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Monotone normality and nabla products

by

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Abstract. Roitman's combinatorial principle Δ is equivalent to monotone normality of the nabla product, $\nabla(\omega+1)^\omega$. If $\{X_n : n \in \omega\}$ is a family of metrizable spaces and $\nabla_n X_n$ is monotonically normal, then $\nabla_n X_n$ is hereditarily paracompact. Hence, if Δ holds then the box product $\square(\omega+1)^\omega$ is paracompact. Large fragments of Δ hold in ZFC, yielding large subspaces of $\nabla(\omega+1)^\omega$ that are 'really' monotonically normal. Countable nabla products of metrizable spaces which are respectively: arbitrary, of size $\leq \mathfrak{c}$, or separable, are monotonically normal under respectively: $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} = \mathfrak{c}$ or the Model Hypothesis.

It is consistent and independent that $\nabla A(\omega_1)^\omega$ and $\nabla(\omega_1+1)^\omega$ are hereditarily normal (or hereditarily paracompact, or monotonically normal). In ZFC neither $\nabla A(\omega_2)^\omega$ nor $\nabla(\omega_2+1)^\omega$ is hereditarily normal.

1. Introduction. Let $\{X_i : i \in I\}$ be a family of topological spaces. (All spaces in this article are Tikhonov.) A *box* is a set $\prod_i U_i$, where each U_i is open in X_i. The *box product*, $\square_i X_i$, is the space with underlying set $\prod_i X_i$ and basis all boxes. Two elements x and y of $\square_i X_i$ are *mod-finite equivalent*, denoted $x \sim y$, if the set $\{i \in I : x(i) \neq y(i)\}$ is finite. The *nabla product*, $\nabla_i X_i$, is the quotient space, $\square_i X_i/{\sim}$.

It is unknown, in ZFC, whether the countable box product $\square[0,1]^\omega$, or even its closed subspace $\square(\omega+1)^\omega$, is normal. This question was asked (orally) for the first time by Tietze sometime in the 1940's. See [12] for a survey of the box product problem. Central to almost all positive results on paracompactness, and hence normality, of box products is a connection with the nabla product due to Kunen [6]: let $\{X_n : n \in \omega\}$ be a family of compact spaces; then $\square_n X_n$ is paracompact if and only if $\nabla_n X_n$ is paracompact. In

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particular, it is now known that under certain set-theoretic assumptions the nabla product $\nabla(\omega+1)^\omega$ is paracompact and so the box product $\Box(\omega+1)^\omega$ is paracompact. These assumptions include the small cardinal conditions, $\mathfrak{b} = \mathfrak{d}$ [2], and $\mathfrak{d} = \mathfrak{c}$ [8, 7], and also the so-called Model Hypothesis [9], which holds in any forcing extension by uncountably many Cohen reals.

In an insightful analysis of the combinatorics behind these consistency results, Roitman [9] extracted a combinatorial principle, Δ. She showed Δ is a consequence of each of the set-theoretic axioms mentioned above, and further claimed that Δ implies the paracompactnes of $\nabla(\omega+1)^\omega$. Unfortunately not all the details for the latter deduction were presented, and the authors and Roitman [11] are unclear how to fill the gap. See Section 4.1 for the definition of Δ, additional notation and more details on the gap.

In Section 4 we close the gap by connecting Δ with *monotone* normality of nabla products. Indeed, (Theorem 20) the property Δ holds if and only if $\nabla(\omega+1)^\omega$ is monotonically normal. Monotonically normal spaces are not automatically paracompact, but (Theorem 8) we show: if $\{X_n : n \in \omega\}$ is a family of metrizable spaces and $\nabla_n X_n$ is monotonically normal, then it is hereditarily paracompact. It follows that if Δ holds, then $\nabla(\omega+1)^\omega$ is monotonically normal, and so hereditarily paracompact, hence $\Box(\omega+1)^\omega$ is paracompact, as Roitman originally claimed.

Recall that a space X is *monotonically normal* if for every pair of disjoint closed sets A, B there is an open set $H(A,B)$ such that

(i) $A \subseteq H(A,B) \subseteq \overline{H(A,B)} \subseteq X \setminus B$ (so $H(\cdot,\cdot)$ separates A from B, and thus witnesses normality), and
(ii) if $A' \subseteq A$ and $B \subseteq B'$, then $H(A',B') \subseteq H(A,B)$ ('monotonicity', the separation respects set inclusion).

An alternative characterization is that for every point x in an open set U there is assigned an open set $G(x,U)$ such that $x \in G(x,U) \subseteq U$, and if $G(x,U) \cap G(y,V) \neq \emptyset$, then $x \in V$ or $y \in U$. Observe that the restriction of a monotone normality operator, $G(\cdot,\cdot)$, to a subspace yields a monotone normality operator for the subspace, and so monotone normality is hereditary. It follows that monotone normality does not transfer from $\nabla(\omega+1)^\omega$ to $\Box(\omega+1)^\omega$. Indeed (see [13]), if $\{X_i : i \in I\}$ is a family of compact or first countable spaces, then $\Box_i X_i$ is not hereditarily normal.

The authors do not know how to prove Δ in ZFC, or to prove that its negation is consistent. In an effort to shed light on this conundrum we have attempted to 'parametrize' the problem: either 'from below' in order to see how close we can get to establishing Δ in ZFC, or 'from above' to determine when natural strengthenings of Δ are false either consistently or in ZFC.

For example, pursuing an idea of Roitman, we characterize in Section 4.3 when certain subspaces A of $\nabla(\omega+1)^\omega$ are monotonically normal in terms of a combinatorial property $\Delta(A)$, where Δ is $\Delta(\nabla(\omega+1)^\omega)$. In particular (see Proposition 16), $\Delta(A)$ is true in ZFC for A consisting of all finite disjoint unions of increasing functions.

In the other direction we have found combinatorial characterizations of when nabla products of certain spaces containing $\omega+1$ as a closed subspace, or otherwise naturally extending $\omega+1$, are monotonically normal. Specifically, observe that $\omega+1$ is the one-point compactification of a countably infinite discrete space. Accordingly, in Section 5, we combinatorially characterize monotone normality of nabla products of spaces of the form $A(\kappa)$, the one-point compactification of a discrete space of size κ. We show that $\nabla A(\omega_2)^\omega$ is not hereditarily normal, and so not monotonically normal, in ZFC, while $\nabla A(\omega_1)^\omega$ is consistently not hereditarily normal. A striking result of Williams [13] is that consistently any countable nabla product of compact spaces of weight (minimal size of a base) no more than $\aleph_1$ is (ω_1-metrizable and so) monotonically normal and hereditarily paracompact. In particular, $\nabla A(\omega_1)^\omega$ is consistently monotonically normal and hereditarily paracompact; and so each of the statements: '$\nabla A(\omega_1)^\omega$ is monotonically normal', '$\nabla A(\omega_1)^\omega$ is hereditarily paracompact' and '$\nabla A(\omega_1)^\omega$ is hereditarily normal' is consistent and independent. These results answer questions of Roitman.

Observing that $\omega+1$ can also be viewed as an ordinal with the order topology, in Section 6, we go on to characterize combinatorially monotone normality of nabla products of ordinals. This yields a finer parametrization than looking at one-point compactifications. Indeed, if $\nabla(\omega+1)^\omega$ is monotonically normal (in other words, Δ holds), then for every n in ω, we also have $\nabla(\omega.n+1)^\omega$ monotonically normal. However, the combinatorial principle characterizing when $\nabla(\omega.\omega+1)^\omega$ is monotonically normal is—at least *formally*—stronger than Δ. By the result of Williams [13] mentioned above, $\nabla(\omega_1+1)^\omega$ is consistently monotonically normal and hereditarily paracompact. We show $\nabla(\omega_1+1)^\omega$ is consistently not hereditarily normal. Thus each of the statements: '$\nabla(\omega_1+1)^\omega$ is monotonically normal', '$\nabla(\omega_1+1)^\omega$ is hereditarily paracompact' and '$\nabla(\omega_1+1)^\omega$ is hereditarily normal' is consistent and independent. In contrast we also show that $\nabla(\omega_2+1)^\omega$ is not hereditarily normal, and so not monotonically normal, in ZFC. Again these results answer questions of Roitman.

In Section 7—thinking of $\omega+1$ as the simplest non-discrete metrizable space—we investigate combinatorial characterizations of monotone normality of nabla products of metrizable spaces. Section 8 concludes the paper with some open problems and potential lines of research.

2. Preliminaries

2.1. Set theory. Recall that $\mathfrak{b}$ is the minimal size of an unbounded set in ω^ω with the mod-finite order, $\leq^*$, and $\mathfrak{d}$ is the minimal size of a cofinal (dominating) set. Further, $\mathfrak{b} = \mathfrak{d}$ if and only if there is a dominating family $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ such that if $\alpha < \beta$, then $f_\alpha \leq^* f_\beta$ (a family of this kind is called a *scale*). We record an additional useful fact, a proof of which can be found in [9].

Lemma 1. *If $\mathcal{G} \subseteq \omega^\omega$, $\mathcal{A} \subseteq \mathcal{P}(\omega)$ and $|\mathcal{G}|, |\mathcal{A}| < \mathfrak{d}$, then there is a function $f \in \omega^\omega$ such that for any $g \in \mathcal{G}$ and $a \in \mathcal{A}$, $|\{n \in a : f(n) > g(n)\}| = \omega$.*

Definition 2 (Roitman [9]). The *Model Hypothesis*, abbreviated MH, is the following statement: For some κ, $H(\omega_1)$ is the increasing union of H_α's, for $\alpha < \kappa$, where each H_α is an elementary submodel of $(H(\omega_1), \in)$ and each $H_\alpha \cap \omega^\omega$ is not dominating.

Here $H(\kappa)$ is the collection of all sets whose transitive closures have size less than κ. In particular, both $\omega^\omega, \mathcal{P}(\omega)$ are subsets of $H(\omega_1)$, and a space of countable weight can be coded as (hence is homeomorphic to) a subset of $H(\omega_1)$.

2.2. Box and nabla products. We will follow Roitman's notation from [9]. For $x \in \square_n X_n$, we write $\overline{x}$ for its mod-finite equivalence class, $[x]_\sim$, in $\nabla_n X_n$. If $x \in \square_n X_n$ or $\overline{x} \in \nabla_n X_n$, and $U = \langle U_n \rangle_{n\in\omega}$ is a sequence of open sets, where $x(n) \in U_n \subseteq X_n$, define the basic neighborhood around x by $N(x, U) := \square_n U_n \subseteq \square_n X_n$ and $N(\overline{x}, U) := \nabla_n U_n \subseteq \nabla_n X_n$. If the X_n's are first countable, a basis of x or $\overline{x}$ is coded by ω^ω as follows: if $\{U_n^k : k \in \omega\}$ is a base at $x(n)$, we will write $N(x, f)$ for $\square_n U_n^{f(n)}$ and $N(\overline{x}, f)$ for $\nabla_n U_n^{f(n)}$, where $f \in \omega^\omega$. Following Roitman, we do not distinguish between elements of $\square_n X_n$ and $\nabla_n X_n$ (x versus $\overline{x}$) if there is no chance of confusion.

A space X is said to be P_κ if the intersection of strictly fewer than κ-many open sets is open. We recall: every nabla product, $\nabla_n X_n$, is a P_{ω_1}-space; and if each X_n is first countable, then $\nabla_n X_n$ is a $P_\mathfrak{b}$-space.

2.3. Monotone normality and halvability. Let A be a subspace of a space X. We say that A is *monotonically normal in* X if for every point x of A and set U open in X containing it, there is assigned an open (in X) set $G(x, U)$ such that $x \in G(x, U) \subseteq U$, and if $G(x, U) \cap G(y, V) \neq \emptyset$, then $x \in V$ or $y \in U$. Observe that for any base for X we only need to define $G(x, U)$ for basic U, and we may assume that $G(x, U)$ is in the base. This will be used frequently in what follows. If A is monotonically normal in some X then clearly A is monotonically normal.

A function F on A is a *neighborhood assignment* (or *neighbornet*) for A (in X) if $F(x)$ is a neighborhood of x for every x in A. (A neighbornet for the whole space is just called a 'neighbornet'.) A neighbornet T for A is *halvable in* X if there is a neighbornet S for A such that: if $S(x) \cap S(y) \neq \emptyset$ then $x \in T(y)$ or $y \in T(x)$. Note that we may assume that every $S(x)$ comes from any given base for X. The subspace A of X is *halvable in* X if every neighbornet of A in X is halvable. A space is *halvable* if it is halvable in itself (every neighbornet can be halved).

Observe that if A is monotonically normal in X then it is halvable in X, and so every monotonically normal space is halvable. The converse is false. For example, every countable (Tikhonov) space is halvable [5], but there are countable spaces that are not monotonically normal (for example, all polynomials with rational coefficients with the topology of pointwise convergence; see [3]). However, it turns out that in certain cases nabla products are monotonically normal if they are halvable, indeed it suffices that just one specific neighbornet be halvable.

LEMMA 3. *Let X be a space with partial order $\preceq$ and neighborhood bases, $\mathcal{B}_x$, for each $x \in X$, such that:*

(a) $\downarrow x = \{y : y \preceq x\}$ *is a neighborhood of x for all x, and*
(b) *if $y \in B \subseteq \downarrow x$, where $B \in \mathcal{B}_x$, then the interval $[y, x] = \{z : y \preceq z \preceq x\}$ is contained in B.*

Let A be a subspace of X. Then A is monotonically normal in X if and only if the neighbornet $T_A(x) = \downarrow x$, for x in A, is halvable in X.

Proof. We only need to show that if S halves $T_A(x) = \downarrow x$, then A is monotonically normal in X. For any element x of A in some $B \in \mathcal{B}_x$, where $B \subseteq \downarrow x$, define $G(x, B) = S(x) \cap B$. We prove that this is a monotone normality operator for A in X. Suppose $z \in G(x, B) \cap G(x', B')$. Then $S(x)$ meets $S(x')$, and suppose without loss of generality that $x' \in \downarrow x$, that is, $x' \preceq x$. As $z \in B \subseteq \downarrow x$, we have $z \preceq x$. Hence, $[z, x]$ is contained in B. But as $z \in B' \subseteq \downarrow x'$, we have $z \preceq x'$. Therefore x' is in $[z, x]$, and so in B. ∎

A space X is *κ-metrizable* (for a cardinal $\kappa \geq \omega_1$) if it has an open base $\mathcal{B} = \{U_{x,\alpha} : \alpha < \kappa,\ x \in X\}$ such that $\{U_{x,\alpha} : \alpha < \kappa\}$ is a neighborhood base at x, and given two points x, y and two ordinals $\alpha \leq \beta < \kappa$ we have (i) if $y \in U_{x,\alpha}$ then $U_{y,\beta} \subseteq U_{x,\alpha}$, and (ii) if $y \notin U_{x,\alpha}$ then $U_{y,\beta} \cap U_{x,\alpha} = \emptyset$. Every κ-metrizable space is paracompact and monotonically normal.

2.4. Not hereditarily normal. We observe that certain spaces are not hereditarily normal. These will be used as test spaces to show certain nabla products are not hereditarily normal. The results are probably folklore, so we sketch just enough of their proofs for the full argument to be reconstructed by the reader.

If λ, κ are cardinals, denote by $D(\kappa)$ the discrete space of size κ and let $L_\lambda(\kappa)$ be the space with underlying set $D(\kappa) \cup \{\kappa\}$, and topology where points of $D(\kappa)$ are isolated and neighborhoods around κ have the form $\{\kappa\} \cup (D(\kappa) \setminus C)$ for $C \subseteq D(\kappa)$ of size less $< \lambda$. Write $A(\kappa)$ for $L_\omega(\kappa)$, the one-point compactification of $D(\kappa)$, and $L(\kappa) = L_{\omega_1}(\kappa)$ the one-point Lindelöfication of $D(\kappa)$.

Lemma 4. *$L(\omega_2)^2$ is not hereditarily normal.*

Proof. More precisely, $Y = L(\omega_2) \times L(\omega_2) \setminus \{(\omega_2, \omega_2)\}$ is not normal, because the sets $H = (L(\omega_2) \setminus \{\omega_2\}) \times \{\omega_2\}$ and $K = \{\omega_2\} \times (L(\omega_2) \setminus \{\omega_2\})$ are disjoint and closed in Y, and cannot be separated by disjoint open sets.

Indeed, suppose U and V are any open neighborhoods around H and K, respectively. For every $(\alpha, \omega_2) \in H$ choose $A_\alpha \in [D(\omega_2)]^\omega$ such that $\{\alpha\} \times (L(\omega_2) \setminus A_\alpha) \subseteq U$, and similarly, for $(\omega_2, \beta) \in K$ choose sets B_β such that $(L(\omega_2) \setminus B_\beta) \times \{\beta\} \subseteq V$. Then there is $\delta \geq \omega_1$ such that for every $\alpha \leq \delta$, $A_\alpha \subseteq \delta$. (To see this, let $M \prec H(\omega_3)$ be an elementary submodel of size ω_1 such that $\{A_\alpha : \alpha < \omega_2\} \in M$, set $\delta = M \cap \omega_2 \in \omega_2$, and now, for $\alpha \in M$, M thinks 'A_α is contained in M', and so does $H(\omega_3)$.) Now a counting argument easily shows U and V meet. ∎

Lemma 5. *Let S be a stationary subset of a regular uncountable cardinal κ. Then, $S \times (S \cup \{\kappa\})$ (as a subspace of $(\kappa+1)^2$) is not normal.*

Proof. Consider the diagonal $H = \{(\alpha, \alpha) : \alpha \in S\}$ and the top edge $K = \{(\alpha, \kappa) : \alpha \in S\}$. Note that H and K are closed disjoint sets. Now, if U, V are neighborhoods of H, K, respectively, a standard Pressing Down Lemma argument shows that U and V must meet. ∎

3. Embeddings into nabla products. The following simple embedding result will be used frequently in what follows. For a space X let X_δ be the *G_δ-modification* of X (the space with underlying set X and topology generated by all G_δ subsets of the space X).

Lemma 6 (Williams [13, Lemma 4.4]). *Let X be a space. Then X_δ embeds as a closed subspace in ∇X^ω via the map $x \mapsto c_x$, where c_x is constantly equal to x.*

However the main technical result of this section is about non-embedding.

Proposition 7. *Let $\{X_n : n \in \omega\}$ be a family of metrizable spaces and S a stationary subset of a regular uncountable cardinal κ. Then S does not embed into $\nabla_n X_n$.*

Since Balogh and Rudin [1] showed that a monotonically normal space is paracompact if and only if it does not contain closed copies of stationary subsets of regular uncountable cardinals, we deduce:

Theorem 8. *Let $\{X_n : n \in \omega\}$ be a family of metrizable spaces. If a subspace A of $\nabla_n X_n$ is monotonically normal then A is hereditarily paracompact.*

Proof of Proposition 7. Suppose, for a contradiction, $\varphi : S \to \nabla_n X_n$ is an embedding. We split the proof into two cases depending on the size of κ. If $\kappa \leq \mathfrak{d}$, then any α in $\mathrm{Lim}(S)$ has $\mathrm{cf}(\alpha) \neq \mathfrak{d}$, so S, and $A = \varphi(S)$, have limit points but no points of character $\mathfrak{d}$, contradicting Lemma 9 below. If $\kappa > \mathfrak{d}$, by Lemma 10 the map φ is eventually constant, thus it cannot be an embedding. ∎

Recall that the *character* of a topological space X at a point x is the cardinality $\chi(x, X)$ of the smallest local base for x. The *tightness* at a point x in X, denoted $t(x, X)$, is the smallest cardinal κ such that whenever $x \in \overline{Y}$ for some $Y \subseteq X$, there exists a subset $Z \subseteq Y$ with $x \in \overline{Z}$ and $|Z| \le \kappa$.

LEMMA 9. *Let $\{X_n : n \in \omega\}$ be a family of first countable spaces. Then for any point $x \in \overline{A} \setminus A$, where $A \subseteq \nabla_n X_n$, we have $t(x, A \cup \{x\}) = \mathfrak{d} = \chi(x, A \cup \{x\})$.*

Proof. It is easy to check that $t(x, A\cup\{x\}) \le \chi(x, A\cup\{x\}) \le \chi(x, \nabla_n X_n) = \mathfrak{d}$ (the last equality holds because a local basis of x can be represented by a dominating family of ω^ω). We only have to prove that $t(x, A \cup \{x\}) \ge \mathfrak{d}$.

If x is a limit point of A, then for infinitely many $n \in \omega$, $x(n)$ is non-isolated in X_n. Thus, without loss of generality we may assume that for all $n \in \omega$, $x(n)$ is non-isolated in X_n. Let $\{B_m(n) : m \in \omega\}$ be a decreasing countable local basis for $x(n)$, for every $n \in \omega$.

Suppose for a contradiction that there is $Z \subseteq A$ with $|Z| < \mathfrak{d}$ and $x \in \mathrm{cl}_{A\cup\{x\}}(Z)$. For every $z \in Z$, there is an infinite set $a_z \subseteq \omega$ such that for $n \in a_z$, $z(n) \ne x(n)$. Also, for every $z \in Z$ there is a function $f_z \in \omega^\omega$ such that for $n \in a_z$, $z(n) \notin B_{f_z(n)}(n)$. Thus, $z \notin N(x, f_z)$. Let $\mathcal{G} = \{f_z : z \in Z\}$ and $\mathcal{A} = \{a_z : z \in Z\}$. By Lemma 1, there is $f \in \omega^\omega$ diagonalizing the families $\mathcal{G}$ and $\mathcal{A}$. Then, for any $z \in Z$, $z \notin N(x, f)$, contradicting that $x \in \mathrm{cl}_{Z\cup\{x\}}(Z)$. ∎

LEMMA 10. *Let $\{(X_n, d_n) : n \in \omega\}$ be a family of metric spaces and S a stationary subset of a regular uncountable cardinal $\kappa > \mathfrak{d}$. Then every continuous function $\varphi : S \to \nabla_n X_n$ is eventually constant.*

Proof. For each $n \in \omega$, write $B_n(a, \varepsilon)$ for the ε-ball around a in X_n with respect to the metric d_n. Let $\{f_\mu : \mu < \mathfrak{d}\} \subseteq \omega^\omega$ be a dominating family. For every $x \in \nabla_n X_n$ and $\mu < \mathfrak{d}$, define

$$N(x, f_\mu) = \nabla_{n\in\omega} B_n\biggl(x(n), \frac{1}{f_\mu(n)}\biggr);$$

observe that $\{N(x, f_\mu) : \mu < \mathfrak{d}\}$ is a local basis at x.

Fix, for the moment, $\mu < \mathfrak{d}$. For every $\alpha \in \mathrm{Lim}(S)$, pick $g_\mu(\alpha) < \alpha$, $g_\mu(\alpha) \in S$, such that $\varphi[(g_\mu(\alpha), \alpha]] \subseteq N(\varphi(\alpha), 2f_\mu)$. Then g_μ is a regressive function and, by the Pressing Down Lemma, there is α_μ in S and a stationary set $S_\mu \subseteq S$ such that for any $\beta \in S_\mu$, $g_\mu(\beta) = \alpha_\mu$.

We claim that for all $\delta, \gamma > \alpha_\mu$, where δ and γ are in S, $\varphi(\gamma)$ is in $N(\varphi(\delta), f_\mu)$. To see this, take any δ and γ strictly larger than α_μ in S.

Since S_μ is stationary, there is a β in $\mathrm{Lim}(S_\mu)$ with $\beta > \max\{\gamma, \delta\}$. Then $\{\varphi(\gamma), \varphi(\delta)\} \subseteq \varphi[(\alpha_\mu, \beta]] = \varphi[(g_\mu(\beta), \beta]] \subseteq N(\varphi(\beta), 2f_\mu)$. By definition of $N(\varphi(\beta), 2f_\mu)$, for all $n \in \omega$ we see that $\varphi(\gamma)(n)$ and $\varphi(\delta)(n)$ are in $B_n(\varphi(\beta)(n), 1/(2f_\mu(n)))$. Then by symmetry and the triangle inequality, for all $n \in \omega$, we get $d_n(\varphi(\gamma)(n), \varphi(\delta)(n)) < 1/f_\mu(n)$. This implies $\varphi(\gamma) \in N(\varphi(\delta), f_\mu)$, as claimed.

Now, as we let μ run over all values below $\mathfrak{d}$, since $\kappa > \mathfrak{d}$, there is a least upper bound α_∞ of $\{\alpha_\mu : \mu < \mathfrak{d}\}$ in S. Notice that by the claim above, for any $\mu < \mathfrak{d}$ and $\gamma, \delta \in S \setminus \alpha_\infty$, we have $\varphi(\gamma) \in N(\varphi(\delta), f_\mu)$, and so $\varphi(\gamma) = \varphi(\delta)$. Hence φ is constant from α_∞ on, as desired. ■

4. Δ and $\nabla(\omega+1)^\omega$

4.1. Roitman's principle Δ. In order to state our parametrized versions of Roitman's Δ principle we introduce some specific definitions and notation, naturally extending those of Roitman, for partial functions.

For any function $x : N \to \omega$, where $N \subseteq \omega$, consider x to be a *partial function* from ω to ω and write $\mathrm{dom}(x)$ for N, the domain of x. We identify a partial function with its graph, which is a subset of $\omega \times \omega$. Then two partial functions, x and y, are *almost equal*, $x =^* y$, if $x \setminus y$ and $y \setminus x$ are both finite. Let $\omega^{\subseteq\omega}$ be the set of all partial functions, including the empty function. Denote by $\omega^{\subset\omega}$ the set of all partial functions whose domain is infinite and co-infinite. For any subset A of $\omega^{\subseteq\omega}$ let $A^* = \{y \in \omega^{\subseteq\omega} : y =^* x$ for some $x \in A\}$. For $k \le \omega$, let c_k be the constant k-valued function in $(\omega+1)^\omega$. Let x be a partial function and $h \in \omega^\omega$; then we say that $x >^* h$ if for all but finitely many $n \in \mathrm{dom}(x)$, $x(n) > h(n)$.

Definition 11. Two partial functions, x and y, *switch* if $|x \setminus y| = |y \setminus x| = \omega$ and $x(n) = y(n)$ for all but finitely many n in $\mathrm{dom}(x) \cap \mathrm{dom}(y)$.

Definition 12. Let A be any subset of $\omega^{\subseteq\omega}$. Then $\Delta(A)$ is the statement: there exists $F : A \to \omega^\omega$ such that if $x, y \in A$ switch, then $x \setminus y \not>^* F(y)$ or $y \setminus x \not>^* F(x)$.

Lemma 13. *Let A and B be subsets of $\omega^{\subseteq\omega}$. Then:*

(1) *if $A \subseteq B \subseteq \omega^{\subseteq\omega}$ then $\Delta(B) \Rightarrow \Delta(A)$, and*
(2) *$\Delta(A) \Leftrightarrow \Delta(B)$ when $A \cap \omega^{\subset\omega} \subseteq B \subseteq (A \cup \omega^\omega \cup \{\emptyset\})^*$.*

Proof. Claim (1) is clear—simply restrict an F witnessing $\Delta(B)$ to A to get a witness of $\Delta(A)$.

For claim (2) it suffices to show $\Delta(A \cap \omega^{\subset\omega}) \Rightarrow \Delta((A \cup \omega^\omega \cup \{\emptyset\})^*)$. Fix F as in $\Delta(A \cap \omega^{\subset\omega})$. Note that if x is in $\omega^\omega \cup \{\emptyset\}$ then it does not switch with *any* y in $\omega^{\subseteq\omega}$. So we can extend F over $\omega^\omega \cup \{\emptyset\}$ completely arbitrarily, and it witnesses $\Delta(A \cup \omega^\omega \cup \{\emptyset\})$. Note that if $x =^* x'$ and $y =^* y'$ then x, y switch if and only if x', y' switch, and similarly for the conclusion of Δ. So we can extend F over A^* in the natural way (if $x \in A$, $x =^* x'$ and $x' \notin A$ then set $F(x') = F(x)$) to get a witness of $\Delta((A \cup \omega^\omega \cup \{\emptyset\})^*)$, as required. ∎

Abbreviate $\Delta(\omega^{\subset\omega})$ to Δ to see this is Roitman's combinatorial principle in [9] and [12]. It is known to be consistently true (under $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} = \mathfrak{c}$, MH, and in any forcing extension obtained by adding cofinally many Cohen reals) but it is unknown if it can be consistently false, or if it is true in ZFC.

In [9] Roitman showed that Δ implies the subspace $\nabla^* = \nabla(\omega+1)^\omega \setminus (\nabla\omega^\omega \cup \{\bar{c}_\omega\})$ of $\nabla(\omega+1)^\omega$ to be paracompact. Then she claimed, without proof, that '∇^* is paracompact if and only if $\nabla(\omega+1)^\omega$ is paracompact'. Here lies the gap: adding isolated points (like those of $\nabla\omega^\omega$) to even the best behaved of spaces (discrete, for example) frequently destroys normality and paracompactness (indeed, many classical counter-examples related to normality have this form).

4.2. $\Delta(A)$ principles that hold in ZFC. While we only know of consistency proofs for Δ equal to $\Delta(\omega^{\subset\omega})$, there are, however, interesting A such that $\Delta(A)$ is true in ZFC. We present here an example.

Let INC be the subset of $\omega^{\subseteq\omega}$ that consists of increasing partial functions (so $x \in \omega^{\subseteq\omega}$ is in INC if whenever $m \leq n$ are in $\mathrm{dom}(x)$ we have $x(m) \leq x(n)$). Now let FI be the subset of $\omega^{\subseteq\omega}$ that consists of those elements that are a finite disjoint union of elements in INC (so x is in FI if $\mathrm{dom}(x)$ can be partitioned into $S_1, \ldots, S_k$ such that each $x{\restriction}S_i$ is in INC). We will show: the combinatorial principle $\Delta(\mathrm{FI})$ is true in ZFC.

First we need to show that elements of FI have a nice representation. Let x be in $\omega^{\subseteq\omega}$. Define $\perp x = \{m \in \mathrm{dom}(x) : \forall n > m,\, x(m) \leq x(n)\}$. Set $x^\perp = x{\restriction}(\perp x)$. Observe that $x^\perp$ is increasing. Set $x_0 = x^\perp$, and inductively, $x_n = (x \setminus \bigcup_{i<n} x_i)^\perp$.

Proposition 14. *Let x be in* FI, *say $x = x^0 \cup \cdots \cup x^\ell$ where each x^i is increasing. Then $x = x_0 \cup \cdots \cup x_\ell$.*

Proof. Evidently each x_k is a subset of x, so we need to show that $x \subseteq x_0 \cup \cdots \cup x_\ell$. We do so by breaking x into finite pieces, ${}^i x$, each of which is contained in $\bigcup_{k\leq\ell} x_k$.

For any $y \in \omega^{\subseteq\omega}$ set $\mathrm{Dec}(y) \subseteq \mathrm{dom}(y)$ such that $\min(\mathrm{dom}(y)) \in \mathrm{Dec}(y)$ and $n \in \mathrm{Dec}(y)$ with $n > \min(\mathrm{dom}(y))$ if and only if for all $m < n$ and $m \in \mathrm{dom}(y)$, $y(m) > y(n)$. Note that $\mathrm{Dec}(y)$ is finite and non-empty, and $y{\restriction}\mathrm{Dec}(y)$ is *strictly* decreasing. Set ${}^0x = x{\restriction}\mathrm{Dec}(x)$, and inductively, ${}^n x = x{\restriction}\mathrm{Dec}(x \setminus \bigcup_{i<n} {}^i x)$. Note that x is the disjoint union of the ${}^i x$'s. Since x is the union of $\ell+1$ increasing functions, the x^i, there cannot be a strictly decreasing portion of x of size greater that $\ell+1$. Hence, for every $i \in \omega$, $0 \leq |{}^i x| - 1 := \ell_i \leq \ell$. Enumerate, in increasing order, $\mathrm{dom}({}^i x) = \langle n^i_j : j \leq \ell_i \rangle$

(hence, ${}^ix(n^i_j) > {}^ix(n^i_{j+1})$). The next claim shows that each ${}^ix$ is contained in $x_0 \cup \cdots \cup x_\ell$, which completes the proof.

Claim. *For every $j \leq \ell_i$ and $i \in \omega$, $(n^i_{\ell_i-j}, {}^ix(n^i_{\ell_i-j}))$ is in $\bigcup_{k\leq j} x_k$.*

We proceed by induction on j. First $j = 0$. Take any $i \in \omega$. The last point $(n^i_{\ell_i}, {}^ix(n^i_{\ell_i}))$ has the property that $\forall m > n^i_{\ell_i}$, ${}^ix(n^i_{\ell_i}) = x(n^i_{\ell_i}) \leq x(m)$. Otherwise, if there is $m > n^i_{\ell_i}$ such that ${}^ix(n^i_{\ell_i}) > x(m)$, pick m_i the minimum $m > n^i_{\ell_i}$ with that property, and we get $(m_i, x(m_i)) \in {}^ix$, which contradicts the construction of ${}^ix$. Hence, $(n^i_{\ell_i-j}, {}^ix(n^i_{\ell_i-j})) \in x^j$.

Now suppose that $\forall m < j$, $\forall i \in \omega$ we have $(n^i_{\ell_i-m}, {}^ix(n^i_{\ell_i-m})) \in \bigcup_{k<j} x_k$. We need to prove that $(n^i_{\ell_i-j}, {}^ix(n^i_{\ell_i-j})) \in x_j$. The point $(n^i_{\ell_i-j}, {}^ix(n^i_{\ell_i-j}))$ has the property

$$\forall m > n^i_{\ell_i-j},\ {}^ix(n^i_{\ell_i-j}) \leq \Big(x \setminus \bigcup_{k\leq j} x_k\Big)(m).$$

Suppose for a contradiction that $m > n^i_{\ell_i-j}$ is the least such that ${}^ix(n^i_{\ell_i-j})$ is strictly greater than $(x \setminus \bigcup_{k\leq j} x_k)(m)$. There is an $i' > i$ such that $m \in \mathrm{dom}({}^{i'}x)$. By inductive hypothesis, we have removed the last j points (from 0 to $j-1$) of ${}^ix$, for all $i \in \omega$. Hence, m has the form $n^{i'}_{\ell_{i'}-j}$. Consider the decomposition $x = x^0 \cup \cdots \cup x^\ell$, where the x^k are in INC. Observe that for every $i \in \omega$ and $k \leq \ell$, $|x^k \cap {}^ix| \leq 1$. Also, notice that for every $s \leq \ell_i - j$, if $n^i_{\ell_i-(j+s)} \in \mathrm{dom}(x^k)$, then $n^{i'}_{\ell_{i'}-j} \notin \mathrm{dom}(x^k)$, because x^k is increasing and $x(n^i_{\ell_i-j}) > x(m)$ (our assumption). Thus, there is a one-to-one correspondence between the last j points of ${}^ix$ and the last $j+1$ points of ${}^{i'}x$, the desired contradiction. ∎

We will say that two elements z, w of $\omega^{\subseteq\omega}$ are *compatible* if for all but finitely many n in $\mathrm{dom}(z) \cap \mathrm{dom}(w)$ we have $z(n) = w(n)$. Now let x be an element of $\omega^{\subseteq\omega}$ with infinite domain. Note that $x^\perp$ also has infinite domain. For each $n \notin \mathrm{dom}(x)$ let n^{+x} be the minimal element of $\mathrm{dom}(x)$ larger than n. Let INC^+ be the set of members of $\omega^{\subseteq\omega}$ that are increasing ($\mathrm{INC} \subseteq \mathrm{INC}^+$). For x in INC^+, define $F(x)$ by: $F(x)(n)$ is $x(n)$ when $n \in \mathrm{dom}(x)$, and is $x(n^{+x}) + 1$ otherwise.

Lemma 15. *Let z, w be compatible elements of $\omega^{\subseteq\omega}$ with infinite domains. If $z \setminus w >^* F(w^\perp)$ and $w \setminus z >^* F(z^\perp)$ then $w^\perp =^* z^\perp$*

Proof. We will show that $z^\perp =^* u^\perp$ where $u = (w \setminus z) \cup z$. Then symmetrically, $w^\perp =^* v^\perp$ where $v = (z \setminus w) \cup w$, hence by compatibility of z, w we have $u =^* v$, and so $z^\perp =^* w^\perp$, as claimed. For $z^\perp =^* u^\perp$ it suffices that $\perp z =^* \perp u$, because u equals z on $\perp z \subseteq \mathrm{dom}(z)$, and so $u^\perp = u{\restriction}(\perp u) =^* z{\restriction}(\perp z) = z^\perp$. Fix N such that for all $n \geq N$ we have $(w \setminus z)(n) > F(z^\perp)(n)$.

First a general observation: if $k \in \mathrm{dom}(w \setminus z)$ and $k \geq N$, then $k \leq k^{+z^\perp}$ and

$$u(k) = (w \setminus z)(k) > F(z^\perp)(k) \geq z^\perp(k^{+z^\perp}) = u(k^{+z^\perp}).$$

Now take any n in $\perp u$ with $n \geq N$. By the observation, n must be in $\mathrm{dom}(z)$ (not $\mathrm{dom}(w \setminus z)$) and (as $z \subseteq u$) clearly n is in $\perp z$.

For the other inclusion, take any $n \geq N$ in $\perp z$. Take any $m \in \mathrm{dom}(u)$, $m \geq n$. If $m \in \mathrm{dom}(z)$ then $u(n) \leq u(m)$ because $n \in \perp z$. Otherwise $m \in \mathrm{dom}(w \setminus z)$, and then by the observation, $u(m) > z^\perp(m^{+z^\perp}) \geq z(n)$ (as $n \in \perp z$), and $z(n) = u(n)$. Either way, $u(n) \leq u(m)$. Thus $n \in \perp u$, as required. ∎

We now show $\Delta(\mathrm{FI})$ is true. From Lemma 13 we deduce that in fact $\Delta((\mathrm{FI} \cup \omega^\omega \cup \{\emptyset\})^*)$ is true. Note that a partial function is in FI^* if it is eventually in FI, or equivalently, if it is the finite union of eventually increasing partial functions.

Proposition 16. *In* ZFC *we have* $\Delta(\mathrm{FI})$.

Proof. Take any x' in FI. Then by Proposition 14, x' is the disjoint union $x'_0 \cup x'_1 \cup \cdots \cup x'_{\ell'}$. Note that $x' =^* x$ where x is either the empty set, or the disjoint union $x_0 \cup x_1 \cup \cdots \cup x_\ell$, and each of $x_0, \ldots, x_\ell$ is in INC^+. Let FI^+ be all x in FI such that all of $x_0, \ldots, x_\ell$ are in INC^+. Then we have just said that $\mathrm{FI} \subseteq (\{\emptyset\} \cup \mathrm{FI}^+)^*$, hence, by Lemma 13, to show $\Delta(\mathrm{FI})$ it suffices to prove $\Delta(\mathrm{FI}^+)$. For x in FI^+ define $F(x)$ to be the maximum of $F(x_0), \ldots, F(x_\ell)$.

Take any x, y in FI^+. They have representations $x = x_0 \cup \cdots \cup x_\ell$ and $y = y_0 \cup \cdots \cup y_m$. Assume, without loss of generality, that $\ell \leq m$. Suppose x and y are compatible, $x \setminus y >^* F(y)$ and $y \setminus x >^* F(x)$. To establish $\Delta(\mathrm{FI}^+)$ we show $x \setminus y$ is finite (so x, y do not switch), because $x_n =^* y_n$ for all $n \leq \ell$, which we verify by induction on n.

Inductively, suppose $x_i =^* y_i$ for all $i < n$. Let $z = x \setminus \bigcup_{i<n} x_i$ and $w = y \setminus \bigcup_{i<n} y_i$. Then $z =^* x \setminus \bigcup_{i<n} y_i$ and $w =^* y \setminus \bigcup_{i<n} x_i$. Note that $z^\perp = x_n$ and $w^\perp = y_n$. Since x and y are compatible, so are z and w. Note that $z \setminus w =^* (x \setminus \bigcup_{i<n} y_i) \setminus (y \setminus \bigcup_{i<n} y_i) = x \setminus y$. Hence, $z \setminus w =^* x \setminus y >^* F(y) \geq F(y_n) = F(w^\perp)$. Symmetrically, $w \setminus z >^* F(z^\perp)$. Thus z and w satisfy the hypotheses of Lemma 15, so $x_n = z^\perp =^* w^\perp = y_n$, as claimed. ∎

4.3. Halvability and monotone normality of ∇. We now connect the combinatorial principle $\Delta(A)$ with the topology of $\nabla(\omega+1)^\omega$. There is a natural bijection between $(\omega+1)^\omega$ and $\omega^{\subseteq\omega}$. Indeed, given $x \in \omega^{\subset\omega}$ we can extend it to x' in $(\omega+1)^\omega$ by giving x' value ω outside the domain of x. Conversely, given x in $(\omega+1)^\omega$ we get an element of $\omega^{\subseteq\omega}$ by restricting it to $N = \{n \in \omega : x(n) \in \omega\}$. *Throughout this section we identify* $\square(\omega+1)^\omega$ *with* $\omega^{\subseteq\omega}$. Recall that if $x \in \square(\omega+1)^\omega$, then we write $\overline{x}$ for its equivalence class, $[x]_\sim$, in $\nabla(\omega+1)^\omega$. We extend this according to our identification, and given x in $\omega^{\subseteq\omega}$, write $\overline{x}$ for $[x']_\sim$. (Note that $\overline{\emptyset} = \overline{c}_\omega$.)

The set $\nabla(\omega+1)^\omega$ has a natural partial order: for x and y in $\omega^{\subseteq\omega}$ write $\overline{y} \preceq \overline{x}$ if for all but finitely many n in $\mathrm{dom}(x)$ we have $y(n) = x(n)$. Note that $\overline{c}_\omega$ is the $\preceq$-largest element of $\nabla(\omega+1)^\omega$. We say that $\overline{x}$ and $\overline{y}$ are *compatible* if they have a common $\preceq$-lower bound. Proof of the next lemma just requires chasing definitions, and is left to the reader.

LEMMA 17. *Let x, y be in $\omega^{\subseteq\omega}$.*

(i) *$\overline{x}$ and $\overline{y}$ are compatible if and only if for all but finitely many n in* $\mathrm{dom}(x) \cap \mathrm{dom}(y)$ *we have $x(n) = y(n)$.*
(ii) *If $\overline{x}$ and $\overline{y}$ are compatible then they have a greatest lower bound, $\overline{z} = \overline{x} \wedge \overline{y}$, where $z = \{(n,k) : x(n) = k = y(n)\} \cup (x \setminus y) \cup (y \setminus x)$.*
(iii) *$\overline{y} \not\preceq \overline{x}$ if and only if $\overline{x}$ and $\overline{y}$ are not compatible or $\overline{x}, \overline{y}$ are compatible but $x \setminus y$ is infinite.*

For any x in $\omega^{\subseteq\omega}$, basic neighborhoods around $\overline{x}$ are of the form $N(\overline{x}, h) = \{\overline{y} \in \nabla(\omega+1)^\omega : \overline{y} \preceq \overline{x}$ and $y \setminus x >^* h\}$, where $h \in \omega^\omega$.

LEMMA 18. *Take any x, y in $\omega^{\subseteq\omega}$ and f_x, f_y in ω^ω.*

(i) (a) *$\overline{x} \in N(\overline{x}, c_0) \subseteq \downarrow\overline{x}$, and* (b) *if $\overline{y} \in N(\overline{x}, f_x)$ and $\overline{y} \preceq \overline{z} \preceq \overline{x}$ then $\overline{z} \in N(\overline{x}, f_x)$.*
(ii) *$N(\overline{x}, f_x) \cap N(\overline{y}, f_y) \neq \emptyset$ if and only if $\overline{x} \wedge \overline{y} \in N(\overline{x}, f_x) \cap N(\overline{y}, f_y)$, if and only if $\overline{x}, \overline{y}$ are compatible, and $y \setminus x >^* f_x$ and $x \setminus y >^* f_y$.*

Proof. Claim (i)(a) is evident. Towards (i)(b), suppose $\overline{y} \preceq \overline{z}$. Then for all but finitely many n in $\mathrm{dom}(z \setminus x) \subseteq \mathrm{dom}(z)$ we have $(z \setminus x)(n) = z(n) = y(n) = (y \setminus x)(n)$. Hence if $y \setminus x >^* f_x$ then also $z \setminus x >^* f_x$, and (i)(b) follows.

For the first equivalence of (ii), note that if $\overline{z}$ is in $N(\overline{x}, f_x) \cap N(\overline{y}, f_y)$ then $\overline{z}$ is $\preceq$-below both $\overline{x}$ and $\overline{y}$. Now apply (i)(b). The second equivalence follows from the definitions. ■

If A is any subset of $\omega^{\subseteq\omega}$, write $\nabla(A)$ for the subspace $\{\overline{x} : x \in A\}$ of $\nabla(\omega+1)^\omega$, set $\nabla^*(A) = \nabla(A \cap \omega^{\subset\omega})$ and set $\nabla^+(A) = \nabla((A \cup \omega^\omega \cup \{\emptyset\})^*)$. Then $\nabla^*(\omega^{\subseteq\omega}) = \nabla(\omega^{\subset\omega})$ is ∇^* from the above, and abbreviate $\nabla(\omega^{\subseteq\omega}) = \nabla^+(\omega^{\subset\omega}) = \nabla(\omega+1)^\omega$ to ∇.

THEOREM 19. *Let A be a subset of $\omega^{\subseteq\omega}$. Then the following are equivalent:*

(1) $\Delta(A)$ *holds,*
(2) $\nabla^*(A)$ *is halvable in* ∇,
(3) $\nabla^+(A)$ *is monotonically normal in* ∇.

Proof. From Lemma 18(i) we see that ∇ with $\preceq$ and the standard basic neighborhoods satisfies conditions (a) and (b) of Lemma 3, and any subspace of ∇ is monotonically normal in ∇ if and only if a specific neighbornet is halvable. Combining this with Lemma 13 we see that to prove the equivalence of (1) through (3) it is sufficient to show: $\Delta(A)$ holds if and only if the neighbornet $T(\overline{x}) = \downarrow\overline{x}$ for x in A is halvable in ∇.

Suppose F is a function from A into ω^ω. Define the neighbornet S of $\nabla(A)$ in ∇ by $S(\overline{x}) = N(\overline{x}, f_x)$ where $f_x = F(x)$. On the other hand, suppose S is a neighbornet of $\nabla(A)$ in ∇. We may assume each $S(\overline{x})$ is basic, say $S(\overline{x}) = N(\overline{x}, f_x)$. Define $F : A \to \omega^\omega$ by $F(x) = f_x$. We show F witnesses $\Delta(A)$ if and only if S halves in ∇ the neighbornet $T(\overline{x}) = \downarrow\overline{x}$ for $\overline{x}$ in A.

First note that 'x and y in A switch', reinterpreted in terms of $\overline{x}$ and $\overline{y}$ via Lemma 17(iii), is equivalent to '$\overline{x}, \overline{y}$ are compatible, but $\overline{y} \not\preceq \overline{x}$ and $\overline{x} \not\preceq \overline{y}$'. Next, taking the contrapositive, '$S(\overline{x}) = N(\overline{x}, F(x))$ halves $T(\overline{x}) = \downarrow\overline{x}$ in ∇' is equivalent to '$\overline{y} \not\preceq \overline{x}$ and $\overline{x} \not\preceq \overline{y}$ imply $N(\overline{x}, F(x)) \cap N(\overline{y}, F(y)) = \emptyset$'.

Now applying Lemmas 17(iii) and 18(ii), we see that '$S(\overline{x}) = N(\overline{x}, F(x))$ halves $T(\overline{x}) = \downarrow\overline{x}$ in ∇' is equivalent to '($\overline{x}, \overline{y}$ not compatible) or ($\overline{x}, \overline{y}$ compatible and $x \setminus y$ infinite and $y \setminus x$ infinite) implies ($\overline{x}, \overline{y}$ not compatible) or ($x \setminus y \not>^* F(y)$ or $y \setminus x \not>^* F(x)$)', which is equivalent to 'if ($\overline{x}, \overline{y}$ compatible and $x \setminus y$ infinite and $y \setminus x$ infinite) then ($x \setminus y \not>^* F(y)$ or $y \setminus x \not>^* F(x)$)', which (by the reinterpretation of switching above) is equivalent to 'F witnesses $\Delta(A)$'. ∎

THEOREM 20. *Let A be a subset of $\omega^{\subseteq\omega}$.*

(1) *If $\Delta(A)$ then $\nabla(A)$ is monotonically normal and hereditarily paracompact.*
(2) *If $\nabla(A)$ is monotonically normal and, whenever $\overline{x}, \overline{y}$ in $\nabla(A)$ are compatible then $\overline{x} \wedge \overline{y}$ is in $\nabla(A)$, then $\Delta(A)$ holds.*

Proof. For (1) note that if $\Delta(A)$ holds, then by the preceding theorem $\nabla^+(A)$ is monotonically normal, so its subspace $\nabla(A)$ is monotonically normal and hereditarily paracompact by Theorem 8.

For (2) assume $\nabla(A)$ is closed under $\wedge$. By Lemma 18(ii), for any $\overline{x}$ and $\overline{y}$ in $\nabla(A)$ we know that one basic open set in $\nabla(A)$, say $N_{\nabla(A)}(\overline{x}, f) = N(\overline{x}, f) \cap \nabla(A)$, meets another, say $N_{\nabla(A)}(\overline{y}, g)$, if and only if they both contain $\overline{x} \wedge \overline{y}$; and so they meet (in $\nabla(A)$) if and only if the corresponding open sets in ∇, $N(\overline{x}, f)$ and $N(\overline{y}, h)$, meet (in ∇). Hence if $\nabla(A)$ is monotonically normal then it is monotonically normal in ∇, and thus, by the preceding theorem, $\Delta(A)$ holds. ∎

From Proposition 16 we deduce:

EXAMPLE 21. Let FI be the family of finite disjoint unions of increasing partial functions. Then, in ZFC, we see that $\nabla(\mathrm{FI})$ is monotonically normal, and hereditarily paracompact.

A space may be monotonically normal for 'trivial' reasons, such as having a topological property that easily implies monotone normality (metrizability, for example) or because it has an especially simple topology (the discrete topology, or one which becomes discrete after removing a single point). Indeed, it is not difficult to check that in ∇ the subspace INC is closed and $\mathrm{INC} \setminus \{\overline{\emptyset}\}$ is discrete, and so monotonically normal 'trivially'. However this is not the case for FI. For x in FI let $\mathrm{ht}(x)$ be the minimal number of partial functions in a representation of x as a disjoint union of increasing partial functions, and set $\mathrm{FI}_n = \{x : \mathrm{ht}(x) = n\}$. Then $\mathrm{FI}_0 = \{\overline{\emptyset}\}$ and $\mathrm{FI}_1 = \mathrm{INC} \setminus \{\overline{\emptyset}\}$, the increasing partial functions with infinite domain. One can verify that the closure of FI_2 contains FI_1. From Lemma 9 it follows that every point of FI_2 has uncountable character in FI. Hence FI is far from being metrizable or topologically trivial.

4.4. Another not hereditarily normal space. Denote by $X(\omega^\omega, \leq^*)$ the subspace $\omega^\omega \cup \{c_\omega\}$ of $\nabla(\omega+1)^\omega$ and write $N(c_\omega, f)_X = N(c_\omega, f) \cap X(\omega^\omega, \leq^*)$, with $f \in \omega^\omega$, the neighborhoods around c_ω in $X(\omega^\omega, \leq^*)$.

THEOREM 22. *The space $L(\omega_1) \times X(\omega^\omega, \leq^*)$ is hereditarily normal if and only if $\mathfrak{b} = \omega_1$.*

Proof. Recall that a subset L of ω^ω (ω^ω with the product topology) is a *K-Luzin* set if it is uncountable and meets every compact of ω^ω in a countable set, or equivalently, for every $g \in \omega^\omega$, the set $\{f \in L : f \leq^* g\}$ is countable. Observe that any uncountable subspace of K-Luzin is K-Luzin, hence the existence of a K-Luzin set is equivalent to $\mathfrak{b} = \omega_1$. Let $p = (\omega_1, c_\omega)$ be the top-right corner of the given product. In $X' = L(\omega_1) \times X(\omega^\omega, \leq^*) \setminus \{p\}$ the top edge, $T = L(\omega_1) \times \{c_\omega\} \setminus \{p\}$, and right edge, $R = \{\omega_1\} \times X(\omega^\omega, \leq^*) \setminus \{p\}$, are disjoint closed sets.

Note that $L(\omega_1) \times X(\omega^\omega, \leq^*)$ is hereditarily normal if and only if X' is hereditarily normal, if and only if there is an open set U containing T whose closure is disjoint from R. We prove there exists such U if and only if there is a K-Luzin set.

For sufficiency, pick f_α such that $\{\alpha\} \times N(c_\omega, f_\alpha)_X \subseteq U$ for $\alpha < \omega_1$, and for $g \in \omega^\omega$ pick a countable $C_g \subseteq D(\omega_1)$ such that $(L(\omega_1) \setminus C_g) \times \{g\} \cap U = \emptyset$.

Let $A = \{f_\alpha : \alpha < \omega_1\}$. The choice of the f_α's can be in such way that they are all distinct, so the enumeration of A is injective. We check that A is K-Luzin. Take any g in ω^ω, then for any α not in C_g, as U and V are disjoint, (α, g) is not in $\{\alpha\} \times N(c_\omega, f_\alpha)_X$, so $f_\alpha \not\leq^* g$. Therefore, $\{\alpha \in \omega_1 : f_\alpha \leq^* g\}$ is contained in C_g, and hence is countable.

For the converse, let $L = \{f_\alpha : \alpha \in \omega_1\} \subseteq \omega^\omega$ be a K-Luzin set. For every $g \in \omega^\omega$, $C_g = \{\alpha \in \omega_1 : f_\alpha \leq^* g\}$ is countable. Hence the open sets

$U = \bigcup_{\alpha\in\omega_1}\{\alpha\} \times N(c_\omega, f_\alpha)_X$ and $V = \bigcup_{g\in\omega^\omega}(L(\omega_1)\setminus C_g)\times\{g\}$ separate T and R. ■

5. Nabla products of $A(\kappa)$'s

5.1. Δ-like characterizations of monotone normality. Denote by $D(\kappa)^{\subseteq\omega}$ the set of partial functions from ω to $D(\kappa)$, and by $D(\kappa)^{\subset\omega}$ the subset of partial functions with infinite and co-infinite domain. Two elements $x, y \in D(\kappa)^{\subseteq\omega}$ *switch* if $|x\setminus y| = |y\setminus x| = \omega$ and $|\{n\in\omega : x(n), y(n)\in D(\kappa)$ and $x(n)\neq y(n)\}| < \omega$.

Definition 23. $\Delta(A(\kappa))$ is the statement: there is $F : D(\kappa)^{\subset\omega} \to ([\kappa]^{<\omega})^\omega$ such that if $x, y \in D(\kappa)^{\subset\omega}$ switch, then $(x\setminus y)(n) \in F(y)(n)$ or $(y\setminus x)(n)\in F(x)(n)$ for infinitely many $n\in\omega$.

Let $\nabla^* A(\kappa) = \{\overline{x} \in \nabla A(\kappa)^\omega : x \in D(\kappa)^{\subset\omega}\}$. For x and y in $D(\kappa)^{\subseteq\omega}$ write $\overline{y} \preceq \overline{x}$ if for all but finitely many n in $\mathrm{dom}(x)$ we have $y(n) = x(n)$. A basic neighborhood of an $\overline{x}$ in $\nabla A(\kappa)^\omega$ is $N(\overline{x}, f) = \{\overline{y}\in\nabla A(\kappa)^\omega : \overline{y}\preceq\overline{x}$ and for all but finitely many $n\in\mathrm{dom}(y\setminus x)$ we have $(y\setminus x)(n)\notin f(n)\}$, where f is in $([\kappa]^{<\omega})^\omega$.

Observe that $\omega+1$ is $A(\aleph_0)$ and that all definitions here reduce in the case $\kappa = \aleph_0$ to those in Section 4. The natural analogues of Lemmas 17 and 18 hold. Their proofs, and that of the following theorem, follow, mutatis mutandis, those for $\Delta(\omega^{\subset\omega})$ and $\nabla(\omega+1)^\omega$ in Section 4, and so are omitted.

Theorem 24. *$\Delta(A(\kappa))$ holds if and only if $\nabla A(\kappa)^\omega$ is monotonically normal, if and only if $\nabla^* A(\kappa)$ is monotonically normal, if and only if $\nabla A(\kappa)^\omega$ is halvable.*

When can we deduce from $\Delta(A(\kappa))$ that $\nabla A(\kappa)^\omega$ is (hereditarily) paracompact? Note that we cannot simply apply Theorem 8. However, for all κ we see that $\nabla A(\kappa)^\omega$ is homeomorphic to its square, and the second author [4] has shown that if the square of a space is monotonically normal then all finite powers are monotonically normal and hereditarily paracompact.

Corollary 25. *If $\Delta(A(\kappa))$ holds then $\nabla A(\kappa)^\omega$ is hereditarily paracompact, and $\square A(\kappa)^\omega$ is paracompact.*

5.2. Not hereditarily normal. Williams' result in [13] that under $\mathfrak{d} = \omega_1$, countable nabla products of compact spaces of weight no more than $\aleph_1$ are ω_1-metrizable, and hence monotonically normal, implies, in particular, that consistently $\nabla A(\omega_1)^\omega$ is monotonically normal. (We leave it to the reader to find a direct proof from Theorem 24.) We now see that this last statement is independent, and ω_1 is the largest cardinal such that $\nabla A(\kappa)^\omega$ can be monotonically normal.

Theorem 26 (Roitman [10]). *$\nabla A(\omega_2)^\omega$ is not hereditarily normal.*

Proof. Since $A(\omega_2)_\delta = L(\omega_2)$, this latter space embeds into $\nabla A(\omega_2)^\omega$. Now, as $\nabla A(\omega_2)^\omega$ is homeomorphic to its square, Lemma 4 applies. ∎

Roitman, in [10], asked: is $\nabla A(\omega_1)^\omega$ consistently not hereditarily normal?

THEOREM 27. *If $\mathfrak{b} > \omega_1$, then $\nabla A(\omega_1)^\omega$ is not hereditarily normal.*

Proof. Since $L(\omega_1) = A(\omega_1)_\delta$, both spaces $L(\omega_1)$ and $\nabla(\omega+1)^\omega$ embed into $\nabla A(\omega_1)^\omega$, and the latter is homeomorphic to its square. Hence, Theorem 22 applies. ∎

REMARK. We observe here that a claim of Roitman is incorrect. Theorem 6.1 and Proposition 6.4 in [9] claim: (1) if $\mathfrak{b} = \mathfrak{d} < \aleph_\omega$ and each X_n is compact and has weight $\leq \mathfrak{d}$, then $\nabla_n X_n$ is $\mathfrak{b}$-metrizable (and hence monotonically normal); and (2) if $\kappa < \mathfrak{b} = \mathfrak{d} < \aleph_\omega$ and the nabla product of countably many compact spaces of weight κ is $\mathfrak{b}$-metrizable, then the nabla product of countably many compact spaces of weight κ^+ is $\mathfrak{b}$-metrizable (and hence monotonically normal). Claim (2) implies claim (1) by finite induction. But both are false. Indeed, the compact spaces $A(\omega_2)$ and ω_2+1 have weight ω_2, but $\nabla A(\omega_2)^\omega$ and $\nabla(\omega_2+1)^\omega$ are not hereditarily normal as shown in Theorems 26 and 34. Hence, they cannot be κ-metrizable. In the attempted proof of claim (2) it is assumed that the nabla product under consideration is $P_{\mathfrak{b}}$, but this is false, in general, when the factors are not first countable.

6. Nabla products of ordinals

6.1. Δ-like characterizations of monotone normality. In this section we uncover a Δ-like combinatorial principle, namely $\Delta(\alpha)$, which characterizes the monotone normality of a nabla product of ordinals, $\nabla\alpha^\omega$. (For an ordinal β, write $\mathrm{Lim}(\beta)$ for the set of limit ordinals of β.)

Basic neigborhoods of an x in $\nabla\alpha^\omega$ have the form $N(x,f) = \{y :$ for all but finitely many n we have $f(n) \leq y(n) \leq x(n)$ if $x(n) \in \mathrm{Lim}(\alpha)$, and $y(n) = x(n)$ if $x(n)$ isolated$\}$, where f is in α^ω and for all but finitely many n, if $x(n)$ is a limit then $f(n) < x(n)$. Define a partial order $\preceq$ on $\nabla\alpha^\omega$ by saying $y \preceq x$ if for all but finitely many n we have $y(n) \leq x(n)$ and if $x(n)$ is isolated then $y(n) = x(n)$. Note that $\nabla\alpha^\omega$ and the above basic neighborhoods and $\preceq$ satisfy conditions (a) and (b) of Lemma 3. Hence for $\nabla\alpha^\omega$ to be monotonically normal it suffices to halve the neighbornet $T(x) = {\downarrow}x = N(x, c_0)$. Next we state the appropriate notion of 'switching' elements in this context and then $\Delta(\alpha)$.

DEFINITION 28. Let α be any ordinal and $x, y \in \nabla\alpha^\omega$. We say that x, y *switch* if for infinitely many n, $x(n) < y(n) \in \mathrm{Lim}(\alpha)$, for infinitely many n, $y(n) < x(n) \in \mathrm{Lim}(\alpha)$, and $\{n \in \omega : x(n), y(n)$ are isolated and $x(n) \neq y(n)\}$ is finite.

Definition 29. $\Delta(\alpha)$ is the statement: there is $F : \nabla\alpha^\omega \to \alpha^\omega$ such that if $x, y \in \nabla\alpha^\omega$ switch, then $y(n) < F(x)(n) < x(n)$ for infinitely many n or $x(n) < F(y)(n) < y(n)$ for infinitely many n.

Now we characterize when $\nabla\alpha^\omega$ is monotonically normal.

Proposition 30. *The following are equivalent:*

(1) $\Delta(\alpha)$ *holds,*
(2) $\nabla\alpha^\omega$ *is halvable,*
(3) $\nabla\alpha^\omega$ *is monotonically normal.*

Proof. By the discussion above, it suffices to show the equivalence of (1) and the following clause:

(2′) *the neighbornet* $T(x) = {\downarrow}x$ *is halvable.*

For (1)⇒(2′), suppose F is a witness of $\Delta(\alpha)$. Define $S(x) = N(x, F(x))$. We check that S halves $T(x) = {\downarrow}x = N(x, c_0)$.

Take any x and y. Suppose $x \notin N(y, c_0)$ and $y \notin N(x, c_0)$. Various cases arise, but in all of them we show $S(x)$ and $S(y)$ are disjoint. If the set $\{n \in \omega : x(n), y(n) \text{ are isolated and } x(n) \neq y(n)\}$ is infinite, then $S(x)$ and $S(y)$ are trivially disjoint. Hence, suppose it is finite. Then the sets $N_y = \{n \in \omega : x(n) < y(n)\}$ and $N_x = \{n \in \omega : y(n) < x(n)\}$ are both infinite. Now, if there are infinitely many $n \in N_y$ such that $y(n)$ is isolated, then $[0, x(n)] \cap \{y(n)\} = \emptyset$, and thus, $S(x)$ and $S(y)$ are disjoint; and likewise if there are infinitely many $n \in N_x$ such that $x(n)$ is isolated. Assume, then, that for all but finitely many $n \in N_y$ and $m \in N_x$, $x(m), y(n) \in \mathrm{Lim}(\alpha)$. That is, x and y switch. By $\Delta(\alpha)$, we have $S(x) \cap S(y) = \emptyset$.

For (2′)⇒(1), consider the neighbornet $T(x) = N(x, c_0)$. Then there is a neighborhood assignment S that halves T. For $x \in \nabla\alpha^\omega$, let $F(x) \in \alpha^\omega$ be such that $N(x, F(x)) \subseteq S(x)$. To see that F satisfies $\Delta(\alpha)$, pick x, y that switch. This implies $x \notin N(y, c_0)$ and $y \notin N(x, c_0)$, hence by halvability, $N(x, F(x)) \cap N(y, F(y)) = \emptyset$. Now it is clear that for infinitely many $n \in \omega$, $y(n) < F(x)(n) < x(n)$ or $x(n) < F(y)(n) < y(n)$. ∎

As we argued for Corollary 25 we deduce:

Corollary 31.

(1) *If* $\Delta(\alpha)$ *holds then* $\nabla\alpha^\omega$ *is hereditarily paracompact.*
(2) *If* $\Delta(\alpha + 1)$ *holds then* $\Box(\alpha + 1)^\omega$ *is paracompact.*

It is important to understand the relationship between $\Delta(\alpha)$ and $\Delta(\beta)$, and especially the strength of Roitman's $\Delta = \Delta(\omega + 1)$. Clearly if $\beta \geq \alpha$ then $\Delta(\beta) \Rightarrow \Delta(\alpha)$ (because monotone normality is hereditary and $\nabla\alpha^\omega$ embeds in $\nabla\beta^\omega$). The next two lemmas give a way to step up.

LEMMA 32. *If α is an ordinal, then* $\nabla(\alpha.\omega)^\omega = \bigoplus\{\nabla_n I_n : (I_n)_n \in \mathcal{I}^\omega\}$ *where* $\mathcal{I} = \{[0,\alpha]\} \cup \{(\alpha.n, \alpha.(n+1)] : n \in \omega\}$.

Proof. The sets in $\mathcal{I}$ form an open partition of $\alpha.\omega$. From 'open' we see that each $\nabla_n I_n$ is open in $\nabla(\alpha.\omega)^\omega$. In turn, from 'partition', and the fact that we take every sequence of members of $\mathcal{I}$, we see that the $\nabla_n I_n$ partition $\nabla(\alpha.\omega)^\omega$. ∎

LEMMA 33. *If* $\Delta(\alpha+1)$ *holds, then* $\Delta(\alpha.\omega)$ *holds.*

Proof. Observe that each $\nabla_n I_n$ from the preceding lemma is homeomorphic to $\nabla(\alpha+1)^\omega$, which is monotonically normal under $\Delta(\alpha+1)$ (Proposition 30). Since a disjoint sum of monotonically normal spaces is monotonically normal, we can apply Proposition 30 again to complete the proof. ∎

6.2. Not hereditarily normal. As seen above, under $\mathfrak{d} = \omega_1$, we have $\nabla\alpha^\omega$ monotonically normal for all $\alpha < \omega_2$. The next two results provide a sharp contrast.

THEOREM 34. *The space* $\nabla(\omega_2+1)^\omega$ *is not hereditarily normal.*

Proof. Let $S = E^{\omega_2}_{\omega_1} = \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$. Then S is a stationary subset of ω_2. Note that $\overline{S} = E^{\omega_2}_{\omega_1} \cup \{\omega_2\}$ and its G_δ-modification $\overline{S}_\delta$ are equal. Observe that S and $\overline{S}$ both embed into $\nabla(\omega_2+1)^\omega$ (Lemma 6). Since $\nabla(\omega_2+1)^\omega$ is homeomorphic to its square, to complete the proof, apply Lemma 5. ∎

THEOREM 35. *If* $\mathfrak{b} > \omega_1$, *then* $\nabla(\omega_1+1)^\omega$ *is not hereditarily normal.*

Proof. Let L be the subspace of ω_1+1 consisting of the isolated points along with ω_1. Then $L_\delta = L$ is homeomorphic to $L(\omega_1)$, and so both $L(\omega_1)$ and $\nabla(\omega+1)^\omega$ embed into $\nabla(\omega_1+1)^\omega$, which is homeomorphic to its square. Hence, Theorem 22 applies. ∎

7. Nabla products of metrizable spaces

7.1. Δ-like characterizations of monotone normality. For this section, $\{(X_n, d_n) : n \in \omega\}$ will be a family of metric spaces. For $x, y \in \nabla_n X_n$ and $f \in \omega^\omega$, define $N(x,f) = \nabla_n B_n(x(n), 1/f(n))$ and $M(x,f;y) = \{n \in \omega : y(n) \notin B_n(x(n), 1/f(n))\}$, where $B_n(a,\varepsilon)$ is $\{a\}$ if a is isolated, and is otherwise the ε-ball in the metric d_n.

We say that $(x,f), (y,g) \in \nabla_n X_n \times \omega^\omega$ *switch* if $M(x,f;y)$ and $M(y,g;x)$ are almost disjoint infinite sets. Observe that switching elements (x,f) and (y,g) imply $y \notin N(x,f)$ and $x \notin N(y,g)$.

DEFINITION 36. Let $\{(X_n, d_n) : n \in \omega\}$ be a family of metric spaces. Then $\Delta((X_n, d_n)_n)$ is the statement: there is $F : \nabla_n X_n \times \omega^\omega \to \omega^\omega$, write

$f_x := F(x, f)$, such that if $(x, f), (y, g) \in \mathrm{dom}(F)$ switch, then $1/f_x(n) + 1/g_y(n) < d_n(x(n), y(n))$ for infinitely many $n \in \omega$.

The conclusion here, namely $1/f_x(n) + 1/g_y(n) < d_n(x(n), y(n))$, implies that $B_n(x(n), 1/f_x(n))$ and $B_n(y(n), 1/g_y(n))$ are disjoint.

PROPOSITION 37.

(1) *If $\Delta((X_n, d_n)_n)$ holds then $\nabla_n X_n$ is monotonically normal.*
(2) *If $\nabla_n X_n$ is monotonically normal, where each X_n is metrizable, then $\Delta((X_n, d_n)_n)$ holds for any choice of compatible metrics d_n for X_n.*

Proof. (1) Let F witness $\Delta((X_n, d_n)_n)$ and define an operator G by letting $G(x, N(x, f))$ be $N(x, \max\{2f, f_x\})$. We prove that G is a monotone normality operator. First observe that $x \in N(x, \max\{2f, f_x\}) \subseteq N(x, f)$. Now, to prove the second property of monotone normality, let $x, y \in \nabla_n X_n$, $f, g \in \omega^\omega$ and assume that $y \notin N(x, f)$ and $x \notin N(y, g)$ then we have to prove that $G(x, f) \cap G(y, g) = \emptyset$. There are three cases for the sets $M(x, f; y)$ and $M(y, g; x)$:

- $M(x, f; y)$ or $M(y, g; x)$ is finite: if $M(x, f; y)$ is finite, then by its definition, $y(n) \in B_n(x(n), 1/f(n))$ for all but finitely many $n \in \omega$. Hence, $y \in N(x, f)$, which is impossible by our assumption.
- $M(x, f; y) \cap M(y, g; x)$ is infinite: let $Z = M(x, f; y) \cap M(y, g; x)$. Then for every $n \in Z$, $y(n) \notin B_n(x(n), 1/f(n))$ and $x(n) \notin B_n(y(n), 1/g(n))$. By triangle inequality, $B_n(x(n), 1/(2f(n))) \cap B_n(y(n), 1/(2g(n))) = \emptyset$ for $n \in Z$. Thus, $G(x, f) \cap G(y, g) = \emptyset$.
- $M(x, f; y)$, $M(y, g; x)$ are infinite almost disjoint sets: this means that $(x, f), (y, g)$ switch. By $\Delta((X_n, d_n)_n)$, $1/f_x(n) + 1/g_y(n) < d_n(x(n), y(n))$ for infinitely many $n \in \omega$. That is, for infinitely many $n \in \omega$, the sets $B_n(x(n), 1/f_x(n))$ and $B_n(y(n), 1/g_y(n))$ are disjoint, which implies that $G(x, f) \cap G(y, g) = \emptyset$.

This concludes the proof of (1).

(2) Now, assume that $\nabla_n X_n$ is monotonically normal with operator G. For each n let d_n be the given compatible metric on X_n. Define $F : \nabla_n X_n \times \omega^\omega \to \omega^\omega$ by $F(x, f) = f_x$ such that $N(x, f_x) \subseteq G(x, N(x, f))$.

We prove that F witnesses $\Delta((X_n, d_n)_n)$. Choose switching elements $(x, f), (y, g) \in \mathrm{dom}(F)$. Then $y \notin N(x, f)$ and $x \notin N(y, g)$. Since G is a monotone normality operator, $G(x, N(x, f)) \cap G(y, N(y, g)) = \emptyset$, which implies $N(x, f_x) \cap N(y, g_y) = \emptyset$. So, $B_n(x(n), 1/f_x(n)) \cap B_n(y(n), 1/g_y(n)) = \emptyset$ for infinitely many $n \in \omega$, and for these n's we have the inequality $1/f_x(n) + 1/g_y(n) < d_n(x(n), y(n))$, as desired. ■

For a sequence $(X_n)_n$ of metrizable spaces, define $\Delta((X_n)_n)$ to mean '$\Delta((X_n, d_n)_n)$ holds for some choice of compatible metrics d_n'. It follows from the preceding result that $\Delta((X_n)_n)$ is equivalent to '$\Delta((X_n, d_n)_n)$ holds for *any* choice of compatible metrics d_n'. Further, for a class $\mathcal{C}$ of spaces, $\Delta(\mathcal{C})$ means '$\Delta((X_n)_n)$ holds for any sequence $(X_n)_n$ of spaces from $\mathcal{C}$'.

Write $\mathcal{M}$ for the class of all metrizable spaces, $\mathcal{M}(\kappa)$ for the class of all metrizable spaces of cardinality $\leq \kappa$, and $\mathcal{SM}$ for the class of separable metrizable spaces.

7.2. Consistency of Δ-like principles

PROPOSITION 38. *If $\mathfrak{b} = \mathfrak{d}$ then $\Delta(\mathcal{M})$ holds, if $\mathfrak{d} = \mathfrak{c}$ then $\Delta(\mathcal{M}(\mathfrak{c}))$ holds, and if* MH *holds then $\Delta(\mathcal{SM})$ holds.*

Proof. We deal with each case in turn.

CASE OF $\mathfrak{b} = \mathfrak{d}$. Let $\{f_\alpha : \alpha < \mathfrak{d}\}$ be a scale such that f_α $\leq^*$-dominates $\{2f_\beta : \beta < \alpha\}$. Let $\downarrow f_\alpha = \{f \in \omega : f \leq^* f_\alpha\}$ and define $F : \nabla_n X_n \times \omega^\omega \to \omega^\omega$ as $F(x, f) = 2f_\alpha$ if and only if α is the least such that $f \in \downarrow f_\alpha$.

Pick switching elements $(x, f), (y, g) \in \nabla_n X_n \times \omega^\omega$. We may assume that $f \in \downarrow f_\alpha$ and $g \in \downarrow f_\beta$ for minimum β, α and $\beta \leq \alpha$. Then $g \leq^* f_\beta \leq^* 2f_\beta \leq^* f_\alpha$. Now, if $n \in M(y, g; x)$ then $x(n) \notin B_n(y(n), g(n))$, that is, $1/g(n) < d_n(y(n), x(n))$. Hence, $1/(2g(n)) + 1/(2f_\alpha(n)) < d_n(y(n), x(n))$, which implies $1/(2f_\beta(n)) + 1/(2f_\alpha(n)) < d_n(y(n), x(n))$, as desired.

CASE OF $\mathfrak{d} = \mathfrak{c}$. If X_n is metrizable of size no more than $\mathfrak{c}$, for $n \in \omega$, then we have $|\nabla_n X_n| = \mathfrak{d}$. Enumerate $\nabla_n X_n \times \omega^\omega = \{(x_\alpha, f_\alpha) : \alpha < \mathfrak{d}\}$. Fix $\alpha < \mathfrak{d}$ and suppose F is constructed satisfying $\Delta((X_n)_n)$ on $\{(x_\beta, f_\beta) : \beta < \alpha\}$ and $F(x_\beta, f_\beta) \geq^* 2f_\beta$ for every $\beta < \alpha$. The sets $\mathcal{F} = \{2f_\beta : \beta < \alpha\}$ and $\mathcal{A} = \{M(x_\beta, f_\beta; x_\alpha) : \beta < \alpha\}$ have size less than $\mathfrak{d}$. Lemma 1 applies, giving $f'_\alpha \in \omega^\omega$ such that $2f_\beta \restriction M(x_\beta, f_\beta; x_\alpha) \not\geq^* f'_\alpha$ for $\beta < \alpha$. Define $F(x_\alpha, f_\alpha) = 2\max\{f_\alpha, f'_\alpha\}$. This construction completes F on $\nabla_n X_n \times \omega^\omega$. To see that F witness $\Delta((X_n)_n)$, pick switching elements $(x_\beta, f_\beta), (x_\alpha, f_\alpha)$, and suppose $\beta < \alpha$. Let $M = \{n \in M(x_\beta, f_\beta; x_\alpha) : f'_\alpha(n) > 2f_\beta\}$, which is infinite. Hence, for $n \in M$, $x_\alpha(n) \notin B_n(x_\beta(n), 1/f_\beta(n))$. It is clear that for $n \in M$, $1/F(x_\beta, f_\beta)(n) + 1/F(x_\alpha, f_\alpha)(n) < d_n(x_\beta(n), x_\alpha(n))$.

CASE OF MH. Every separable metrizable space embeds into the Hilbert cube $[0, 1]^\omega$, which is isomorphic to a subset of $H(\omega_1)$. Hence if $(X_n)_n$ is a sequence of separable metrizable spaces, then we can suppose $\nabla_n X_n \subseteq H(\omega_1)$.

Let H_α be as in MH (Definition 2) and f_α be a witness that $H_\alpha \cap \omega^\omega$ is not dominating. We may assume that $H_\alpha \subseteq H_{\alpha+1}$ and that $f_\alpha \in H_{\alpha+1}$. Define $F : \nabla_n X_n \times \omega^\omega \to \omega^\omega$ as $F(x, f) = 2f_\alpha$ if and only if α is the least such that $(x, f) \in H_\alpha$. Choose switching elements $(x, f), (y, g) \in \mathrm{dom}(F)$. Then $(x, f) \in H_\beta, (y, g) \in H_\alpha$ for minimum $\beta \leq \alpha$. Since the H_α's are elementary submodels, $2g, 2f, M(x, f; y), M(y, g; f)$ are in H_α. Also, for any $h \in H_\alpha \cap \omega^\omega$ and $a \in H_\alpha \cap [\omega]^\omega$, $h \restriction a \not\geq^* f_\alpha$. Hence, $2g \restriction M(y, g; x) \not\geq^* f_\alpha$, which implies that there is an infinite set $M \subseteq M(y, g; x)$ such that for

$n \in M$, $2g(n) \leq f_\alpha(n)$ and $x(n) \notin B_n(y(n), 1/g(n))$. As a consequence, for every $n \in M$, $1/f_\alpha(n) + 1/(2g(n)) < d_n(x(n), y(n))$. We conclude that for $n \in M$, $1/F(x,f)(n) + 1/F(y,g)(n) < d_n(x(n), y(n))$. ∎

8. Open problems. The most basic open question, related to this paper, is that of Roitman:

QUESTION 39. *Is* $\neg\Delta$ *consistent? Or is* Δ *true in* ZFC*?*

Suppose $\Delta = \Delta(\omega + 1)$ were true in ZFC, so $\nabla(\omega + 1)^\omega$ is monotonically normal. Then it seems implausible to the authors that $\nabla(\omega.\omega + 1)^\omega$ would not also be monotonically normal, in other words $\Delta(\omega.\omega + 1)$ could be consistently false. If that is correct then, in ZFC, it should be possible to deduce $\Delta(\omega.\omega + 1)$ from $\Delta(\omega + 1)$.

PROBLEM 40. *Show, in* ZFC, *that* $\Delta(\omega + 1) \Rightarrow \Delta(\omega.\omega + 1)$.

We know $\nabla A(\omega_1)^\omega$ is monotonically normal under $\mathfrak{d} = \omega_1$; and that $\nabla A(\omega_1)^\omega$ monotonically normal implies both Δ and $\mathfrak{b} = \omega_1$. This leaves a gap.

PROBLEM 41. *Do* $\mathfrak{b} = \omega_1$ *and* Δ *imply that the space* $\nabla A(\omega_1)^\omega$ *is monotonically normal? Alternatively, if* $\nabla A(\omega_1)^\omega$ *is monotonically normal, is it true that* $\mathfrak{d} = \omega_1$*?*

We have seen that Roitman's Δ is equivalent to $\nabla(\omega + 1)^\omega$ being monotonically normal. Hence Δ is a sufficient condition for $\nabla(\omega + 1)^\omega$ to be hereditarily paracompact. A natural question is then: is it necessary? But a more fundamental problem is to find necessary *combinatorial* conditions for $\nabla(\omega + 1)^\omega$ to be paracompact. That would open a path to showing the independence of the box product problem.

PROBLEM 42. *Find combinatorial properties* (*in the style of* Δ) *implied by '*$\nabla(\omega + 1)^\omega$ *is paracompact'.*

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