

Weak type $(1, 1)$ estimates for maximal functions along 1-regular sequences of integers

by

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Abstract. We show the pointwise convergence of the averages

$$\mathcal{A}_N f(x) = \frac{1}{\#\mathbf{B}_N} \sum_{n \in \mathbf{B}_N} f(x+n)$$

for $f \in \ell^1(\mathbb{Z})$ where $\mathbf{B}_N = \mathbf{B} \cap [1, N]$, and \mathbf{B} is a 1-regular sequence of integers, for example $\mathbf{B} = \{\lfloor n \log n \rfloor : n \in \mathbb{N}\}$.

1. Introduction. Let (X, \mathcal{B}, μ, T) be an ergodic dynamical system, that is, (X, \mathcal{B}, μ) is a σ -finite measure space with a measurable and measure preserving transformation $T : X \rightarrow X$. The classical Birkhoff theorem [1] says that for any $f \in L^p(X, \mu)$, $p \geq 1$, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x)$$

exists for μ -almost all $x \in X$. This classical result motivated others to study ergodic averages of the form

$$(1.1) \quad \frac{1}{N} \sum_{n=1}^N f(T^{a_n} x)$$

for various unbounded sets $\{a_n : n \in \mathbb{N}\}$ of integers and $f \in L^p(X, \mu)$, $p \in [1, \infty)$. In particular, in his PhD thesis Wierdl [9, Theorem 4.4] proved that the averages corresponding to $a_n = \lfloor n \log n \rfloor$ converge μ -almost everywhere for functions in $L^p(X, \mu)$, $p \in (1, \infty)$. On the other hand, Rosenblatt [7, Remark 27] showed that for every aperiodic dynamical system (X, \mathcal{B}, μ, T) there is a function in $L^p(X, \mu)$, $p \geq 1$, such that the averages along the orbit of the sequence $a_n = n \lfloor \log n \rfloor$ do not converge on a set of positive

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measure. The latter motivated Rosenblatt and Wierdl [8, Conjecture 4.1] to conjecture that for all sets $\{a_n : n \in \mathbb{N}\}$ of integers having zero Banach density, and any aperiodic dynamical system (X, \mathcal{B}, μ, T) , there is a function $f \in L^1(X, \mu)$ such that the ergodic averages (1.1) do not converge μ -almost everywhere.

However, in [3] Buczolich constructed inductively a sophisticated example disproving the conjecture. Nowadays, thanks to Mirek's article [6], a wide class of concrete examples modeled on certain c -regular sequences, $c \in (1, 30/29)$, is known for which the Rosenblatt–Wierdl conjecture does not hold.

In this paper, we show how to complete the picture by covering the case $c = 1$, thus answering a question posed in [6]. In particular, we find that the ergodic averages for the sequence $a_n = \lfloor n \log n \rfloor$ converge μ -almost everywhere for all functions in $L^1(X, \mu)$, which is in sharp contrast with Rosenblatt's observation [7, Remark 27].

Let us highlight the difference between discrete and continuous averaging operators. In the 1980s Bourgain [2] established the pointwise convergence of the ergodic averages along the sequence $a_n = n^2$ for functions in $L^p(X, \mu)$, $p > 1$. At that time it was not clear what one should expect at the endpoint $p = 1$. The continuous counterpart has the form

$$(1.2) \quad \frac{1}{t} \int_0^t f(x - y^2) \, dy.$$

Observe that by a simple change of variables, we get

$$\begin{aligned} \left| \frac{1}{t} \int_0^t f(x - y^2) \, dy \right| &= \left| \frac{1}{2t} \int_0^{t^2} f(x - y) \frac{dy}{\sqrt{y}} \right| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{2t} \int_{2^{-n-1}t^2}^{2^{-n}t^2} |f(x - y)| \frac{dy}{\sqrt{y}} \\ &\leq \sum_{n=0}^{\infty} 2^{-n/2} \cdot \sup_{r>0} \frac{1}{r} \int_0^r |f(x - y)| \, dy. \end{aligned}$$

Therefore, by the classical Hardy–Littlewood maximal inequality, the maximal function corresponding to the averages (1.2) satisfies weak type $(1, 1)$ estimates. This might suggest that the same holds true for the discrete case, but in 2010 Bucholich and Mauldin [4] showed that the maximal function corresponding to the averages along $a_n = n^2$ is *not* of weak type $(1, 1)$. This illustrates that the phenomena occurring in the discrete setting may completely differ from what is known for the continuous analogues, mainly due to their arithmetic nature.

Before we formulate our result, let us recall the class of c -regular functions we are interested in. Denote by \mathcal{L} the family of slowly varying functions $\ell : [x_0, \infty) \rightarrow (0, \infty)$ such that

$$\ell(x) = \exp\left(\int_{x_0}^x \frac{\vartheta(t)}{t} dt\right)$$

where $\vartheta \in \mathcal{C}^2([x_0, \infty))$ is a real function satisfying

$$\lim_{x \rightarrow \infty} \vartheta(x) = 0, \quad \lim_{x \rightarrow \infty} x\vartheta'(x) = 0, \quad \lim_{x \rightarrow \infty} x^2\vartheta''(x) = 0.$$

In \mathcal{L} we distinguish a subfamily \mathcal{L}_0 consisting of slowly varying functions $\ell : [x_0, \infty) \rightarrow (0, \infty)$ such that

$$\ell(x) = \exp\left(\int_{x_0}^x \frac{\vartheta(t)}{t} dt\right)$$

where $\vartheta \in \mathcal{C}^2([x_0, \infty))$ is a positive decreasing real function satisfying

$$\lim_{x \rightarrow \infty} \vartheta(x) = 0, \quad \lim_{x \rightarrow \infty} \frac{x\vartheta'(x)}{\vartheta(x)} = 0, \quad \lim_{x \rightarrow \infty} \frac{x^2\vartheta''(x)}{\vartheta(x)} = 0,$$

and for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that $1 \leq C_\varepsilon \vartheta(x)x^\varepsilon$ and $\lim_{x \rightarrow \infty} \ell(x) = \infty$. Finally, for every $c \in [1, 30/29)$ let \mathcal{R}_c be the family of increasing, convex, regularly-varying functions $h : [x_0, \infty) \rightarrow [1, \infty)$ of the form

$$h(x) = x^c L(x)$$

where $L \in \mathcal{L}$. If $c = 1$ we require $L \in \mathcal{L}_0$.

We are interested in sets \mathbf{B} of integers of the form

$$\mathbf{B} = \{[h(m)] : m \in \mathbb{N}\}$$

where h is a fixed function belonging to \mathcal{R}_c . In view of [6, Lemma 2.14], there is $\delta > 0$ so that

$$\#\mathbf{B}_N = \varphi(N)(1 + \mathcal{O}(N^{-\delta}))$$

where $\mathbf{B}_N = \mathbf{B} \cap [1, N]$, and φ is the inverse function to h . We study the pointwise convergence of the ergodic averages

$$\mathcal{A}_N f(x) = \frac{1}{\#\mathbf{B}_N} \sum_{n \in \mathbf{B}_N} f(T^n x).$$

In view of [6, Section 4], it is enough to show the weak type (1, 1) maximal estimates for \mathcal{A}_N . Thanks to the Calderón transference principle [5] we reduce the problem to the model dynamical system, that is, the integers \mathbb{Z} with the counting measure and the shift operator. In this context, for a function

$f \in \ell^1(\mathbb{Z})$ we set

$$\mathcal{A}_N f(x) = \frac{1}{\#\mathbf{B}_N} \sum_{n \in \mathbf{B}_N} f(x+n).$$

2. The result. Our aim is to show the following theorem.

THEOREM 2.1. *Let $c = 1$. There is $C > 0$ such that for all $f \in \ell^1(\mathbb{Z})$,*

$$(2.1) \quad \sup_{\lambda > 0} \lambda \cdot \#\left\{x \in \mathbb{Z} : \sup_{N \in \mathbb{N}} |\mathcal{A}_N f(x)| > \lambda\right\} \leq C \|f\|_{\ell^1}.$$

In view of [6, Section 5], to obtain (2.1) it is sufficient to prove [6, Lemma 5.1], which is the subject of the following lemma. In fact, its proof is valid not only for $c = 1$ but for the whole range of $c \in [1, 30/29)$.

LEMMA 2.2. *There is $C > 0$ such that for all $N \in \mathbb{N}$ and $x \in \mathbb{Z}$ such that $0 < |x| \leq \varphi(N)$, we have*

$$(2.2) \quad K_N * \check{K}_N(x) \leq C/N$$

where

$$K_N = \frac{1}{\#\mathbf{B}_N} \sum_{N/4 < n \leq N} \delta_n \mathbf{1}_{\mathbf{B}}(n)$$

and $\check{K}_N(x) = K_N(-x)$.

Proof. Since $K_N * \check{K}_N$ is symmetric, we can restrict attention to $0 < x < \varphi(N)$. Then

$$(2.3) \quad \begin{aligned} K_N * \check{K}_N(x) &= \frac{1}{(\#\mathbf{B}_N)^2} \sum_{N/4 < n, m \leq N} \delta_n * \delta_{-m}(x) \mathbf{1}_{\mathbf{B}}(n) \mathbf{1}_{\mathbf{B}}(m) \\ &= \frac{1}{(\#\mathbf{B}_N)^2} \sum_{N/4 < n, m \leq N} \delta_{n-m}(x) \mathbf{1}_{\mathbf{B}}(n) \mathbf{1}_{\mathbf{B}}(m) \\ &= \frac{1}{(\#\mathbf{B}_N)^2} \sum_{N/4 < n, n+x \leq N} \mathbf{1}_{\mathbf{B}}(n) \mathbf{1}_{\mathbf{B}}(x+n). \end{aligned}$$

Hence, our aim is to estimate the cardinality of the set

$$M(x, N) = \left\{ \frac{1}{4}N < n \leq N : \frac{1}{4}N < n+x \leq N \text{ and } n, n+x \in \mathbf{B} \right\}.$$

Notice that $n \in \mathbf{B}_N \setminus \mathbf{B}_{N/4}$ implies that there is $m \in \mathbb{N}$ such that $n = \lfloor h(m) \rfloor$, thus

$$\frac{1}{4}N < h(m) \leq N+1, \quad \text{that is, } \varphi(N/4) < m \leq \varphi(2N).$$

Next, $n+x \in \mathbf{B}_N \setminus \mathbf{B}_{N/4}$ translates into

$$n+x = \lfloor h(k) \rfloor \quad \text{for some } k \in \mathbb{N} \text{ with } \varphi(N/4) < k \leq \varphi(2N).$$

Hence,

$$\begin{aligned} x &\leq h(k) - n \leq h(k) - h(m) + 1, \\ x &> h(k) - n - 1 \geq h(k) - h(m) - 1. \end{aligned}$$

Therefore, the cardinality of $M(x, N)$ is bounded by the number of pairs $(k, m) \in \mathbb{N}^2$ such that

$$(2.4) \quad \begin{cases} \varphi(N/4) \leq k, m \leq \varphi(2N), \\ x - 1 \leq h(k) - h(m) < x + 1. \end{cases}$$

To improve the counting in [6, Lemma 5.1], for a given $m \in \mathbb{N}$ satisfying

$$\varphi(N/4) \leq m \leq \varphi(2N),$$

we estimate the number of $0 < s \leq \varphi(2N)$ such that

$$x - 1 \leq h(m + s) - h(m) \leq x + 1.$$

Let $g(s) = h(m + s) - h(m)$. Observe that g is unbounded, increasing and $g(0) = 0$. Therefore, there are $s_1 \leq s_2$ such that

$$g(s_1 - 1) < x - 1 \leq g(s_1) \leq g(s_2) \leq x + 1 < g(s_2 + 1).$$

Our aim is to estimate $s_2 - s_1 + 1$. Let us compute the difference $g(s_2 + 1) - g(s_1)$. By the mean value theorem there is $\xi \in [0, 1]$ such that

$$\begin{aligned} g(s_2 + 1) - g(s_1) &= h(m + s_2 + 1) - h(m) - h(m + s_1) + h(m) \\ &= h(m + s_2 + 1) - h(m + s_1) = h'(m + s_1 + \xi). \end{aligned}$$

Hence ⁽¹⁾,

$$g(s_2 + 1) - g(s_1) \simeq h'(\varphi(N)) \simeq N/\varphi(N),$$

where the last estimate follows by [6, Lemma 2.1]. Thus

$$s_2 - s_1 + 1 \simeq \varphi(N)/N,$$

and so

$$\#M(x, N) \leq C\varphi(N) \frac{\varphi(N)}{N}$$

for some $C > 0$. Now, by (2.3),

$$K_N * \check{K}_N(x) \leq \frac{C}{(\#\mathbf{B}_N)^2} \frac{\varphi(N)^2}{N},$$

which together with $\#\mathbf{B}_N \simeq \varphi(N)$ leads to (2.2). ■

⁽¹⁾ We write $A \simeq B$ if there are absolute constants $C_2 \geq C_1 > 0$ such that $C_1 A \leq B \leq C_2 A$.

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