.1841</td>
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<tr>
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<th>36</th>
<th>37</th>
<th>38</th>
<th>39</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>0.1820</td>
<td>0.1809</td>
<td>0.1799</td>
<td>0.1788</td>
<td>0.1777</td>
<td>0.1766</td>
<td>0.1755</td>
</tr>
</tbody>
</table>

In the opposite direction, a lower bound for \( \varpi' = \varpi'(n') \) can be computed for \( n' \) assuming that \( \log n' \) is in \( I_j \). To do so, we split the interval \( I_j \) into 210 subintervals of length \( \frac{1}{210} \) that we call \( I_{j, j_1} \) where \( 1 \leq j_1 \leq 210 \). We use Lemma \ref{lem:4.7}(i) with \( \varphi = 1 \) termwise to compute

\[
\min_{z \in I_{j, j_1}} \sum_{i=1}^{44} \min_{\alpha_i \in \mathbb{Z}} \left\{ \frac{(\alpha_i + 1)44 \log p_i - \log n_{44}}{z} - 1 \right\}
\]

and take the minimum over \( j_1 \) to get the lower bound for \( \varpi'(n') \) for \( \log n' \) in \( I_j \). We record the result in Table 7.

Also, we verify that for each \( j \in \{40, \ldots, 118\} \) we have \( \varpi'(j) > w(j) \), thereby implying that there exist no \( n' \) with \( \log n' \) in \( I_j \).
All this gives rise to a new concept that will be crucial for the remainder of the proof. This is the difference between the upper and lower bounds for \( \omega' \). Here the difference when \( j = 1 \), that is, \( 0.2137 - 0.1814 = 0.0323 \), is too large for us. In fact, it will be convenient to work with a slightly different concept. Consider the function defined on primes \( p \) by

\[
\epsilon_j(p) := \min_{z \in I_j} \min_{\alpha \in \mathbb{Z}} \left( \frac{(\alpha + 1)44 \log p - \log n_{44}}{z} - 1 \right).
\]

The value of \( \epsilon_j(p) \) is computed by using Lemma 4.7(i). For each \( j \in \{1, \ldots, 39\} \) we sum the \( \epsilon_j(p_i) \) for \( i \in \{1, \ldots, 44\} \) and subtract the result from the upper bound \( w(j) \). We call these values \( \delta' (= \delta'(j)) \) and record them in Table 8.

### Table 7

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega' )</td>
<td>0.1814</td>
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<td>0.1810</td>
<td>0.1808</td>
<td>0.1804</td>
<td>0.1802</td>
<td>0.1800</td>
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<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega' )</td>
<td>0.1797</td>
<td>0.1797</td>
<td>0.1798</td>
<td>0.1800</td>
<td>0.1800</td>
<td>0.1800</td>
<td>0.1798</td>
<td>0.1797</td>
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<table>
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<tr>
<th>( j )</th>
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<th>19</th>
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<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega' )</td>
<td>0.1797</td>
<td>0.1798</td>
<td>0.1800</td>
<td>0.1800</td>
<td>0.1800</td>
<td>0.1798</td>
<td>0.1796</td>
<td>0.1792</td>
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<tr>
<th>( j )</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
<th>29</th>
<th>30</th>
<th>31</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega' )</td>
<td>0.1788</td>
<td>0.1784</td>
<td>0.1781</td>
<td>0.1779</td>
<td>0.1777</td>
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<th>36</th>
<th>37</th>
<th>38</th>
<th>39</th>
</tr>
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<tbody>
<tr>
<td>( \omega' )</td>
<td>0.1769</td>
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<td>0.1763</td>
<td>0.1761</td>
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### Table 8

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<tbody>
<tr>
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<td>0.03422</td>
<td>0.03353</td>
<td>0.03283</td>
<td>0.03203</td>
<td>0.03142</td>
<td>0.03083</td>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
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<tbody>
<tr>
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<td>0.02936</td>
<td>0.02848</td>
<td>0.02753</td>
<td>0.02638</td>
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<th>18</th>
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<th>21</th>
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<tbody>
<tr>
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<td>0.02075</td>
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<table>
<thead>
<tr>
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<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \delta' )</td>
<td>0.01561</td>
<td>0.01481</td>
<td>0.01398</td>
<td>0.01340</td>
<td>0.01279</td>
<td>0.01216</td>
<td>0.01133</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>( j )</th>
<th>29</th>
<th>30</th>
<th>31</th>
<th>32</th>
<th>33</th>
<th>34</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta' )</td>
<td>0.01053</td>
<td>0.00964</td>
<td>0.00874</td>
<td>0.00795</td>
<td>0.00705</td>
<td>0.00618</td>
<td>0.00547</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>( j )</th>
<th>36</th>
<th>37</th>
<th>38</th>
<th>39</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta' )</td>
<td>0.00458</td>
<td>0.00392</td>
<td>0.00331</td>
<td>0.00258</td>
</tr>
</tbody>
</table>
The value $\delta'$ is to be interpreted as an upper bound on the extra error that can be produced by $n'$.

9.3. A first verification. We want to make some direct verifications to prove that $\lambda(n') \geq 1$ is impossible if the exponent vector of $n'$ is of a certain type. Consider the sets

$$J_{\delta}(p, j) := \left\{ \left\lfloor \frac{(1 - \delta)(10639.8 + j) + \log n_{44}}{44 \log p} \right\rfloor - 1, \ldots, \left\lceil \frac{(1 + \delta)(10640.8 + j) + \log n_{44}}{44 \log p} \right\rceil - 1 \right\}.$$ (9.3)

The set $J_{\delta}(p, j)$ has the property that if

$$\left| \frac{(\alpha + 1)44 \log p - \log n_{44}}{\log n} - 1 \right| \leq \delta$$

then $\alpha \in J_{\delta}(p, j)$.

We divide the verifications into two distinct types. Type 1 concerns the sets

$$S_j(\delta) := J_{\delta}(p_1, j) \times \cdots \times J_{\delta}(p_{44}, j).$$

We take $\delta = 0.011$ for $j \in \{1, \ldots, 4\}$ and $\delta = 0.01$ for $j \in \{5, \ldots, 14\}$. Also, to speed up the process, we consider the termwise union of $S_1(0.011)$, \ldots, $S_4(0.011)$ to get a new set, say $S_1$, so that $S_1 = J_{0.011}(2,1) \cup \cdots \cup J_{0.011}(2,4) \times \cdots \times \cdots$. We do the same with $S_5(0.01), \ldots, S_{14}(0.01)$ to get $S_2$. These sets have respectively 92160 and 53760 elements. For each vector $v = (\alpha_1, \ldots, \alpha_{44})$ in each of these two sets, we take one of the 946 possible choices of two elements in a set of 44 elements, say $(i_1, i_2)$, and construct the new set

$$\{\alpha_1\} \times \cdots \times \{\alpha_{i_1-1}\} \times J_{\epsilon_j(p_{i_1})+\delta'(j)}(p_{i_1}, j) \times \{\alpha_{i_1+1}\} \times \cdots \times \{\alpha_{i_2-1}\}$$

$$\times J_{\epsilon_j(p_{i_2})+\delta'(j)}(p_{i_2}, j) \times \{\alpha_{i_2+1}\} \times \cdots \times \{\alpha_{44}\}.$$ We verify that all these exponent vectors $v$ give rise to an integer $n$ such that $\lambda(n) < 1$, $\log n < 10640.8$ or $n = n_*$.

Type 2 concerns the sets

$$S'_j(\delta) := J_{\epsilon_j(p_1)+\delta}(p_1, j) \times \cdots \times J_{\epsilon_j(p_{44})+\delta}(p_{44}, j).$$

This time, we take $\delta = 0.0055$ for $j \in \{1, \ldots, 6\}$, $\delta = 0.0054$ for $j \in \{7, \ldots, 9\}$ and $j \in \{10, \ldots, 13\}$, $\delta = 0.005$ and $j \in \{14, \ldots, 19\}$, $\delta = 0.0044$ for $j \in \{20, \ldots, 23\}$, $\delta = 0.004$ for $j \in \{24, 25, 26\}$, $\delta = 0.0035$ for $j \in \{27, 28, 29\}$ and $\delta = 0.003$ for $j \in \{30, \ldots, 39\}$. Again, to speed up the process, we consider the termwise unions the same way, so that we have $S'_1 = J_{\epsilon_1(2)+0.0055}(2,1) \cup \cdots \cup J_{\epsilon_6(2)+0.0055}(2,6) \times \cdots$, and the same for $S'_2, \ldots, S'_8$. These sets have respectively 98304, 73728, 49152, 49152, 32768,
32768, 32768 and 24576 elements. For each vector $v$, we do the same as for type 1.

At the end of these verifications, we know that there are at least three entries in the exponent vector that produce a large error, and this occurs in both type 1 and 2.

9.4. Reducing the upper bound for $\delta'$. Our strategy begins with a lower bound for $\varpi'_1$ and $\varpi'_2$. For each $j$, there are four cases to consider depending on the position of the $x_i$ (4.1) compared to $\mu$ (4.2). Indeed, we have seen in the previous section, with the type 1 verification, that there are at least three $x_i$ that are far from $\mu$ but this does not tell us where they are.

For this reason, we will classify the cases with a signature $s$ in $\{0, 1, 2, 3\}$ that tells us how many of these $x_i$ are less than $\mu$. Also, any lower bound for $\varpi'_1$ and $\varpi'_2$ will in fact be a lower bound for the pair $(\varpi'_1, \varpi'_2)$ as a function of both $s$ and the total number $m$ of $x_i$ that are less than or equal to $\mu$. This number $m$ can be shown to take the values we recorded in Table 9.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, \ldots, 6}$</td>
<td>${11, \ldots, 33}$</td>
</tr>
<tr>
<td>${7}$</td>
<td>${12, \ldots, 33}$</td>
</tr>
<tr>
<td>${8, \ldots, 12}$</td>
<td>${12, \ldots, 32}$</td>
</tr>
<tr>
<td>${13}$</td>
<td>${13, \ldots, 32}$</td>
</tr>
<tr>
<td>${14}$</td>
<td>${13, \ldots, 31}$</td>
</tr>
</tbody>
</table>

To do so, we use Table 7 and verify that $z_{m, \varpi'(j)} < 10639.8 + j$ (see Section 9.2) for all the values of $m$ not listed in Table 9.

Also, the definitions of $\varpi'_1$ and $\varpi'_2$ in (4.3) and (4.4) include the exact value of $\varpi'$, something that we cannot know precisely. So we assume an interval containing the value of $\varpi'$, and look for a contradiction. More precisely, we will assume, for $j \in \{1, \ldots, 14\}$, that $\varpi'$ belongs to $W(j) := [w(j) - 0.01, w(j)]$.

To get these lower bounds, we fix $j$, $m$ and a signature $s$. Then, we split the interval $I_j$ into 30 subintervals $I_{j,r_1}$ ($r_1 = 1, \ldots, 30$) of equal length, and the interval $W(j)$ into 80 subintervals $W(j, r_2)$ ($r_2 = 1, \ldots, 80$) of equal length.

We fix $I_{j,r_1}$ and $W(j, r_2)$ and begin with $\varpi'_1$. First, we focus on the $s$ points $x_i$ that are less than $\mu$. We will show that in this case the minimum of

$$
\left| \frac{(\alpha + 1)44 \log p - \log n_{44}}{x} - 1 + \frac{\varpi'}{2m} \right|
$$

is attained with $\alpha := [\frac{(1-\delta)(10639.8+j)+\log n_{44}}{44 \log p}] - 2$ and, as in Lemma 4.7(ii), at the extremity of the intervals $I_{j,r_1}$ and $W(j, r_2)$. In fact, from the definition of $J_8$, it is enough to show that $\frac{(\alpha+1)44 \log p - \log n_{44}}{x} - 1 + \frac{\varpi'}{2m} < 0$ and this follows from.
\[
\left( \frac{\alpha + 1}{44} \right) \log p - \log n_{44} \frac{x}{2m} - 1 + \frac{\varpi'}{2m}
\]

\[
= \frac{\left( \frac{1}{44} \log p \right)^{\left(1 - \delta(10639.8 + j) + \log n_{44}\right)} - 1}{44 \log p} \frac{\left( \frac{1}{44} \log p \right)^{\left(1 - \delta(10639.8 + j) + \log n_{44}\right)} + \xi - 1}{44 \log p} \frac{x}{2m} - 1 + \frac{\varpi'}{2m}
\]

\[
= \frac{1}{44 \log p} \frac{(1 - \delta)(10639.8 + j) - (1 - \xi)44 \log p}{2m} - 1 + \frac{\varpi'}{2m} \leq -\delta + \frac{\varpi'}{2m} < 0
\]

since \(2m\delta \geq 0.22 > \varpi'\) from our choices, where \(0 \leq \xi < 1\) and both \(\varpi'\) and \(x\) are seen as fixed. We keep the \(s\) smallest such values among the 44 primes.

Then, we compute the minimum value of (9.4), without any constraint on \(\alpha\), using Lemma 4.7(i). We keep the \(m - s\) smallest ones among the 44 prime numbers. We sum the \(m\) values we have kept so far and we take the minimum of the \(30 \cdot 80 = 2400\) possible values of \((r_1, r_2)\)—this is the required lower bound for \(\varpi'_1\) in \(I(j)\) with these values of \(m\) and \(s\). We do the same for \(\varpi'_2\) with \(3 - s\) values of \(x_i\) greater than \(\mu\) along with the choice \(\alpha := \left( \frac{1 + \delta(10640.8 + j) + \log n_{44}}{44 \log p} \right)\) instead and the function

\[
(9.5) \quad \left| \frac{(\alpha + 1)}{44 \log p} \log p - \log n_{44} \frac{x}{2m} - 1 - \frac{\varpi'}{2(44 - m)} \right|
\]

as in the definition of \(\varpi'_2\) in (4.4). The proof is similar. We obtain the value for \(\varpi'_2\) in \(I_j\) with parameters \(3 - s\) and \(44 - m\). We keep the pair \((\varpi'_1, \varpi'_2)\)

\((= (\varpi'_1(j, m, s), \varpi'_2(j, m, s)))\). For example, here is the output we get as lower bound with \(j = 1\) and \(m = 11\):

\((0.01029642154, 0.093154520284), (0.010296421544, 0.089438737225), (0.011179764497, 0.087104430865), (0.012637967643, 0.085223479629)\)

for \((\varpi'_1, \varpi'_2)\) when \(s = 0, 1, 2, 3\) respectively. Note that these values are just stated as an example, they are sensitive to the way the program is executed.

Now, using the same reasoning as (9.1), we are led to consider

\[
(9.6) \quad v_{m, m_1, m_2}(z, w, w_1, w_2)
\]

\[
:= \beta(n_{44})^{\frac{1}{44}} \log 44 \left( 1 + \frac{\log n_{44}}{z} - \frac{w}{2m} - \frac{w_1}{2m_1} \right)^{\frac{m_1}{44}} \left( 1 + \frac{\log n_{44}}{z} - \frac{w}{2m} + \frac{w_1}{2m_1} \right)^{\frac{m_1}{44}} \left( 1 + \frac{\log n_{44}}{z} - \frac{w}{2(44 - m)} - \frac{w_2}{2m_2} \right)^{\frac{m_2}{44}} \left( 1 + \frac{\log n_{44}}{z} + \frac{w}{2(44 - m)} - \frac{w_2}{2m_2} \right)^{1 - \frac{m + m_2}{44}} - \frac{44 \log 44}{z}
\]
for fixed values of $m_1 \in \{1, \ldots, m - 1\}$ and $m_2 \in \{1, \ldots, 43 - m\}$. We also define $z_{j,m,m_1,m_2,s}$ implicitly by

$$v_{j,m,m_1,m_2}(z_{j,m,m_1,m_2,s}, w(j) - 0.01, \varpi_1(j, m, s), \varpi_2(j, m, s)) = 1$$

and verify that

$$\max_m \max_{s \in \{0, \ldots, 3\}} \max_{m_1 \in \{1, \ldots, m - 1\}} \max_{m_2 \in \{1, \ldots, 43 - m\}} z_{j,m,m_1,m_2,s} < 10639.8 + j,$$

where the maximum is taken over the values of $m$ appearing in Table 9. We justify that $z_{j,m,m_1,m_2,s}$ is the appropriate choice by using Lemma 4.6(i,ii,iv) (there is a condition to verify in part (ii)). We find that $v_{m,m_1,m_2}$ would be smaller for larger values of the variables. This is the contradiction we were looking for. So we have in fact $\varpi' \notin W(j)$ and a new upper bound for $\delta'$ recorded in Table 10.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta'$</td>
<td>0.02422</td>
<td>0.02353</td>
<td>0.02283</td>
<td>0.02203</td>
<td>0.02142</td>
<td>0.02083</td>
<td>0.02005</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$j$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta'$</td>
<td>0.01936</td>
<td>0.01848</td>
<td>0.01753</td>
<td>0.01638</td>
<td>0.01536</td>
<td>0.01436</td>
<td>0.01340</td>
</tr>
</tbody>
</table>

9.5. The last verification. Our strategy of verification begins with a preliminary computation. We use type 2 computations that we did previously to prove that at least three points $x_i$ defined in (4.1) are far from $\mu$ defined in (4.2). We first want to show that, among the 13244 possibilities of triplets of primes, at most a few hundreds can produce these three values of $x_i$.

To do so, we fix $j$ and split the interval $I_j$ into 25 subintervals $I_{j,r}$ of equal length. We also fix a triplet $(q_1, q_2, q_3)$. Now, type 2 computations reveal that the exponent of a prime $p \in \{q_1, q_2, q_3\}$ that divides exactly $n'$ is not in $J_{\delta + j(p)}(p, j)$ where $\delta = \delta(j)$ can be found in Section 9.3. We are thus in the exact situation of Lemma 4.7(ii). So we compute and sum the three minimal errors, we take the minimum over $r = 1, \ldots, 25$ and call this minimum $\zeta (= \zeta(q_1, q_2, q_3))$. If

$$\zeta - \epsilon_j(q_1) - \epsilon_j(q_2) - \epsilon_j(q_3) > \delta'(j)$$

then the triplet $(q_1, q_2, q_3)$ is rejected. Otherwise, we keep

$$(q_1, q_2, q_3, \zeta(q_1, q_2, q_3) - \epsilon_j(q_1) - \epsilon_j(q_2) - \epsilon_j(q_3))$$

to the last verification in a set $T(j)$, say. The value of $\delta'$ is picked from Table 10 if $j \leq 14$ and from Table 8 if $15 \leq j \leq 39$.

Now, for the very last verification, after all the $T(j)$ have been computed, we use a new idea. We assume that $j$ is fixed. For a prime $p$ in a fixed vector
(q_1, q_2, q_3, \rho) \in T(j), we observe that it is enough to check the integers n with the exponent in J_{\delta'(j)+\epsilon_j(p)}(p, j). Then, for the remaining 41 primes p, it is enough to verify with the exponent in the set
\[ J_{\delta'(j)/2-\rho/2+\epsilon_j(p)}(p, j) \]
for all but one prime p for which it can be in J_{\delta'(j)-\rho+\epsilon_j(p)}(p, j).

With these observations in mind, we design an algorithm. We compute the largest fourth component in any of the vectors in T(j) and call it t. Then we consider only the vectors such that the fourth component is in \[ [t - u/1000, t - (u - 1)/1000] \]
for all but one prime p for which it can be in J_{\delta'(j)} - \rho + \epsilon_j(p)(p, j).

With \( u \) fixed, we store in memory all the vectors in
\[ J_{\delta'(j)/2-(t-u/1000)/2+\epsilon_j(2)(2, j)} \times \cdots \times J_{\delta'(j)/2-(t-u/1000)/2+\epsilon_j(193)(193, j)} \]
to which we add two dimensions: one is the value of \( \tau(n) \) for the integer n with this exponent vector, whereas the other is its logarithm. Then, for all such vectors, only four exponents have to be modified at each verification and thus the last two facts need only a small adjustment to be used to compute the value of \( \lambda \) in each case. So, for each vector of 46 dimensions, for each exponent in J_{\delta'(j)+\epsilon_j(p)}(p, j) of each prime p in each triplet (in a vector) in T_u(j) and for each exponent in J_{\delta'(j)-\rho+\epsilon_j(p)}(p, j) of any other of the 41 primes p we compute the corresponding value of \( \lambda \). We try each value of \( u \) and then all the values of j.

After all these verifications, no value of \( n' \) has been found. This is the contradiction we were searching for and thus \( n^* \) is the largest number n such that \( \lambda(n) > 1 \) and \( \omega(n) \geq 44 \). The proof is complete.

10. Final remarks. One can show that
\[ \sum_{n \leq x} \left| \lambda(n) - \frac{\log \log x \log \log \log x}{\log x} \right|^2 \ll \frac{x \log \log x (\log \log \log x)^2}{\log^2 x}, \]
from which we conclude that for almost all \( n \leq x, \)
\[ \lambda(n) = (1 + o(1)) \frac{\log \log x \log \log \log x}{\log x} \quad (x \to \infty). \]

On the other hand, we can show that there are infinitely many n for which \( \lambda(n) > 1 \). Indeed, to any set \( S \) of primes satisfying
\[ \prod_{p \in S} \frac{\log k}{\log p} > 1 \quad \text{and} \quad \#S = k, \]
we can associate a sequence of integers \( l_1, l_2, \ldots \) such that their exponent at each prime factor, and then the associated \( \theta_i \) as defined in Corollary 4.5 is as close as possible to the optimal value as defined in Lemma 4.4. Precisely,
for each \( p \in S \), we can choose \( l_j \) to be an integer for which the exponent of \( p \), \( \alpha_p \), is the closest integer to \( \frac{\log z_j}{k \log p} \) for a fixed large \( z_j \). One verifies that

\[
\lambda(l_j) \to \left( \prod_{p \in S} \frac{\log k}{\log p} \right)^{1/k} (z_j \to \infty).
\]

Finally, we can also show that the set of limit points of \( \lambda(n) \) is the interval

\[ [0, \beta(6)^{1/6} \log 6] = [0, 1.145206 \ldots] \]

and that there exists a positive constant \( \eta \) such that

\[
\# \{ n \leq x : \lambda(n) \geq 1 \} = \eta \log^{43} x + O(\log^{42} x) \quad (x \to \infty).
\]

Moreover, we have

\[
\sup_{\omega(n) = k} \lambda(n) = 1 - \frac{\log \log k - 1}{\log k}
+ \frac{(\log \log k)^{2} - 3 \log \log k}{\log^{2} k} + O\left( \frac{1}{\log^{2} k} \right) \quad (k \to \infty).
\]

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REFERENCES


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