STUDIA MATHEMATICA

Online First version

On the Uniform Mazur Intersection Property

by

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Dedicated to Professor Bor-Luh (Peter) Lin on his 85th birthday

Abstract. We show that a Banach space X has the Uniform Mazur Intersection Property (UMIP) if and only if every $f \in S(X^*)$ is a uniformly w*-semidenting point of $B(X^*)$. We also prove an analogous result for the uniform version of the w*-MIP.

1. Introduction. The Mazur Intersection Property (MIP)—every closed bounded convex set is the intersection of closed balls containing it—is an extremely well studied property in Banach space theory. MathSciNet search with keywords "Mazur('s) Intersection Property" returns 53 hits. The paper [8] is an excellent survey of the MIP, its generalisations and variants. A complete characterisation was obtained in [7], the most well-known criterion stating that the w*-denting points of $B(X^*)$ are norm dense in $S(X^*)$. The paper [7] also considered the property in dual spaces that every w*-compact convex set is the intersection of balls (w*-MIP). In [4], Chen and Lin introduced the notion of w*-semidenting points and showed that a Banach space X has the MIP if and only if every $f \in S(X^*)$ is a w*-semidenting point of $B(X^*)$. Among subsequent papers, we mention [6].

A much less studied uniform version of the MIP (UMIP or UI) was introduced by Whitfield and Zizler [9]. MathSciNet finds only one citation of [9], namely [8], in the last 30+ years. Characterisations similar to [7] were also obtained, but an analogue of the w*-denting point criterion was missing, which is perhaps a reason for its being less pursued.

In this paper, we show that a Banach space X has the UMIP if and only if every $f \in S(X^*)$ is a uniformly w*-semidenting point of $B(X^*)$, thus filling a long felt gap. In the process, we present simpler proofs of some

2020 *Mathematics Subject Classification*: Primary 46B20.

Key words and phrases: Mazur Intersection Property, w*-(semi)denting points.

Received 29 November 2020; revised 24 December 2020.

Published online 15 March 2021.

DOI: 10.4064/sm201129-4-1

characterisations in [9]. We also introduce a w*-version of the UMIP in the spirit of [7] and obtain similar characterisations.

We hope that our results will also be a small step towards answering the long standing open question whether the UMIP implies that the space is super-reflexive.

2. Characterisation of the UMIP. We consider real Banach spaces only. Let X be a Banach space. For $x \in X$ and $r > 0$, we denote by $B(x,r)$ the *open ball* $\{y \in X : \|x-y\| < r\}$ and by $B[x,r]$ the *closed ball* $\{y \in X : \|x-y\| \leq r\}$. We denote by $B(X)$ the *closed unit ball* $\{x \in X : \|x\| \leq 1\}$ and by $S(X)$ the *unit sphere* $\{x \in X : \|x\| = 1\}$.

For $x \in S(X)$, we denote by $D(x)$ the set $\{f \in S(X^*) : f(x) = 1\}$. Any selection of D is called a *support mapping.*

For a bounded set $C \subseteq X$ and $x \in X$, let

$$d(x,C) = \inf\{\|x-z\| : z \in C\},$$
$$\operatorname{diam}(C) = \sup\{\|x-y\| : x,y \in C\}.$$

Definition 2.1 ([9]). We say that a Banach space X has the *Uniform Mazur Intersection Property* (*UMIP*) if for every $\varepsilon > 0$ and $M(\varepsilon) \geq 2$, there is $K(\varepsilon) > 0$ such that whenever a closed convex set $C \subseteq X$ and a point $p \in X$ are such that $\operatorname{diam}(C) \leq M(\varepsilon)$ and $d(p,C) \geq \varepsilon$, there is a closed ball $B \subseteq X$ of radius $\leq K(\varepsilon)$ such that $C \subseteq B$ and $d(p,B) \geq \varepsilon/2$.

Remark 2.2. In the original definition in [9], $M(\varepsilon) = 1/\varepsilon$.

Definition 2.3. A *slice* of $B(X)$ determined by $f \in S(X^*)$ is a set of the form

$$S(B(X),f,\delta) := \{x \in B(X) : f(x) > 1-\delta\}$$

for some $0 < \delta < 1$. For $x \in S(X)$, $S(B(X^*),x,\delta)$ is called a *w^*-slice* of $B(X^*)$.

We say that $x \in S(X)$ is a *denting point* of $B(X)$ if for every $\varepsilon > 0$, x is contained in a slice of $B(X)$ of diameter less than ε. A *w^*-denting point* of $B(X^*)$ is defined similarly.

For w*-semidenting points, we use an equivalent definition.

Definition 2.4. $f \in S(X^*)$ is said to be a *w^*-semidenting point* of $B(X^*)$ if for every $\varepsilon > 0$, there exists a w*-slice $S(B(X^*),x,\delta) \subseteq B(f,\varepsilon)$. A semidenting point of $B(X)$ can be defined similarly.

We begin with a variant of Phelps' Parallel Hyperplane Lemma [6, Lemma 2.1]. We include the well-known proof as we will use one of its steps later.

Lemma 2.5. *For a normed linear space X, if $\{x \in B(X) : f(x) > \varepsilon\} \subseteq \{x \in X : g(x) > 0\}$ for $f, g \in S(X^*)$ and $\varepsilon > 0$, then $\|f - g\| \leq 2\varepsilon$.*

Proof. By the given condition, if $x \in B(X)$ and $g(x) = 0$, then $|f(x)| \leq \varepsilon$. That is, $\|f|_{\ker(g)}\| \leq \varepsilon$. By the Hahn–Banach Theorem, there exists $h \in X^*$ such that $\|h\| \leq \varepsilon$ and $h \equiv f$ on $\ker(g)$. It follows that $f - h = tg$ for some $t \in \mathbb{R}$. Then $\|f - tg\| = \|h\| \leq \varepsilon$. Now, if $y \in \{x \in B(X) : f(x) > \varepsilon\}$, then $g(y) > 0$ and

$$\varepsilon < f(y) = (f - tg)(y) + tg(y) \leq \|f - tg\| + tg(y) \leq \varepsilon + tg(y).$$

It follows that $t > 0$. Furthermore,

$$|1 - t| \leq \big|\|f\| - \|tg\|\big| \leq \|f - tg\| \leq \varepsilon.$$

Thus,

$$\|f - g\| \leq \|f - tg\| + |1 - t| \leq \varepsilon + \varepsilon = 2\varepsilon. \ \blacksquare$$

The theorem below is equivalent to [5, Theorem 2.1] with a much simpler proof.

Theorem 2.6. *Let X be a Banach space. Let $A \subseteq X^{**}$ be a bounded set. Then there exists a closed ball $B^{**} \subseteq X^{**}$ with centre in X such that $A \subseteq B^{**}$ and $0 \notin B^{**}$ if and only if $d(0, A) > 0$ and there is a w^*-slice of $B(X^*)$ contained in*

$$\{f \in B(X^*) : x^{**}(f) > 0 \textit{ for all } x^{**} \in A\}.$$

Proof. Let $A \subseteq B^{**}[x_0, r]$ and $0 \notin B^{**}[x_0, r]$. Then $\|x_0\| > r$. Clearly, $d(0, A) \geq \|x_0\| - r > 0$.

Let $S = \{f \in B(X^*) : f(x_0) > r\}$. Then S is a w*-slice of $B(X^*)$. And if $g \in S$, then for any $x^{**} \in A$,

$$g(x_0 - x^{**}) \leq \|x_0 - x^{**}\| \leq r \quad \text{and hence,} \quad x^{**}(g) \geq g(x_0) - r > 0.$$

Thus,

$$S \subseteq \{f \in B(X^*) : x^{**}(f) > 0 \text{ for all } x^{**} \in A\}.$$

Conversely, let $d = d(0, A) > 0$ and let $x_0 \in S_X$ and $0 < \varepsilon < 1$ be such that

$$\{f \in B(X^*) : f(x_0) > \varepsilon\} \subseteq \{f \in B(X^*) : x^{**}(f) > 0 \text{ for all } x^{**} \in A\}.$$

Let $M = \sup\{\|x^{**}\| : x^{**} \in A\}$. By the proof of Lemma 2.5, for all $x^{**} \in A$, there exists $t \in \mathbb{R}$ such that $1 - \varepsilon \leq t \leq 1 + \varepsilon$ and

$$\left\| \frac{t x^{**}}{\|x^{**}\|} - x_0 \right\| \leq \varepsilon.$$

Then for $\lambda \geq M/(1-\varepsilon)$,

$$\|x^{**} - \lambda x_0\| \leq \left\| x^{**} - \frac{\|x^{**}\|}{t} x_0 \right\| + \left| \frac{\|x^{**}\|}{t} - \lambda \right| \leq \frac{\varepsilon \|x^{**}\|}{t} + \lambda - \frac{\|x^{**}\|}{t}$$
$$= \lambda - \frac{\|x^{**}\|}{t}(1-\varepsilon) \leq \lambda - \frac{d(1-\varepsilon)}{1+\varepsilon}.$$

Therefore,

$$A \subseteq B^{**}\left[\lambda x_0, \lambda - \frac{d(1-\varepsilon)}{1+\varepsilon}\right] \quad \text{and clearly,} \quad 0 \notin B^{**}\left[\lambda x_0, \lambda - \frac{d(1-\varepsilon)}{1+\varepsilon}\right]. \ \blacksquare$$

REMARK 2.7. From Theorem 2.6, it can be shown that if every $f \in S(X^*)$ is a w*-semidenting point of $B(X^*)$, then X has the MIP. See (b)⇒(a) in Theorem 2.11 below.

DEFINITION 2.8. For $\varepsilon, \delta > 0$ and $x \in S(X)$, denote

$$d_1(x,\delta) = \sup_{0<\lambda<\delta,\, y \in B(X)} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda},$$
$$d_2(x,\delta) = \operatorname{diam}\big(S(B(X^*), x, \delta)\big),$$
$$d_3(x,\delta) = \operatorname{diam}\big(D\big(S(X) \cap B(x,\delta)\big)\big).$$

DEFINITION 2.9 ([9]). For $\varepsilon, \delta > 0$, define

$$M_{\varepsilon,\delta}(X) = \left\{ x \in S(X) : \sup_{0<\|y\|<\delta} \frac{\|x+y\| + \|x-y\| - 2}{\|y\|} < \varepsilon \right\}.$$

In other words, $M_{\varepsilon,\delta}(X) = \{x \in S(X) : d_1(x,\delta) < \varepsilon\}$.

The lemma below is quantitatively more precise than [9, Lemma 1]. It follows from [1, Lemma 2.1]. We include the details for completeness.

LEMMA 2.10. *For any $\alpha, \delta > 0$, we have*

(i) $d_2(x,\alpha) \leq d_1(x,\delta) + 2\alpha/\delta$,
(ii) $d_3(x,\delta) \leq d_2(x,\delta)$,
(iii) $d_1(x,\delta) \leq d_3(x, 2\delta)$.

Proof. (i) Fix α, $\delta > 0$. Let $d_2 = d_2(x,\alpha)$. For each $n \geq 1$, we can choose $f_n, g_n \in S(B(X^*), x, \alpha)$ such that $\|f_n - g_n\| > d_2 - 1/n$. Choose $y_n \in B(X)$ such that $(f_n - g_n)(y_n) > d_2 - 1/n$. Then

$$\frac{\|x+\delta y_n\| + \|x - \delta y_n\| - 2}{\delta} \geq \frac{f_n(x+\delta y_n) + g_n(x - \delta y_n) - 2}{\delta} \geq d_2 - \frac{1}{n} - \frac{2\alpha}{\delta}.$$

Thus, $d_1(x,\delta) \geq d_2 - 2\alpha/\delta$.

(ii) Let $y \in S(X) \cap B(x,\delta)$ and $f_y \in D(y)$. Then

$$0 \leq 1 - f_y(x) = f_y(y-x) \leq \|y-x\| < \delta.$$

Thus, $f_y \in S(B(X^*), x, \delta)$. Therefore, $D(S(X) \cap B(x,\delta)) \subseteq S(B(X^*), x, \delta)$.

(iii) Let $\lambda \leq \delta$. Observe that for any $y \in B(X)$,

$$\left\| \frac{x \pm \lambda y}{\|x \pm \lambda y\|} - x \right\| \leq \left\| \frac{x \pm \lambda y}{\|x \pm \lambda y\|} - (x \pm \lambda y) \right\| + \lambda = \big|1 - \|x \pm \lambda y\|\big| + \lambda$$
$$= \big|\|x\| - \|x \pm \lambda y\|\big| + \lambda \leq 2\lambda.$$

Let

$$f \in D\left(\frac{x + \lambda y}{\|x + \lambda y\|} \right) \quad \text{and} \quad g \in D\left(\frac{x - \lambda y}{\|x - \lambda y\|} \right).$$

Then

$$\frac{\|x + \lambda y\| + \|x - \lambda y\| - 2}{\lambda} = \frac{f(x + \lambda y) + g(x - \lambda y) - 2}{\lambda}$$
$$\leq (f - g)(y) \leq \|f - g\|. \blacksquare$$

We now come to our main theorem. The conditions (c)–(e) are reformulations of (ii)–(iv) of [9, Theorem 1] in the language of [7, Theorem 2.1]. We give a self-contained proof of (a)⇔(b), our main result. The proofs of (c)⇒(e) and (c)⇔(d) are essentially from [9, Theorem 1], which we include for the sake of completeness and because we will need them in the next section. For the rest, our arguments are either new or simpler.

THEOREM 2.11. *For a Banach space X, the following are equivalent:*

(a) *X has the UMIP.*

(b) *Every $f \in S(X^*)$ is* uniformly w*-semidenting*, i.e., given $\varepsilon > 0$, there exists $0 < \delta < 1$ such that for any $f \in S(X^*)$, there exists $x \in S(X)$ such that*

$$S(B(X^*), x, \delta) \subseteq B(f, \varepsilon).$$

(c) *The duality map is* uniformly quasicontinuous*, i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $f \in S(X^*)$, there exists $x \in S(X)$ such that $D(S(X) \cap B(x, \delta)) \subseteq B(f, \varepsilon)$.*

(d) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that every support mapping maps a δ-net in $S(X)$ to an ε-net in $S(X^*)$.*

(e) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $f \in S(X^*)$, there exists $x \in M_{\varepsilon,\delta}(X)$ such that $D(x) \subseteq B(f, \varepsilon)$.*

Proof. (a)⇒(b). We use an idea from [6, Lemma 2.2(ii)].

Let $0 < \varepsilon < 1$ and $M(\varepsilon) \geq 2$ be given. Choose $K = K(\varepsilon)$ as given by (a) for $\varepsilon/3$. We may assume that $K \geq 1$.

Let $f \in S(X^*)$. Consider $C := \{x \in B(X) : f(x) \geq \varepsilon/3\}$. Then $d(0, C) \geq \varepsilon/3$. Also, $\operatorname{diam}(C) \leq 2 \leq M(\varepsilon)$.

So, we have $B = B[x_0, r]$ containing C, $r \leq K$ and $d(0, B) \geq \varepsilon/6$. Since $d(0, B[x_0, r]) \geq \varepsilon/6$, it follows that $\|x_0\| \geq r + \varepsilon/6 > r + \varepsilon/9$.

CLAIM. $S := S(B(X^*), x_0/\|x_0\|, 1 - K/(K + \varepsilon/9)) \subseteq B(f, \varepsilon)$, *that is,* (b) *holds for $x = x_0/\|x_0\|$ and $\delta = 1 - K/(K + \varepsilon/9)$.*

Let $g \in S$. Since $r/(r+\varepsilon/9) \le K/(K+\varepsilon/9)$, $g(x_0/\|x_0\|) > r/(r+\varepsilon/9)$, which implies $g(x_0) > r\|x_0\|/(r+\varepsilon/9) > r$. That is, $\inf g(B) > 0$. So,

$$\{x \in B(X) : f(x) > \varepsilon/3\} \subseteq C \subseteq B \subseteq \{x : g(x) > 0\}.$$

By Lemma 2.5, $\big\|f - g/\|g\|\big\| \le 2\varepsilon/3$. Also, $g \in S$ implies $\|g\| \ge g(x_0/\|x_0\|) > K/(K+\varepsilon/9)$. Hence,

$$\|f-g\| \le \big\|f - g/\|g\|\big\| + (1-\|g\|) \le 2\varepsilon/3 + \varepsilon/(9K+\varepsilon) < \varepsilon.$$

This proves the Claim.

(b)⇒(a). Given $\varepsilon > 0$ and $M(\varepsilon) \ge 2$, it suffices to show that there is $K(\varepsilon) > 0$ such that whenever a closed convex set $C \subseteq X$ satisfies $\operatorname{diam}(C) \le M(\varepsilon)$ and $d(0,C) \ge \varepsilon$, there is a closed ball $B \subseteq X$ of radius $\le K(\varepsilon)$ such that $C \subseteq B$ and $d(0,B) \ge \varepsilon/2$.

Let $\varepsilon > 0$ be given. Let $L = M(\varepsilon) + \varepsilon$. Choose $0 < \delta < 1$ for $\varepsilon/(4L)$ obtained from (b). Let $K = L/\delta + 1$. We will show that this K works.

Case I: $C \setminus B[0, M(\varepsilon)+\varepsilon/2] \ne \emptyset$. Choose $z \in C \setminus B[0, M(\varepsilon)+\varepsilon/2]$. Then $C \subseteq B[z, M(\varepsilon)]$, $d(0, B[z, M(\varepsilon)]) \ge \varepsilon/2$ and $M(\varepsilon) \le K$.

Case II: $C \subseteq B[0, M(\varepsilon)+\varepsilon/2]$. Define $D = \overline{C + \frac{\varepsilon}{2}B(X)}$. Then $D \subseteq B[0, L]$ and $d(0, D) \ge \varepsilon/2$, and hence D is disjoint from $B(0, \varepsilon/2)$. By the separation theorem, there exists $f \in S(X^*)$ such that $\inf f(D) \ge \varepsilon/2$. By choice of δ, there exists $x_0 \in S(X)$ such that

$$S(B(X^*), x_0, \delta) \subseteq B(f, \varepsilon/(4L)).$$

It follows that if $g \in B(X^*) \cap B(f, \varepsilon/(4L))$ and $z \in D$, then $g(z) \ge f(z) - \|f-g\|\,\|z\| \ge \varepsilon/2 - \varepsilon/4 = \varepsilon/4 > 0$. Therefore,

$$\begin{aligned} S(B(X^*), x_0, \delta) &\subseteq B(X^*) \cap B(f, \varepsilon/(4L)) \\ &\subseteq \{g \in B(X^*) : g(x) > 0 \text{ for all } x \in D\}. \end{aligned}$$

By the proof of Theorem 2.6, for $K \ge \lambda \ge L/\delta$,

$$D \subseteq B\left[\lambda x_0, \lambda - \frac{d(0,D)\delta}{2-\delta}\right].$$

It follows that

$$C \subseteq B = B\left[\lambda x_0, \lambda - \frac{\varepsilon}{2} - \frac{d(0,D)\delta}{2-\delta}\right] \quad \text{and} \quad d(0,B) = \frac{\varepsilon}{2} + \frac{d(0,D)\delta}{2-\delta} \ge \frac{\varepsilon}{2}.$$

(b)⇒(c). As observed in the proof of Lemma 2.10(ii), $D(S(X) \cap B(x,\delta)) \subseteq S(B(X^*), x, \delta)$.

(c)⇒(e). Let $\varepsilon > 0$ be given. Then from (c) for $\varepsilon/3$, there exists $\delta > 0$ such that for any $f \in S(X^*)$, there exists $x \in S(X)$ such that $D(S(X) \cap B(x, 2\delta)) \subseteq B(f, \varepsilon/3)$. This implies $d_3(x, 2\delta) \le 2\varepsilon/3 < \varepsilon$. By Lemma 2.10(iii), $d_1(x,\delta) < \varepsilon$, i.e., $x \in M_{\varepsilon,\delta}(X)$. Finally, $D(x) \subseteq B(f, \varepsilon/3) \subseteq B(f, \varepsilon)$.

(e)⇒(b). Let $\varepsilon > 0$ be given. By (e), there exists $\delta > 0$ such that for every $f \in S(X^*)$, there is an $x \in M_{\varepsilon/4,\delta}(X)$ such that $D(x) \subseteq B(f, \varepsilon/4)$. So, $d_1(x, \delta) < \varepsilon/4$. By Lemma 2.10(i), $d_2(x, \alpha) < \varepsilon/2$ for $\alpha < \delta\varepsilon/8$. That is, $\operatorname{diam}(S(B(X^*), x, \alpha)) < \varepsilon/2$. Now, $D(x) \subseteq S(B(X^*), x, \alpha)$. Hence for any $g \in S(B(X^*), x, \alpha)$ and $f_x \in D(x)$,

$$\|f - g\| \le \|f - f_x\| + \|f_x - g\| \le \varepsilon/4 + \varepsilon/2 < \varepsilon.$$

Therefore,

$$S(B(X^*), x, \alpha) \subseteq B(f, \varepsilon).$$

(c)⇒(d) is obvious.

(d)⇒(c). If (c) does not hold, then there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $f_\delta \in S(X^*)$ such that for all $x \in S(X)$, there exist $z_x \in S(X)$ and $f_{z_x} \in D(z_x)$ such that $\|z_x - x\| < \delta/2$ and $\|f_\delta - f_{z_x}\| \ge \varepsilon$. Then $\{z_x : x \in S(X)\}$ is a δ-net, but $\{f_{z_x} : x \in S(X)\}$ is not an ε-net in $S(X^*)$. ■

3. Characterisation of the w*-UMIP. In this section, we study the w*-version of the UMIP defined above.

Definition 3.1. We say that a dual Banach space X^* has the *w*-Uniform Mazur Intersection Property* (*w*-UMIP*) if for every $\varepsilon > 0$ and $M(\varepsilon) \ge 2$, there is $K(\varepsilon) > 0$ such that whenever a w*-compact convex set $C \subseteq X^*$ and a point $f \in X^*$ are such that $\operatorname{diam}(C) \le M(\varepsilon)$ and $d(f, C) \ge \varepsilon$, there is a closed ball $B \subseteq X^*$ of radius $\le K(\varepsilon)$ such that $C \subseteq B$ and $d(f, B) \ge \varepsilon/2$.

For $f \in S(X^*)$, the inverse duality mapping is defined as

$$D^{-1}(f) = \begin{cases} \{x \in S(X) : f(x) = 1\} & \text{if } f \in D(S(X)), \\ \emptyset & \text{if } f \notin D(S(X)). \end{cases}$$

The fact that $D^{-1}(f) = \emptyset$ for some $f \in S(X^*)$ introduces some technicalities in the discussion of the w*-UMIP.

Definition 3.2. For $\varepsilon, \delta > 0$ and $f \in S(X^*)$, denote

$$d_1^*(f, \delta) = \sup_{0<\lambda<\delta,\, g\in B(X^*)} \frac{\|f + \lambda g\| + \|f - \lambda g\| - 2}{\lambda},$$

$$d_2^*(f, \delta) = \operatorname{diam}\big(S(B(X), f, \delta)\big),$$

$$d_3^*(f, \delta) = \operatorname{diam}\big(D^{-1}\big(D(S(X)) \cap B(f, \delta)\big)\big).$$

Lemma 3.3. *For any $\alpha, \delta > 0$, we have*

(i) $d_2^*(f, \alpha) \le d_1^*(f, \delta) + 2\alpha/\delta$,

(ii) $d_3^*(f, \delta) \le d_2^*(f, \delta)$,

(iii) $d_1^*(f, \alpha) \le \dfrac{d_3^*(f, \delta) + 2\alpha}{1 - \alpha^2}$ *if* $\alpha < \sqrt{\delta + 1} - 1$.

Proof. (i) follows from Lemma 2.10(i).

(ii) holds since $D^{-1}(D(S(X)) \cap B(f,\delta)) \subseteq S(B(X), f, \delta)$.

(iii) Let $d_3^* = d_3^*(f,\delta)$. Choose $0 < \alpha < \sqrt{\delta+1} - 1$. Then $\alpha^2 + 2\alpha < \delta$. Let $g \in S(X^*)$, $0 < \lambda < \alpha$. As before,

$$\left\| \frac{f \pm \lambda g}{\|f \pm \lambda g\|} - f \right\| \leq 2\lambda.$$

Find $h_1, h_2 \in D(S(X))$ such that

$$\left\| \frac{f + \lambda g}{\|f + \lambda g\|} - h_1 \right\| \leq \lambda^2 \quad \text{and} \quad \left\| \frac{f - \lambda g}{\|f - \lambda g\|} - h_2 \right\| \leq \lambda^2$$

and $x_1, x_2 \in S(X)$ such that $h_i \in D(x_i)$, $i = 1, 2$.

Observe that $\|h_i - f\| \leq \lambda^2 + 2\lambda \leq \alpha^2 + 2\alpha < \delta$, i.e., $h_1, h_2 \in D(S(X)) \cap B(f,\delta)$. Thus $\|x_1 - x_2\| \leq d_3^*$. Now,

$$0 \leq 1 - x_1\left(\frac{f + \lambda g}{\|f + \lambda g\|}\right) = x_1\left(h_1 - \frac{f + \lambda g}{\|f + \lambda g\|}\right) \leq \left\| \frac{f + \lambda g}{\|f + \lambda g\|} - h_1 \right\| \leq \lambda^2.$$

So, $x_1(f + \lambda g) \geq (1 - \lambda^2)\|f + \lambda g\|$. Similarly, $x_2(f - \lambda g) \geq (1 - \lambda^2)\|f - \lambda g\|$. Therefore,

$$\begin{aligned}
\frac{\|f + \lambda g\| + \|f - \lambda g\| - 2}{\lambda} &\leq \frac{x_1(f + \lambda g) + x_2(f - \lambda g) - 2(1 - \lambda^2)}{\lambda(1 - \lambda^2)} \\
&= \frac{(x_1 + x_2)(f) - 2 + \lambda(x_1 - x_2)(g) + 2\lambda^2}{\lambda(1 - \lambda^2)} \\
&\leq \frac{\|x_1 - x_2\| + 2\lambda}{1 - \lambda^2} \leq \frac{d_3^* + 2\lambda}{1 - \lambda^2} \leq \frac{d_3^* + 2\alpha}{1 - \alpha^2}
\end{aligned}$$

(since $(d_3^* + 2\lambda)/(1 - \lambda^2)$ is increasing in λ). Thus,

$$d_1^*(f,\alpha) \leq \frac{d_3^* + 2\alpha}{1 - \alpha^2}. \blacksquare$$

Remark 3.4. The proof of (iii) is adapted from that of [2, Lemma 2]. Notice that unlike the proof of [7, Lemma 3.1], our proof uses just the Bishop–Phelps Theorem and not Bollobás' extensions.

We now come to our characterisation theorem. Though the proof is similar to that of Theorem 2.11, there are added technicalities as pointed out earlier.

Theorem 3.5. *For a Banach space X, the following are equivalent:*

(a) *X^* has the w^*-UMIP.*

(b) *Every $x \in S(X)$ is* uniformly semidenting, *i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in S(X)$, there exists $f \in S(X^*)$ such that*

$$S(B(X), f, \delta) \subseteq B(x, \varepsilon).$$

(c) *The inverse duality map is* uniformly quasicontinuous, *i.e., given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any* $x \in S(X)$, *there exists* $f \in D(S(X))$ *such that* $D^{-1}(D(S(X)) \cap B(f,\delta)) \subseteq B(x,\varepsilon)$.

(d) *Given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that every support mapping on* X^* *that maps* $D(S(X))$ *into* $S(X)$ *maps any* $\Delta \subseteq D(S(X))$ *that is a* δ*-net in* $S(X^*)$ *to an* ε*-net in* $S(X)$.

(e) *Given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for every* $x \in S(X)$, *there exists* $f \in M_{\varepsilon,\delta}(X^*) \cap D(S(X))$ *such that* $D^{-1}(f) \subseteq B(x,\varepsilon)$.

Proof. (a)⇒(b) is similar to (a)⇒(b) in Theorem 2.11.

(b)⇒(a). Define $D = C + \frac{\varepsilon}{2}B(X^*)$. Then D is w*-compact, and $d(0,D) \geq \varepsilon/2$. Therefore, D is disjoint from the w*-compact convex set $(3\varepsilon/8)B(X^*)$. Hence there exists $x \in S(X)$ such that $\inf \hat{x}(D) \geq 3\varepsilon/8$.

The rest of the argument is similar to (b)⇒(a) in Theorem 2.11.

(b)⇒(c). Let $\varepsilon > 0$ be given. By (b), there exists $\delta > 0$ such that for any $x \in S(X)$, there exists $g \in S(X^*)$ such that

$$S(B(X), g, 2\delta) \subseteq B(x,\varepsilon).$$

As in [3, Lemma 3.2.6], if we choose $0 < \eta < \delta/4$, then for any $h \in B(g,\eta)$,

$$S(B(X), h, \delta) \subseteq S(B(X), g, 2\delta).$$

By the Bishop–Phelps Theorem, there exists $f \in D(S(X)) \cap B(g,\eta)$. Then

$$S(B(X), f, \delta) \subseteq S(B(X), g, 2\delta) \subseteq B(x,\varepsilon).$$

And

$$D^{-1}(D(S(X)) \cap B(f,\delta)) \subseteq S(B(X), f, \delta).$$

Therefore, (c) holds.

(c)⇒(e). Let $\varepsilon > 0$ be given. Then from (c) for $\varepsilon/4$, there exists $\delta > 0$ such that for any $x \in S(X)$, there exists $f \in D(S(X))$ with

$$D^{-1}(D(S(X)) \cap B(f,\delta)) \subseteq B(x,\varepsilon/4).$$

This implies $d_3^*(f,\delta) \leq \varepsilon/2$. By Lemma 3.3(iii),

$$d_1^*(f,\alpha) \leq \frac{d_3^*(f,\delta) + 2\alpha}{1-\alpha^2} \leq \frac{\varepsilon/2 + 2\alpha}{1-\alpha^2} < \varepsilon$$

$$\text{if } \alpha < \sqrt{\delta+1} - 1 \text{ and } \frac{\varepsilon/2 + 2\alpha}{1-\alpha^2} < \varepsilon.$$

That is, $f \in M_{\varepsilon,\alpha}(X^*)$. Clearly, the choice of α depends only on ε. And $D^{-1}(f) \subseteq B(x,\varepsilon/4) \subseteq B(x,\varepsilon)$.

(e)⇒(b). Let $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ obtained from (e) for $\varepsilon/4$. So, given $x \in S(X)$, there is $f \in M_{\varepsilon/4,\delta}(X^*) \cap D(S(X))$ such that $D^{-1}(f) \subseteq B(x,\varepsilon/4)$. As $f \in M_{\varepsilon/4,\delta}(X^*)$, $d_1^*(f,\delta) < \varepsilon/4$. By Lemma 3.3, $d_2^*(f,\alpha) < \varepsilon/2$ for $0 < \alpha < \varepsilon\delta/8$. That is, $\operatorname{diam}(S(B(X), f, \alpha)) < \varepsilon/2$.

Also, $D^{-1}(f) \subseteq B(x, \varepsilon/4)$ and $f \in D(S(X))$. So for any $y \in S(B(X), f, \alpha)$ and $z \in D^{-1}(f)$,

$$\|x-y\| \leq \|x-z\| + \|y-z\| \leq \varepsilon/4 + \varepsilon/2 < \varepsilon.$$

Therefore,

$$S(B(X), f, \alpha) \subseteq B(x, \varepsilon).$$

(c)⇒(d) is obvious.

(d)⇒(c). If (c) does not hold, then there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x_\delta \in S(X)$ such that for all $f \in D(S(X))$, there exists $g_f \in D(S(X))$ and $y_f \in D^{-1}(g_f)$ with $\|f - g_f\| < \delta/2$ and $\|x_\delta - y_f\| \geq \varepsilon$.

Define $\Delta = \{g_f : f \in D(S(X))\}$. Clearly, Δ forms a $\delta/2$-net in $D(S(X))$. Since $D(S(X))$ is dense in $S(X^*)$, Δ forms a δ-net in $S(X^*)$. Define a support mapping $\Phi : D(S(X)) \to S(X)$ as follows:

- If $g \in \Delta$, we choose any $f \in D(S(X))$ such that $g = g_f$ and define $\Phi(g) = y_f$. Observe that, by definition of Δ, there is at least one such $f \in D(S(X))$.
- If $g \in D(S(X)) \setminus \Delta$, define $\Phi(g) = y$ for some $y \in D^{-1}(g)$.

Now, as $\|x_\delta - \Phi(g)\| \geq \varepsilon$ for all $g \in \Delta$, $\Phi(\Delta)$ does not form an ε-net in $S(X)$. So, (d) does not hold. ∎

Acknowledgements. We thank the referee for a careful reading of our paper and for some detailed and helpful comments that have considerably improved the paper's readability.

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