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Some results on the lattice of closed ideals of $\mathcal{L}^r(X)$ for X of the form $(\bigoplus_i \ell_p^i)_q$

by

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Abstract. We study the lattice of closed (order and algebra) ideals of $\mathcal{L}^r(X)$ when X is a Banach lattice of the form $(\bigoplus_i \ell_p^i)_q$ ($p \in [1,\infty]$, $q \in [1,\infty) \cup \{0\}$ & $p \neq q$). We show that for every such X, $\mathcal{L}^r(X)$ has a unique maximal (order and algebra) ideal. For $1 < p < \infty$ and $q \in \{0,1\}$, we show, in particular, that the lattice of closed (order and algebra) ideals of $\mathcal{L}^r(X)$ contains at least five distinct ideals.

1. Introduction. It is well-known that if X is a Banach lattice then the linear span of the positive operators on X, under composition and endowed with the so-called regular norm (see below for definitions), is a Banach algebra. We shall denote the latter space by $\mathcal{L}^r(X)$. It is also well-known that, when X is Dedekind complete, $\mathcal{L}^r(X)$ is a Banach lattice with respect to the order induced by the cone of positive operators.

An important feature of any Banach algebra is its lattice of closed (two-sided) ideals. In the case of $\mathcal{L}^r(X)$, it has become apparent through the work of various authors that the right kind of algebra ideal to consider are those which are also order ideals. Accordingly, from here onwards, whenever applied to the algebra $\mathcal{L}^r(X)$, the term *ideal*, without any prefixes, will always mean order and (two-sided) algebra ideal.

Interest in understanding the structure of the lattice of closed ideals of $\mathcal{L}^r(X)$ goes at least as far back as the work of Arendt and Sourour [ASo] and of Arendt and Schwarz [ASc], where it is shown that for $X = \ell_p$ $(1 \le p < \infty)$ and $X = c_0$, the algebra $\mathcal{L}^r(X)$ has a unique non-trivial proper closed (order and algebra) ideal, namely, the norm-closure of the ideal of finite-rank operators on X. These results can be thought of as the order counterpart of the classical result of Gohberg, Markus and Feldman, regarding the lattice of closed ideals of the algebras $\mathcal{B}(\ell_p)$ $(1 \le p < \infty)$ and $\mathcal{B}(c_0)$. Apart from ℓ_p $(1 \le p < \infty)$ and c_0,

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we are unaware of any other Banach lattice X for which the lattice of closed ideals of $\mathcal{L}^r(X)$ has been completely described, or of any attempt at describing the structure of the lattice of closed ideals of $\mathcal{L}^r(X)$, for an atomic X, since [ASc].

We should mention here, though, that individual classes of compact-like operators between Banach lattices (which could be thought of as analogues of operator ideals, in the sense of Pietsch, but with respect to the spaces $\mathcal{L}^r(X,Y)$; see below for definition) have been extensively studied since the seminal work of Dodds and Fremlin [DF]. These results, although relevant to our research, are not the kind of results this note is concerned with, and are probably too many for us to survey here in a proper and fair way. Instead, results about various classes of regular operators, required in the present work, will be explained when needed. The interested reader, unfamiliar with the theory of regular operators on Banach lattices, will find in the classical monographs [AB], [Sc], [MN] and [Za] an excellent place to start.

On the other hand, much research has been carried out over the past two decades into the structure of the lattice of closed two-sided algebra ideals of $\mathcal{B}(X)$ since Laustsen, Loy and Read added, in [LLR], the space $(\bigoplus_i \ell_2^i)_0$ to the above list of Banach spaces for which the lattice of closed ideals of $\mathcal{B}(X)$ is completely understood. Although little has grown the list since then (the lattice of closed ideals of $\mathcal{B}((\bigoplus_i \ell_2^i)_1)$ is completely described in [LSZ] and, to our knowledge, no other Banach lattice has been added to the list), significant advances have been made in other directions, in particular, in connection with Banach spaces of the forms $(\bigoplus_i \ell_p^i)_q$ and $\ell_p \oplus \ell_q$ $(1 \le p \ne q \le \infty)$.

This note is a first step towards bringing our current knowledge about the lattice of closed ideals of $\mathcal{L}^r(X)$, at least for atomic X, up to speed with that of $\mathcal{B}(X)$. Our main concern will be Banach lattices of the form $(\bigoplus_i \ell_p^i)_q$. However, the results of the note apply, more generally, to any Banach lattice of the form $(\bigoplus_i \ell_p^{n_i})_q$ with (n_i) unbounded (see the end of the next section for a sketch of the argument).

It should also be noted that, unlike in the Banach space situation, two Banach lattices $(\bigoplus_i \ell_p^i)_q$ and $(\bigoplus_i \ell_s^i)_t$ are isometrically Riesz isomorphic if and only if $p = q$ and $s = t$, so for instance, $(\bigoplus_i \ell_2^i)_p$ and ℓ_p are isomorphic as Banach spaces, but not as Banach lattices, unless $p = 2$. In turn, by [Wi2, Theorem 3], $\mathcal{L}^r((\bigoplus_i \ell_p^i)_q)$ and $\mathcal{L}^r((\bigoplus_i \ell_s^i)_t)$ are isomorphic as Banach lattice algebras (i.e., Banach algebras which are also Banach lattices and whose positive cones are closed under the algebra products) if and only if $p = q$ and $s = t$.

The note is organized as follows. We start with some background material and terminology, in the next section. In Section 3, we show that the algebras $\mathcal{L}^r((\bigoplus_i \ell_p^i)_q)$ $(p \in [1,\infty],\ q \in [1,\infty) \cup \{0\}$ and $p \ne q)$ have unique maximal ideals and also unique maximal algebra ideals (see our convention above regarding the use of the term *ideal*). Finally, in Section 4, we show that for every $1 < p < \infty$ the lattice of closed ideals of $\mathcal{L}^r((\bigoplus_i \ell_p^i)_0)$ (resp. $\mathcal{L}^r((\bigoplus_i \ell_p^i)_1)$) contains at least five distinct ideals. Furthermore, we show that every closed non-trivial proper ideal, different from the ideal of approximable regular operators, must contain the norm-closure of the ideal of regular operators that factor through c_0 (resp. ℓ_1) with regular factors.

2. Background and terminology. Given a Banach space X, we write X' for its topological dual and $X_{[\rho]}$ for the closed ball in X centered at the origin and of radius ρ. Given a subset S of X we denote by $\mathrm{sp}(S)$ its linear span. If E is a subspace of X, we write $\overline{E}$ for its norm-closure in X. Furthermore, if $(x_i)_{i\in\mathbb{N}}$ is a sequence in X, we write $[x_i]_{i\in\mathbb{N}}$ for the norm-closure of its linear span. In the case of a finite sequence $(x_i)_{i=1}^n$, we may write also $[x_i]_{i=1}^n$ for $\mathrm{sp}(\{x_i : 1 \le i \le n\})$. If X is a Banach lattice, we denote by X_+ its positive cone.

Recall a complex Banach lattice X is defined as the complexification $X_\mathbb{R} + iX_\mathbb{R}$ of a real Banach lattice $X_\mathbb{R}$, with $|\cdot|$ and $\|\cdot\|$ given by $|\xi + i\eta| := \sup_{\theta\in[0,2\pi]} |(\cos\theta)\xi + (\sin\theta)\eta|$ ($\xi, \eta \in X_\mathbb{R}$) and $\|x\| := \big\||x|\big\|_{X_\mathbb{R}}$ ($x \in X$), respectively. It follows that if $E \subset X$ is an order ideal, then $E = E_\mathbb{R} + iE_\mathbb{R}$, where $E_\mathbb{R} \subset X_\mathbb{R}$ is the image of E under the natural projection onto the real component of X.

For a linear map $T : X \to Y$, and a linear subspace $E \subset X$ (resp. $F \subset Y$, containing the image of T), we write $T|_E$ (resp. $T|^F$) for its restriction to E (resp. corestriction to F, i.e., the map $x \mapsto Tx$, $X \to F$). We write $T(X)$ for the image of T, and if X and Y are Banach spaces, the topological adjoint of T is denoted by T'. Given $\phi \in X'$ and $\xi \in Y$, we write $\phi\otimes\xi$ for the linear map from X to Y defined by $(\phi\otimes\xi)(x) := \phi(x)\xi$ ($x \in X$). The identity map on a vector space X is denoted by id_X, or just by id if X is clear from context.

By a *positive* linear map between Banach lattices X and Y, we shall mean a linear map $T : X \to Y$ such that $T(X_+) \subseteq Y_+$. We shall write $\mathcal{L}^r(X, Y)$ for the linear span of the positive linear maps from X to Y. As customary, elements of $\mathcal{L}^r(X, Y)$ will be called *regular operators*. For any pair X, Y of Banach lattices, $\mathcal{L}^r(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$ (:= the Banach space of all bounded linear operators from X to Y). Endowed with the so-called regular norm $\|T\|_r := \inf\{\|S\| : S \in \mathcal{B}(X, Y) \text{ and } |Tx| \le S|x|\,(x \in X)\}$ ($T \in \mathcal{L}^r(X, Y)$), $\mathcal{L}^r(X, Y)$ becomes a Banach space; furthermore, when Y is Dedekind complete, $(\mathcal{L}^r(X, Y), \|\cdot\|_r)$ is a Banach lattice and $\|T\|_r = \big\||T|\big\|$ ($T \in \mathcal{L}^r(X, Y)$) (see for instance, [MN, Proposition 1.3.6] and [Ar, Section 1]). Unless otherwise stated, we shall always assume $\mathcal{L}^r(X, Y)$ to be endowed with the norm $\|\cdot\|_r$. If $X = Y$, we shall write $\mathcal{L}^r(X)$ instead of $\mathcal{L}^r(X, X)$.

Since the main concern of the note is with those algebra ideals of $\mathcal{L}^r(X)$ which are also order ideals, we can restrict ourselves to the case of real Banach lattices—the complex case can be readily obtained from the real case by complexification. To explain this, recall that if X and Y are complex Banach lattices such that $\mathcal{L}^r(X_{\mathbb{R}}, Y_{\mathbb{R}})$ is a Banach lattice, then $\mathcal{L}^r(X, Y)$ is isometrically order isomorphic to the complexification of $\mathcal{L}^r(X_{\mathbb{R}}, Y_{\mathbb{R}})$ via the map $T \mapsto (\mathrm{Re} \circ T)|_{X_{\mathbb{R}}} + i(\mathrm{Im} \circ T)|_{X_{\mathbb{R}}}$, $\mathcal{L}^r(X, Y) \to \mathcal{L}^r(X_{\mathbb{R}}, Y_{\mathbb{R}}) + i\mathcal{L}^r(X_{\mathbb{R}}, Y_{\mathbb{R}})$, where $\mathrm{Re} : Y \to Y_{\mathbb{R}}$, $\xi + i\eta \mapsto \xi$, and $\mathrm{Im} : Y \to Y_{\mathbb{R}}$, $\xi + i\eta \mapsto \eta$. When $X = Y$, the previous map is also an algebra isomorphism, provided $\mathcal{L}^r(X_{\mathbb{R}}) + i\mathcal{L}^r(X_{\mathbb{R}})$ is endowed with the algebra product $(S_1 + iS_2)(T_1 + iT_2) := (S_1T_1 - S_2T_2) + i(S_1T_2 + S_2T_1)$ $(S_1, S_2, T_1, T_2 \in \mathcal{L}^r(X_{\mathbb{R}}))$. One can easily check that $\mathcal{I}$ is an ideal of $\mathcal{L}^r(X_{\mathbb{R}}) + i\mathcal{L}^r(X_{\mathbb{R}})$ if and only if $\mathcal{I} = \mathcal{I}_{\mathbb{R}} + i\mathcal{I}_{\mathbb{R}}$ and $\mathcal{I}_{\mathbb{R}}$ is an ideal of $\mathcal{L}^r(X_{\mathbb{R}})$, so the complexification procedure defines a bijective (inclusion preserving) correspondence between the ideals of $\mathcal{L}^r(X_{\mathbb{R}})$ and those of its complexification (which in turn are in bijective correspondence with those of $\mathcal{L}^r(X)$).

Recall a subset A of a Banach lattice X is *semicompact* (or *almost order bounded*) if for every $\varepsilon > 0$ there exists $u \in X_+$ such that $\|(|\xi| - u)_+\| < \varepsilon$ (for $\xi \in A$). In turn, an operator $T : X \to Y$ from a Banach space X to a Banach lattice Y is said to be *semicompact* if it maps norm-bounded sets to semicompact ones. It is easy to see that every compact operator is semicompact. Given Banach lattices X, Y, we define $\mathcal{SK}^r(X, Y)$ to be the linear span of the positive semicompact operators from X to Y. (If Y is Dedekind complete, this definition is equivalent to the one given in [SW].) One easily checks that if Y is Dedekind complete, then $\mathcal{SK}^r(X, Y)$ is a norm-closed order ideal of $\mathcal{L}^r(X, Y)$. Furthermore, $\mathcal{SK}^r(X, Y)$ is closed under composition with regular maps from the left and from the right, i.e., if E and F are Banach lattices, $U \in \mathcal{L}^r(Y, F)$, $V \in \mathcal{L}^r(E, X)$ and $T \in \mathcal{SK}^r(X, Y)$, then $UTV \in \mathcal{SK}^r(E, F)$.

Next, we present a result on representation of regular operators on separable atomic Banach lattices. It will simplify our arguments in subsequent sections by reducing them to a situation in which the matrix representations of the operators involved, with respect to a fixed 1-unconditional basis, are block diagonal (see below for definition). We note here that a similar result to Lemma 2.1 below first appeared in [SSTT] (see Lemma 4.4 of the latter). Before we can state the result, though, we need to introduce some more terminology.

We shall write $\mathcal{A}^r(X)$ for the norm-closure in $\mathcal{L}^r(X)$ of the ideal of finite-rank operators on X. If X is a Banach space with a normalized 1-unconditional basis (x_i), then $T \in \mathcal{L}^r(X)$ is said to be *block diagonal* with respect to (x_i) if there is a strictly increasing sequence $(n_i) \subset \mathbb{N}$ such that

$(P_{n_i} - P_{n_{i-1}})T = T(P_{n_i} - P_{n_{i-1}})$ $(i \in \mathbb{N})$, where (P_i) is the sequence of natural projections with respect to (x_i) and $P_{n_0} := 0$. In this note, the term *block diagonal* will always be used in connection with the unit vector basis of X, so, to simplify, we omit mention of the basis. Given an operator $T \in \mathcal{L}^r(X)$, we write $\mathcal{A}_T$ for the closed algebra ideal of $\mathcal{L}^r(X)$ generated by T.

LEMMA 2.1. *Let X be a Banach space with a normalized 1-unconditional Schauder basis (x_i) and let $T \in \mathcal{L}^r(X)$. Let (P_j) be the sequence of natural projections with respect to (x_i), let (x_i^*) be the corresponding sequence of biorthogonal functionals and suppose $\lim_j \|x_i^* \circ T \circ (\mathrm{id} - P_j)\| = 0$ $(i \in \mathbb{N})$. Then, for every $\varepsilon > 0$, T admits a representation of the form $D_1 + D_2 + A$, where $A \in \mathcal{A}^r(X)_{[\varepsilon]}$, and D_1 and D_2 are both block diagonal with respect to (x_i). Furthermore, $\mathcal{A}_T = \overline{\mathcal{A}_{D_1} + \mathcal{A}_{D_2}}$ and if (q_i) is a disjoint sequence of finite-rank band projections with respect to (x_i) then D_2 can be chosen so that, for some subsequence (q_{i_k}), $q_{i_k} D_2 = 0 = D_2 q_{i_k}$ $(k \in \mathbb{N})$.*

REMARK 2.2. Note that if (x_i) is shrinking then $\lim_j \|x_i^* \circ T \circ (\mathrm{id} - P_j)\| = \lim_j \|x_i^* \circ T\big|_{[x_k]_{k>j}}\| = 0$ for every $T \in \mathcal{L}^r(X)$ and $i \in \mathbb{N}$.

Proof of Lemma 2.1. Let $\varepsilon > 0$ be arbitrary. For each $i \in \mathbb{N}$, choose $l_i > i$, so that $\|(\mathrm{id} - P_{l_i})Tx_i\| < 2^{-i-1}\varepsilon$ and $\|x_i^* \circ (T(\mathrm{id} - P_{l_i}))\| < 2^{-i-1}\varepsilon$ (which is possible because $\lim_j \|x - P_j x\| = 0$ $(x \in X)$ and $\lim_j \|x_i^* \circ T \circ (\mathrm{id} - P_j)\| = 0$). Set

$$A := \sum_i x_i^* \otimes (\mathrm{id} - P_{l_i})Tx_i + \sum_i x_i^* \circ T(\mathrm{id} - P_{l_i}) \otimes x_i,$$

and $S := T - A$. It is clear that $A \in \mathcal{A}^r(X)_{[\varepsilon]}$. Furthermore, the matrix representation of S with respect to (x_i), i.e., the infinite matrix whose (k, j)th entry is $x_k^*(Sx_j)$ $(k, j \in \mathbb{N})$, has all its rows and columns in c_{00} (the space of finitely non-zero sequences), for

$$x_k^*(Ax_j) = x_k^*((\mathrm{id} - P_{l_j})Tx_j) + x_k^*(T(\mathrm{id} - P_{l_k})x_j) = x_k^*(Tx_j)$$

if either k is fixed and $j > l_k$ (in which case, also $l_j > k$) or if j is fixed and $k > l_j$ (in which case, also $l_k > j$).

Let (q_i) be as in the hypotheses and define increasing sequences of positive integers, (m_i), (n'_i) and (n_i), as follows. Set $i_0 := m_0 := n_0 := n'_0 := 0$ and $P_0 := 0$. Then, for each $k \in \mathbb{N}$, if $i_0, \dots, i_{k-1}$, $n'_0, \dots, n'_{k-1}$, $n_0, \dots, n_{k-1}$ and $m_0, \dots, m_{k-1}$ have already been chosen, choose $i_k > i_{k-1}$ so that $P_{m_{k-1}} q_{i_k} = 0$, choose $n'_k > m_{k-1}$ so that $P_{n'_k} q_{i_k} = q_{i_k}$, choose $n_k > n'_k$ so that

$$(\mathrm{id} - P_{n_k}) S P_{n'_k} = 0 = P_{n'_k} S (\mathrm{id} - P_{n_k}), \tag{2.1}$$

and choose $m_k > n_k$ so that

$$(\mathrm{id} - P_{m_k}) S P_{n_k} = 0 = P_{n_k} S (\mathrm{id} - P_{m_k}). \tag{2.2}$$

Let $P : X \to X$, $x \mapsto \sum_{k \text{ even}} (P_{n_k} - P_{n_{k-1}})x$, and define

$$D_1 := PSP + (\mathrm{id} - P)S(\mathrm{id} - P) \quad \text{and} \quad D_2 := S - D_1.$$

Note that then $D_2 = PS(\mathrm{id} - P) + (\mathrm{id} - P)SP$. Furthermore, note from the definition of the sequences (n_k), (m_k) and (n'_k) above that $n_1 > n'_1 > m_0 = n_0 = 0$ and $n_{k+1} > n'_{k+1} > m_k > n_k$ $(k \in \mathbb{N})$.

First we check D_1 is block diagonal. Note that if k is even, then $PP_{n_k} = PP_{m_k}$ and $P_{n_k}(\mathrm{id} - P) = P_{n'_k}(\mathrm{id} - P)$, so

$$\begin{aligned}
&(\mathrm{id} - P_{n_k})D_1 P_{n_k} \\
&\qquad = P(\mathrm{id} - P_{n_k})SP_{n_k}P + (\mathrm{id} - P)(\mathrm{id} - P_{n_k})SP_{n_k}(\mathrm{id} - P) \\
&\qquad = P(\mathrm{id} - P_{m_k})SP_{n_k}P + (\mathrm{id} - P)(\mathrm{id} - P_{n_k})SP_{n'_k}(\mathrm{id} - P) = 0,
\end{aligned}$$

the last equality following by (2.1)–(2.2). If k is odd, then $P_{n_k}P = P_{n'_k}P$ and $(\mathrm{id} - P)P_{n_k} = (\mathrm{id} - P)P_{m_k}$, so

$$\begin{aligned}
&(\mathrm{id} - P_{n_k})D_1 P_{n_k} \\
&\qquad = P(\mathrm{id} - P_{n_k})SP_{n_k}P + (\mathrm{id} - P)(\mathrm{id} - P_{n_k})SP_{n_k}(\mathrm{id} - P) \\
&\qquad = P(\mathrm{id} - P_{n_k})SP_{n'_k}P + (\mathrm{id} - P)(\mathrm{id} - P_{m_k})SP_{n_k}(\mathrm{id} - P) = 0,
\end{aligned}$$

the last equality again following by (2.1)–(2.2). This shows $(\mathrm{id} - P_{n_k})D_1 P_{n_k} = 0$ $(k \in \mathbb{N})$. The proof that $P_{n_k}D_1(\mathrm{id} - P_{n_k}) = 0$ $(k \in \mathbb{N})$ is completely analogous. Combining both sets of equalities, one readily finds that $P_{n_k}D_1 = P_{n_k}D_1P_{n_k} = D_1P_{n_k}$ $(k \in \mathbb{N})$, and in turn that D_1 is block diagonal.

As for D_2, first note that if k is even, then $PP_{m_k} = PP_{n_{k+1}}$ and $P_{m_k}P = P_{n_k}P_{m_k}P$, so

$$\begin{aligned}
&(\mathrm{id} - P_{m_k})D_2 P_{m_k} \\
&\quad = P(\mathrm{id} - P_{m_k})SP_{m_k}(\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{m_k})SP_{m_k}P \\
&\quad = P(\mathrm{id} - P_{n_{k+1}})SP_{n'_{k+1}}P_{m_k}(\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{m_k})SP_{n_k}P_{m_k}P = 0,
\end{aligned}$$

where we have taken into account that $n'_{k+1} > m_k$ (so $P_{m_k} = P_{n'_{k+1}}P_{m_k}$), and the last equality follows from (2.1) and (2.2). On the other hand, if k is odd, then $P_{m_k}(\mathrm{id} - P) = P_{n_k}(\mathrm{id} - P)$ and $(\mathrm{id} - P)P_{m_k} = (\mathrm{id} - P)P_{n_{k+1}}$, so

$$\begin{aligned}
&(\mathrm{id} - P_{m_k})D_2 P_{m_k} \\
&\qquad = P(\mathrm{id} - P_{m_k})SP_{m_k}(\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{m_k})SP_{m_k}P \\
&\qquad = P(\mathrm{id} - P_{m_k})SP_{n_k}(\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{n_{k+1}})SP_{n'_{k+1}}P_{m_k}P = 0,
\end{aligned}$$

the last equality following by (2.1) and (2.2). Similarly, one verifies that $P_{m_k}D_2(\mathrm{id} - P_{m_k}) = 0$ $(k \in \mathbb{N})$, so $P_{m_k}D_2 = D_2P_{m_k}$ $(k \in \mathbb{N})$, and therefore D_2 is block diagonal too.

It remains to see that $q_{i_k} D_2 = 0 = D_2 q_{i_k}$ $(k \in \mathbb{N})$. To this end, we first show that

$$(\mathrm{id} - P_{n'_k}) D_2 P_{n'_k} = 0 = P_{n'_k} D_2 (\mathrm{id} - P_{n'_k}) \quad (k \in \mathbb{N}). \tag{2.3}$$

Indeed, if k is even, then one sees that $PP_{m_{k-1}} = PP_{n'_k} P_{m_{k-1}}$ (or equivalently $P(\mathrm{id} - P_{n'_k}) = P(\mathrm{id} - P_{n'_k})(\mathrm{id} - P_{m_{k-1}})$), and $P_{n'_k}(\mathrm{id} - P) = P_{n_{k-1}}(\mathrm{id} - P)$, and $(\mathrm{id} - P)P_{n'_k} = (\mathrm{id} - P)P_{n_k}$, so

$$\begin{aligned}
&(\mathrm{id} - P_{n'_k}) D_2 P_{n'_k} \\
&\qquad = P(\mathrm{id} - P_{n'_k}) S P_{n'_k} (\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{n'_k}) S P_{n'_k} P \\
&\qquad = P(\mathrm{id} - P_{n'_k})(\mathrm{id} - P_{m_{k-1}}) S P_{n_{k-1}} (\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{n_k}) S P_{n'_k} P,
\end{aligned}$$

while if k is odd, then $PP_{n'_k} = PP_{n_k}$, $(\mathrm{id} - P)P_{m_{k-1}} = (\mathrm{id} - P)P_{n'_k} P_{m_{k-1}}$ (or equivalently $(\mathrm{id} - P)(\mathrm{id} - P_{n'_k}) = (\mathrm{id} - P)(\mathrm{id} - P_{n'_k})(\mathrm{id} - P_{m_{k-1}})$) and $P_{n'_k} P = P_{n_{k-1}} P$, so

$$\begin{aligned}
&(\mathrm{id} - P_{n'_k}) D_2 P_{n'_k} \\
&\qquad = P(\mathrm{id} - P_{n'_k}) S P_{n'_k} (\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{n'_k}) S P_{n'_k} P \\
&\qquad = P(\mathrm{id} - P_{n_k}) S P_{n'_k} (\mathrm{id} - P) + (\mathrm{id} - P)(\mathrm{id} - P_{n'_k})(\mathrm{id} - P_{m_{k-1}}) S P_{n_{k-1}} P.
\end{aligned}$$

Hence, by (2.1) and (2.2) (and in the case $k = 1$, the fact that $P_{n_0} = 0$), one concludes from the above identities that $(\mathrm{id} - P_{n'_k}) D_2 P_{n'_k} = 0$ $(k \in \mathbb{N})$. The proof that $P_{n'_k} D_2 (\mathrm{id} - P_{n'_k}) = 0$ $(k \in \mathbb{N})$ is completely analogous.

It follows from (2.3) that $P_{n'_k} D_2 = D_2 P_{n'_k}$ $(k \in \mathbb{N})$. Combining this with the fact that $P_{m_k} D_2 = D_2 P_{m_k}$ $(k \in \mathbb{N})$, established earlier, one obtains $(P_{n'_k} - P_{m_{k-1}}) D_2 = D_2 (P_{n'_k} - P_{m_{k-1}})$ $(k \in \mathbb{N})$. Also, $(P_{n'_k} - P_{m_{k-1}})P = (P_{n'_k} - P_{m_{k-1}})$ if k is even, while $(P_{n'_k} - P_{m_{k-1}})P = 0$ if k is odd. Combining all these facts, one finally deduces that

$$\begin{aligned}
q_{i_k} D_2 &= q_{i_k} (P_{n'_k} - P_{m_{k-1}}) D_2 = q_{i_k} (P_{n'_k} - P_{m_{k-1}}) D_2 (P_{n'_k} - P_{m_{k-1}}) \\
&= q_{i_k} (P_{n'_k} - P_{m_{k-1}})(PS(\mathrm{id} - P) + (\mathrm{id} - P)SP)(P_{n'_k} - P_{m_{k-1}}) = 0
\end{aligned}$$

(the first equality follows by our choice of the sequences (i_k) and (n'_k)). The proof that $D_2 q_{i_k} = 0$ $(k \in \mathbb{N})$ is completely analogous.

Thus, $T = D_1 + D_2 + A$ is a decomposition of T with the required properties. This readily implies that $\mathcal{A}_T \subseteq \overline{\mathcal{A}_{D_1} + \mathcal{A}_{D_2}}$. On the other hand, since $D_1 - PTP - (\mathrm{id} - P)T(\mathrm{id} - P)$ and $D_2 - PT(\mathrm{id} - P) - (\mathrm{id} - P)TP$ are both in $\mathcal{A}^r(X)$ (see the definitions of S, D_1, D_2 above) and $\mathcal{A}^r(X) \subseteq \mathcal{A}_T$, one concludes that $D_1, D_2 \in \mathcal{A}_T$, and in turn that $\overline{\mathcal{A}_{D_1} + \mathcal{A}_{D_2}} \subseteq \mathcal{A}_T$. So $\mathcal{A}_T = \overline{\mathcal{A}_{D_1} + \mathcal{A}_{D_2}}$, as claimed. ■

As indicated in the introduction, our results apply more generally to Banach lattices of the form $\mathcal{X} = (\bigoplus_i \ell_p^{n_i})_q$ with (n_i) unbounded. It can be shown that such an $\mathcal{X}$ is necessarily isomorphic as a Banach lattice to $(\bigoplus_i \ell_p^i)_q$. For the reader's convenience, we finish this section with a brief sketch of the argument involved. To this end, let $X := (E \oplus E \oplus \cdots)_q$ with $E = (\bigoplus_i \ell_p^i)_q$, and let $Y := (\bigoplus_i \ell_p^{m_i})_q$ with (m_i) unbounded. Then X is isometrically Riesz isomorphic to both $X \oplus_q X$ and $(X \oplus X \oplus \cdots)_q$, Y is isometrically Riesz isomorphic to a projection band of X, and X is isometrically Riesz isomorphic to a projection band of Y. One can then apply Pełczyński's decomposition method (see for instance [LT, p. 54]) to show X is isomorphic as a Banach lattice to Y (it is very easy to see that all isomorphisms in Pełczyński's method can be chosen to be Riesz isomorphisms). The desired conclusion follows readily from this.

Further terminology and notation will be introduced as needed. Lastly, recall that the term *ideal*, without any prefixes, whenever applied to $\mathcal{L}^r(X)$, stands for order and algebra ideal.

3. Maximal ideals of the algebras $\mathcal{L}^r((\bigoplus_i \ell_p^i)_q)$. First, recall from [dePS] that given an order bounded operator between Banach lattices, $T : X \to Y$ say, the *measure of non-semicompactness* of T is defined as

$$\rho(T) := \inf\{M \geq 0 : \rho(TD) \leq M\rho(D) \text{ for all norm-bounded } D \subset X\},$$

where

$$\rho(D) := \inf\{\varepsilon > 0 : \exists u \in X_+ \text{ such that } \|(|d| - u)_+\| \leq \varepsilon \, (d \in D)\}.$$

It will be relevant to our work below that $\rho(T) = 0$ if and only if T is semicompact (see [dePS]).

Given a sequence (X_i) of Banach spaces and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, recall that the ultraproduct $(X_i)_{\mathcal{U}}$ is the quotient of the Banach space $X := (\bigoplus_i X_i)_\infty$ by its closed subspace $\{(x_i) \in X : \lim_{\mathcal{U}} \|x_i\| = 0\}$. If $X_i = E$ for every $i \in \mathbb{N}$, we write $E_{\mathcal{U}}$ for the corresponding ultraproduct. The equivalence class of a sequence $(x_i) \in X$ is denoted $(x_i)_{\mathcal{U}}$. Endowed with the quotient norm, $\|(x_i)_{\mathcal{U}}\| := \lim_{\mathcal{U}} \|x_i\|$ ($(x_i)_{\mathcal{U}} \in (X_i)_{\mathcal{U}}$), $(X_i)_{\mathcal{U}}$ becomes a Banach space. Furthermore, if each X_i is a Banach lattice, then so is $(X_i)_{\mathcal{U}}$ for the order determined by the cone $\{(x_i)_{\mathcal{U}} : (x_i) \in X_+\}$. With respect to this order, for every pair $(x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}} \in (X_i)_{\mathcal{U}}$, one sees that $(x_i)_{\mathcal{U}} \wedge (y_i)_{\mathcal{U}} = (x_i \wedge y_i)_{\mathcal{U}}$ and $(x_i)_{\mathcal{U}} \vee (y_i)_{\mathcal{U}} = (x_i \vee y_i)_{\mathcal{U}}$. Lastly, given a bounded sequence of operators $(T_i : X_i \to Y_i)_{i\in\mathbb{N}}$, recall that $(T_i)_{\mathcal{U}}$ is defined as the operator $(T_i)_{\mathcal{U}} : (X_i)_{\mathcal{U}} \to (Y_i)_{\mathcal{U}}$, $(x_i)_{\mathcal{U}} \mapsto (T_i x_i)_{\mathcal{U}}$.

The main result of the section, concerning the lattice of closed ideals of the algebra $\mathcal{L}^r((\bigoplus_i \ell_p^i)_q)$, is the following.

THEOREM 3.1. *Let $X := (\bigoplus_i \ell_p^i)_q$ with $p \in [1,\infty]$, $q \in [1,\infty) \cup \{0\}$ and $p \neq q$. For each $i \in \mathbb{N}$, let $p_i : X \to \ell_p^i$ and $\imath_i : \ell_p^i \to X$ be the ith coordinate* *projection and embedding, respectively, and let $\mathfrak{U}$ be the collection of all free ultrafilters on $\mathbb{N}$.*

(I) *Suppose $p < q < \infty$ or $q = 0$ and let*

$$\mathcal{R} := \mathrm{sp}\,\{T \in \mathcal{L}^r(X)_+ : \rho((p_i T)_{\mathcal{U}}) = 0 \ (\mathcal{U} \in \mathfrak{U})\}.$$

Then

$$\mathcal{M}_X^{\mathrm{oa}} := \big\{T \in \mathcal{R} : ST \in \mathcal{R} \ (S \in \mathcal{L}^r(X))\big\}$$

is the only maximal ideal of $\mathcal{L}^r(X)$. Further, if $\imath : X \to (\bigoplus_i \ell_p^i)_\infty =: X_\infty$ is the formal inclusion map, then

$$\mathcal{R} = \{T \in \mathcal{L}^r(X) : \imath T \in \mathcal{SK}^r(X, X_\infty)\},$$

and if $q = 0$, then

$$\mathcal{M}_X^{\mathrm{oa}} = \mathcal{R} = \{T \in \mathcal{L}^r(X) : \kappa_X T \in \mathcal{SK}^r(X, X'')\},$$

where κ_X stands for the canonical embedding of X into X''.

(II) *Suppose $1 \le q < p \le \infty$ and let*

$$\mathcal{L} := \mathrm{sp}\,\{T \in \mathcal{L}^r(X)_+ : \rho((\imath_i' T')_{\mathcal{U}}) = 0 \ (\mathcal{U} \in \mathfrak{U})\}.$$

Then

$$\mathcal{M}_X^{\mathrm{oa}} := \{T \in \mathcal{L} : TS \in \mathcal{L} \ (S \in \mathcal{L}^r(X))\}$$

is the only maximal ideal of $\mathcal{L}^r(X)$. Further, if $\imath : X' \to (\bigoplus_i \ell_{p'}^i)_\infty =: X'_\infty$ is the formal inclusion, then

$$\mathcal{L} = \big\{T \in \mathcal{L}^r(X) : \imath T' \in \mathcal{SK}^r(X', X'_\infty)\big\},$$

and if $q = 1$, then

$$\mathcal{M}_X^{\mathrm{oa}} = \mathcal{L} = \{T \in \mathcal{L}^r(X) : T' \in \mathcal{SK}^r(X')\}.$$

In proving the theorem, we shall make use of the following facts.

Lemma 3.2. *Let (X_i) and (Y_i) be sequences of Banach lattices, and let $(T_i : X_i \to Y_i)_{i \in \mathbb{N}}$ be a sequence of regular maps, bounded with respect to the regular norm, and let $\mathcal{V}$ be a free ultrafilter on $\mathbb{N}$. Let $\mathcal{X} := (X_i)_{\mathcal{V}}$, let $\mathcal{Y} := (Y_i)_{\mathcal{V}}$ and let $\mathcal{T} := (T_i)_{\mathcal{V}}$. If (x_n) and (f_n) are disjoint sequences in $\mathcal{X}_{[1]}$ and $\mathcal{Y}'_{[1]}$, respectively, such that $\inf_n f_n(\mathcal{T} x_n) =: \delta > 0$, then there is a strictly increasing sequence $(j_n) \subset \mathbb{N}$ such that, for every $n \in \mathbb{N}$, there are disjoint sequences $(\xi_{j_n,i})_{i=1}^n \subset (X_{j_n})_{[1]}$ and $(\phi_{j_n,i})_{i=1}^n \subset (Y'_{j_n})_{[1]}$ such that*

$$\min\{\phi_{j_n,i}(T_{j_n} \xi_{j_n,i}) : 1 \le i \le n\} > \delta/2.$$

If only (x_n) (resp. (f_n)) is disjoint, then the same conclusion holds except for the disjointness of the sequences $(\phi_{j_n,i})_{i=1}^n$ (resp. $(\xi_{j_n,i})_{i=1}^n$).

Proof. Let $x_n = (x_{j,n})_{\mathcal{V}} \in \mathcal{X}_{[1]}$ and $f_n = (f_{j,n})_{\mathcal{V}} \in \mathcal{Y}'_{[1]}$ $(n \in \mathbb{N})$ be such that

$$\inf_n f_n(\mathcal{T} x_n) =: \delta > 0,$$

so

$$\lim_{\mathcal{V}} f_{j,n}(T_j x_{j,n}) = (f_{j,n})_{\mathcal{V}}((T_j)_{\mathcal{V}}(x_{j,n})_{\mathcal{V}}) = f_n(\mathcal{T} x_n) \geq \delta \quad (n \in \mathbb{N}).$$

We shall assume, as we may, that $\sup_{j,n} \|x_{j,n}\| \leq 1$ and $\sup_{j,n} \|f_{j,n}\| \leq 1$.

Suppose first (x_n) and (f_n) are both disjoint. Then, for $m \neq n$,

$$\lim_{\mathcal{V}} \big\| |x_{j,m}| \wedge |x_{j,n}| \big\| = \big\| |x_m| \wedge |x_n| \big\| = 0,$$
$$\lim_{\mathcal{V}} \big\| |f_{j,m}| \wedge |f_{j,n}| \big\| = \big\| |f_m| \wedge |f_n| \big\| = 0,$$

and we can define a nested sequence $(M_n) \subset \mathcal{V}$ as follows. Set $K := \sup_j \|T_j\|$ and $\delta_n := \delta/(nK2^{n+3})$ $(n \in \mathbb{N})$. First, choose $M_1 \in \mathcal{V}$ so that $f_{j,1}(T_j x_{j,1}) \geq 3\delta/4$ $(j \in M_1)$, and in general, if $M_1, \ldots, M_{n-1} \in \mathcal{V}$ have been chosen, choose $M_n \subseteq M_{n-1}$ so that, for every $j \in M_n$,

$$f_{j,n}(T_j x_{j,n}) \geq \frac{3\delta}{4}, \quad \max_{k<n} \big\| |x_{j,k}| \wedge |x_{j,n}| \big\| \leq \delta_n, \quad \max_{k<n} \big\| |f_{j,k}| \wedge |f_{j,n}| \big\| \leq \delta_n.$$

Choose a strictly increasing sequence $(j_n) \subset \mathbb{N}$ so that $j_n \in M_n$ $(n \in \mathbb{N})$, and define $(\xi_{j_n,i})_{i=1}^n \subset X_{j_n}$ and $(\phi_{j_n,i})_{i=1}^n \subset Y'_{j_n}$ by

$$\xi_{j_n,i_\pm} := x_{j_n,i_\pm} - x_{j_n,i_\pm} \wedge \sum_{\substack{1 \leq k \leq n \\ k \neq i}} |x_{j_n,k}| \quad (1 \leq i \leq n),$$

$$\phi_{j_n,i_\pm} := f_{j_n,i_\pm} - f_{j_n,i_\pm} \wedge \sum_{\substack{1 \leq k \leq n \\ k \neq i}} |f_{j_n,k}| \quad (1 \leq i \leq n).$$

That $(\xi_{j_n,i})_{i=1}^n$ and $(\phi_{j_n,i})_{i=1}^n$ are disjoint is clear. Furthermore,

$$\begin{aligned} \|\xi_{j_n,i} - x_{j_n,i}\| &= \Big\| |x_{j_n,i}| \wedge \sum_{k \leq n\,:\, k \neq i} |x_{j_n,k}| \Big\| \\ &\leq \sum_{1 \leq k < i} \big\| |x_{j_n,i}| \wedge |x_{j_n,k}| \big\| + \sum_{i < k \leq n} \big\| |x_{j_n,i}| \wedge |x_{j_n,k}| \big\| \\ &\leq i\delta_i + \sum_{i < k \leq n} \delta_k \leq \frac{\delta}{8K}, \end{aligned}$$

and similarly $\|\phi_{j_n,i} - f_{j_n,i}\| \leq \delta/(8K)$ $(1 \leq i \leq n,\ n \in \mathbb{N})$. It follows from these estimates and the fact that $f_{j_n,i}(T_{j_n} x_{j_n,i}) \geq 3\delta/4$ that

$$\begin{aligned} \phi_{j_n,i}(T_{j_n} \xi_{j_n,i}) &\geq f_{j_n,i}(T_{j_n} x_{j_n,i}) \\ &\qquad - |\phi_{j_n,i}(T_{j_n}(\xi_{j_n,i} - x_{j_n,i}))| - |(\phi_{j_n,i} - f_{j_n,i})(T_{j_n}(x_{j_n,i}))| \\ &\geq \frac{3\delta}{4} - 2K\frac{\delta}{8K} \geq \frac{\delta}{2} \quad (1 \leq i \leq n), \end{aligned}$$

as required.

If only (x_n) (resp. (f_n)) is disjoint, then one chooses $(M_n) \subset \mathcal{V}$ in such a way that $f_{j,n}(T_j x_{j,n}) \geq 3\delta/4$ and $\max_{k<n} \big\| |x_{j,k}| \wedge |x_{j,n}| \big\| \leq \delta_n$

(resp. $\max_{k<n} \||f_{j,k}| \wedge |f_{j,n}|\| \le \delta_n$) ($j \in M_n$); then define (j_n) and $(\xi_{j_n,i})_{i=1}^n$ (resp. $(\phi_{j_n,i})_{i=1}^n$) as above, and let $\phi_{j_n,i} := f_{j_n,i}$ (resp. $\xi_{j_n,i} := x_{j_n,i}$) ($1 \le i \le n$). The rest of the argument, except for some obvious changes, remains essentially the same. ■

The second fact is merely an observation. Its proof is straightforward and will be omitted. Recall that an associative algebra $\mathcal{A}$ is said to be a *Riesz algebra* if it also a Riesz space whose positive cone is closed under multiplication, and is said to be *simple* if its only algebra ideals are $\{0\}$ and $\mathcal{A}$ itself.

LEMMA 3.3. *Let* $\mathcal{A}$ *be a Riesz algebra and suppose* $\mathcal{R}$ (*resp.* $\mathcal{L}$) *is a proper order and right* (*resp. left*) *algebra ideal of* $\mathcal{A}$ *such that any proper ideal of* $\mathcal{A}$ *is contained in* $\mathcal{R}$ (*resp.* $\mathcal{L}$). *If* $\mathcal{A}$ *is not simple then* $\mathcal{I} := \{r \in \mathcal{R} : ar \in \mathcal{R}\ (a \in \mathcal{A})\}$ (*resp.* $\mathcal{I} := \{l \in \mathcal{L} : la \in \mathcal{L}\ (a \in \mathcal{A})\}$) *is the only maximal ideal of* $\mathcal{A}$.

The last fact that we shall need is a restricted version of [Le, Proposition 1.1], adapted to our present needs:

LEMMA 3.4. *Let* E *be either* ℓ_p $(1 \le p < \infty)$ *or* c_0, *let* (P_i) *be the sequence of natural projections associated with the unit vector basis* (e_i) *of* E, *let* $E_i := P_i(E)$ $(i \in \mathbb{N})$, *and let* $(T_i : E_i \to E_i)_{i\in\mathbb{N}}$ *be a bounded sequence of linear maps such that* $\inf\{e_j^*(T_i e_j) : 1 \le j \le i,\ i \in \mathbb{N}\} =: \delta > 0$, *where* (e_j^*) *is the sequence of biorthogonal functionals corresponding to* (e_j). *There are then a strictly increasing sequence* $(n_k) \subset \mathbb{N}$ *and bounded sequences of operators* $(A_k : E_{n_k} \to E_k)_{k\in\mathbb{N}}$ *and* $(B_k : E_k \to E_{n_k})_{k\in\mathbb{N}}$ *such that* $\mathrm{id}_{E_k} = A_k T_{n_k} B_k$ $(k \in \mathbb{N})$.

Proof. Let $E = \ell_p$ $(1 \le p < \infty)$, and for each $i \in \mathbb{N}$, let $I_i : E_i \to E$ be the natural inclusion. As a dual space, E can be endowed with the weak* topology w^*. Choose a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define $R, S \in \mathcal{L}^r(E)_+$ by

$$Rx := w^*\text{-}\lim_{i,\mathcal{U}} I_i T_i^+ P_i x \quad \text{and} \quad Sx := w^*\text{-}\lim_{i,\mathcal{U}} I_i T_i^- P_i x \qquad (x \in E),$$

where T_i^+ and T_i^- stand for the positive and negative parts of T_i $(i \in \mathbb{N})$. Set $T := R - S$. Then $e_i^*(Te_i) = e_i^*(w^*\text{-}\lim_{j,\mathcal{U}} I_j T_j e_i) = \lim_{j,\mathcal{U}} e_i^*(T_j e_i) \ge \delta$ $(i \in \mathbb{N})$. It is known that in this situation there are $A, B \in \mathcal{L}^r(E)$ such that $ATB = \mathrm{id}_E$. [The latter is essentially due to Pełczyński [Pe]. For the reader's convenience, we outline here briefly the argument. Take $0 < \varepsilon < 2^{-1}\delta$ arbitrary and set $i_1 := 1$ and $n_0 := 0$. Then, for each $k \in \mathbb{N}$, if $i_1, \dots, i_k$ and $n_1, \dots, n_{k-1}$ are known, choose $n_k > n_{k-1}$ so that $\|(\mathrm{id}_E - P_{n_k}) Te_{i_k}\| \le 2^{-k}\varepsilon$, and then choose $i_{k+1} > i_k$ so that $\|P_{n_k} Te_{i_{k+1}}\| < 2^{-k-1}\varepsilon$. Set $\alpha_k := \delta e_{i_k}^*(Te_{i_k})^{-1}$ $(k \in \mathbb{N})$, and define $A := \sum_k \alpha_k e_{i_k}^* \otimes e_k$ and $B := \sum_k e_k^* \otimes e_{i_k}$. One verifies that $\|ATB - \delta\,\mathrm{id}_E\|_r < 2\varepsilon$. The rest is clear.] It follows that for every $x \in E$,

$$x = \lim_i AP_i TBx = \lim_i AP_i\Big(w^*\text{-}\lim_{j,\mathcal{U}} I_j T_j P_j Bx\Big) = \lim_i \lim_{j,\mathcal{U}} AP_i I_j T_j P_j Bx$$

(the last equality is valid because each P_i is weak*-norm-continuous), and so, for each $k \in \mathbb{N}$, there are i_k and j_k with $\|P_k(AP_{i_k}I_{j_k}T_{j_k}P_{j_k}B)I_k - \mathrm{id}_{E_k}\|_r \le 2^{-1}$. The desired result follows from this, if we set

$$A_k := (P_kAP_{i_k}I_{j_k}T_{j_k}P_{j_k}BI_k)^{-1}P_kAP_{i_k}I_{j_k} \quad \text{and} \quad B_k := P_{j_k}BI_k \quad (k \in \mathbb{N}).$$

The case $p = 0$ follows readily on applying the result for $p = 1$ to the sequence of adjoints (T_i'). ∎

Proof of Theorem 3.1. Throughout, we write E_n for the nth direct summand of X and $(e_{i,n})_{i=1}^n$ for its unit vector basis. Infinite sums of operators in $\mathcal{L}^r(X)$ are to be understood in the sense of the strong operator topology.

(I) Let $X := (\bigoplus_i \ell_p^i)_q$ with $1 \le p < \infty$ and either $p < q < \infty$ or $q = 0$. Set

$$\mathcal{A} := \{T \in \mathcal{L}^r(X) : \imath T \in \mathcal{SK}^r(X, X_\infty)\},$$
$$\mathcal{B} := \mathrm{sp}\,\{T \in \mathcal{L}^r(X)_+ : \rho((p_iT)_\mathcal{U}) = 0 \ (\mathcal{U} \in \mathfrak{U})\}.$$

First note that $\mathcal{A}$ is a proper order and right algebra ideal of $\mathcal{L}^r(X)$. Indeed, if $R, S \in \mathcal{A}$ and $T \in \mathcal{L}^r(X)$ then $\imath ST, \imath R + \imath S \in \mathcal{SK}^r(X, X_\infty)$, so $ST, R + S \in \mathcal{A}$. If $|T| \le S \in \mathcal{A}$ then $\imath T_\pm \le \imath S$, and hence $T \in \mathcal{A}$. Lastly, $\imath \notin \mathcal{SK}^r(X, X_\infty)$ (and hence $\mathrm{id}_X \notin \mathcal{A}$). The latter follows on noting that if $y = (y_n) \in \prod_n E_n$ is such that $\sup_n \|(|x_n| - y_n)_+\|_p = \|(|\imath(x)| - y)_+\| \le \varepsilon < 1$ for every $x = (x_n) \in X_{[1]}$, then $(e_{i,n} - y_n)_+ \le \varepsilon e_{i,n}$ for every $1 \le i \le n$ $(n \in \mathbb{N})$, and in turn $\|y_n\|_p \ge (1 - \varepsilon)n^{1/p}$ $(n \in \mathbb{N})$.

Let $S \in \mathcal{B}_+$ be arbitrary, and for each $\varepsilon > 0$ and $i \in \mathbb{N}$, set

$$t_i(\varepsilon) := \inf\,\{\|\phi\| : \phi \in E_i \text{ and } \|(|(p_iS)(x)| - \phi)_+\| \le \varepsilon \ (x \in X_{[1]})\}.$$

Suppose towards a contradiction that $\{t_i(\varepsilon) : i \in \mathbb{N}\}$ is unbounded for some ε. Choose an infinite subset I of $\mathbb{N}$ so that $\{t_i(\varepsilon) : i \in I\}$ is unbounded and strictly increasing on I, and choose $\mathcal{U} \in \mathfrak{U}$, containing I. Since $S \in \mathcal{B}$, we see that $(p_iS)_\mathcal{U}$ is semicompact, and therefore there is a positive $f = (f_i)_\mathcal{U} \in (E_i)_\mathcal{U}$ such that, for every $(x_i)_\mathcal{U} \in (X_\mathcal{U})_{[1]}$,

$$\big\|\big(|(p_iS)_\mathcal{U}((x_i)_\mathcal{U})| - f\big)_+\big\| = \lim_\mathcal{U} \big\|\big(|(p_iS)(x_i)| - f_i\big)_+\big\| < \varepsilon. \tag{3.1}$$

Clearly, we can assume $\|f_i\| < t_i(\varepsilon)$ $(i \in I)$. Then, for every $i \in I$, there exists $x_i \in X_{[1]}$ such that $\|(|(p_iS)(x_i)| - f_i)_+\| > \varepsilon$, while, by (3.1), $\mathcal{U}$ must contain an infinite subset $J \subset I$ such that $\|(|(p_iS)(x_i)| - f_i)_+\| < \varepsilon$ $(i \in J)$, a plain contradiction. Thus, $S \in \mathcal{A}$. That $\mathcal{B} \subseteq \mathcal{A}$ follows readily.

Now let $S \in \mathcal{A}_+$ and $\varepsilon > 0$ be arbitrary, and let $(y_i) \in X_\infty$ be such that $\sup_i \|(|p_iSx| - y_i)_+\|_p < \varepsilon$ $(x \in X_{[1]})$. Fix $\mathcal{U} \in \mathfrak{U}$ and let $(x_i)_\mathcal{U} \in (X_\mathcal{U})_{[1]}$. Clearly, we can assume $x_i \in X_{[1]}$ $(i \in \mathbb{N})$, and so

$$\big\|\big(|(p_iS)_\mathcal{U}((x_i)_\mathcal{U})| - (y_i)_\mathcal{U}\big)_+\big\| = \lim_\mathcal{U} \|(|p_iSx_i| - y_i)_+\|_p \le \varepsilon,$$

which proves that $(p_i S)_{\mathcal{U}}$ is semicompact, and hence that $S \in \mathcal{B}$. Thus, also $\mathcal{A} \subseteq \mathcal{B}$.

We show next that every proper ideal of $\mathcal{L}^r(X)$ is contained in $\mathcal{B}$. For this, it will clearly suffice to show that if $T \in \mathcal{L}^r(X)_+ \setminus \mathcal{B}$ then $\mathcal{A}_T = \mathcal{L}^r(X)$ (for if $T \notin \mathcal{B}$ then $|T| \notin \mathcal{B}$). So let $T \in \mathcal{L}^r(X)_+ \setminus \mathcal{B}$ and let $\mathcal{V} \in \mathfrak{U}$ be such that $\rho((p_i T)_{\mathcal{V}}) > 0$. Set $\mathcal{X} := X_{\mathcal{V}}$, $\mathcal{E} := (E_i)_{\mathcal{V}}$ and $\mathcal{T} := (p_i T)_{\mathcal{V}}$. We consider separately the cases $1 < p < \infty$ and $p = 1$:

• $1 < p < \infty$. Then $\mathcal{E}$ is a reflexive L^p-space while $\mathcal{X}$ satisfies an upper p-estimate, or more precisely, for every finite disjoint sequence $x_1, \dots, x_n \in \mathcal{X}$,

$$\Big\|\sum_{i=1}^n x_i\Big\| \le \Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p},$$

which follows easily from the fact that X has the same property. (To see this last fact, let $(\xi_i)_{i=1}^n$ be a disjoint sequence in X and note that if $q = 0$, then

$$\Big\|\sum_i \xi_i\Big\|_X^p = \sup_k \sum_i \|p_k \xi_i\|_p^p \le \sum_i \Big(\sup_k \|p_k \xi_i\|_p\Big)^p = \sum_i \|\xi_i\|_X^p,$$

and if $q > p$ and $r = q/p$, then

$$\Big\|\sum_i \xi_i\Big\|_X^p = \Big(\sum_k \Big(\sum_i \|p_k \xi_i\|_p^p\Big)^r\Big)^{1/r} \le \sum_i \Big(\sum_k \|p_k \xi_i\|_p^{pr}\Big)^{1/r} = \sum_i \|\xi_i\|_X^p,$$

where the inequality can be easily established by induction on the number of vectors in the sequence $(\xi_i)_{i=1}^n$.) The fact that $\mathcal{X}$ satisfies a non-trivial upper estimate implies in particular that $\mathcal{X}'$ is order continuous (see [Do, Theorem 2.3]).

Let $0 < \delta < \rho((p_i T)_{\mathcal{V}})$. By [dePS, Theorem 2.3], there are disjoint sequences $(x_n) \subset \mathcal{X}_{[1]}$ and $(f_n) \subset \mathcal{E}'_{[1]}$ such that $\inf_n f_n(\mathcal{T} x_n) > \delta$. In turn, by Lemma 3.2, there is a strictly increasing sequence $(j_n) \subset \mathbb{N}$, and for every $n \in \mathbb{N}$, disjoint sequences $(\xi_{j_n,i})_{i=1}^n \subset X_{[1]}$ and $(\phi_{j_n,i})_{i=1}^n \subset (E'_{j_n})_{[1]}$ such that

$$\phi_{j_n,i}(p_{j_n} T(\xi_{j_n,i})) > \delta/2 \quad (1 \le i \le n). \tag{3.2}$$

For each $n \in \mathbb{N}$, let $(e^*_{i,n})_{i=1}^n \subset E'_n$ be the sequence of biorthogonal functionals corresponding to $(e_{i,n})_{i=1}^n$, and let

$$R_n := \sum_{i=1}^n \phi_{j_n,i} \otimes e_{i,n} \quad \text{and} \quad S_n := \sum_{i=1}^n e^*_{i,n} \otimes \xi_{j_n,i}.$$

Then, for every $\xi = \sum_j \alpha_j e_{j,j_n} \in (E_{j_n})_+$,

$$\begin{aligned}\big\||R_n|(\xi)\big\|_p &= \Big\|\sum_i |\phi_{j_n,i}|(\xi) e_{i,n}\Big\|_p \\ &\le \Big(\sum_i \Big\|\sum_{j \in \operatorname{supp} \phi_{j_n,i}} \alpha_j e_{j,j_n}\Big\|_p^p\Big)^{1/p} \le \|\xi\|_p,\end{aligned}$$

and for every $\eta = \sum_i \alpha_i e_{i,n} \in (E_n)_+$,

$$\big\||S_n|(\eta)\big\|_X = \Big\|\sum_i e^*_{i,n}(\eta)|\xi_{j_n,i}|\Big\|_X \le \Big(\sum_i |\alpha_i|^p \|\xi_{j_n,i}\|_X^p\Big)^{1/p} \le \|\eta\|_p,$$

where the penultimate inequality follows from the fact that X satisfies an upper p-estimate with constant 1. (It is precisely at this point of the proof that the fact that $E_n = \ell_p^n$ is used.) It follows that $\|R_n\|_r, \|S_n\|_r \le 1$ ($n \in \mathbb{N}$). Furthermore,

$$e^*_{i,n}(R_n p_{j_n} T S_n(e_{i,n})) = \phi_{j_n,i}(p_{j_n} T(\xi_{j_n,i})) \quad (1 \le i \le n).$$

By Lemma 3.4, applied to the sequence $(R_n p_{j_n} T S_n)_{n\in\mathbb{N}}$, there are a strictly increasing sequence $(n_k) \subset \mathbb{N}$ and bounded sequences of operators $(A_k : E_{j_{n_k}} \to E_k)_{k\in\mathbb{N}}$ and $(B_k : E_k \to X)_{k\in\mathbb{N}}$ such that $\mathrm{id}_{E_k} = A_k p_{j_{n_k}} T B_k$ ($k \in \mathbb{N}$). Let $D_1, D_2 \in \mathcal{L}^r(X)$ be block diagonal operators with $T - D_1 - D_2 =: A \in \mathcal{A}^r(X)$ and $(\sum_j \imath_{j_{n_{k_l}}} p_{j_{n_{k_l}}}) D_2 = 0$ for some subsequence $(j_{n_{k_l}})$ of (j_{n_k}) (such operators exist by Lemma 2.1). Let $Q := \sum_j \imath_{j_{n_{k_l}}} p_{j_{n_{k_l}}}$. We can assume $\|A\|_r < (4 \sup_k \|A_k\|_r \sup_k \|B_k\|_r)^{-1}$. Then $\mathrm{id}_{E_{k_l}} = A_{k_l} p_{j_{n_{k_l}}} T B_{k_l} = A_{k_l} p_{j_{n_{k_l}}} D_1 B_{k_l} + A_{k_l} p_{j_{n_{k_l}}} A B_{k_l}$, and so $\|A_{k_l} p_{j_{n_{k_l}}} D_1 B_{k_l} - \mathrm{id}_{E_{k_l}}\|_r \le 4^{-1}$ ($l \in \mathbb{N}$). For each l, let $\pi_l : E_{k_l} \to E_l$ and $\gamma_l : E_l \to E_{k_l}$ be norm-1 positive linear maps such that $\pi_l \gamma_l = \mathrm{id}_{E_l}$; let $L_l := \pi_l A_{k_l}$ and $M_l := B_{k_l}(A_{k_l} p_{j_{n_{k_l}}} D_1 B_{k_l})^{-1} \gamma_l$, so

$$\mathrm{id}_{E_l} = L_l p_{j_{n_{k_l}}} D_1 M_l, \quad \|L_l\|_r \le \|A_{k_l}\|_r \quad \text{and} \quad \|M_l\|_r \le 2\|B_{k_l}\|_r.$$

To simplify the notation, let $m_l := j_{n_{k_l}}$ ($l \in \mathbb{N}$).

Let (P_n) be the sequence of natural projections with respect to the unit vector basis of X. We construct a sequence of band projections $(Q_s)_{s\in\mathbb{N}} \subset \mathcal{L}^r(X)$ as follows: first, set $l_1 := 1$, choose j_1 so that $P_{j_1} D_1 = D_1 P_{j_1}$ and $p_{m_{l_1}} P_{j_1} = p_{m_{l_1}}$, and set $Q_1 := P_{j_1}$; in general, if $l_1, \dots, l_{s-1}$ and $j_1, \dots, j_{s-1}$ have been chosen, choose l_s so that $p_{m_{l_s}} P_{j_{s-1}} = 0$, choose $j_s > j_{s-1}$ so that

$$P_{j_s} D_1 = D_1 P_{j_s} \quad \text{and} \quad p_{m_{l_s}} P_{j_s} = p_{m_{l_s}},$$

and set $Q_s := P_{j_s} - P_{j_{s-1}}$. Define

$$U := \sum_s \imath_{l_s} L_{l_s} p_{m_{l_s}} \quad \text{and} \quad V := \sum_s Q_s M_{l_s} p_{l_s}.$$

One easily checks $\|U\|_r \le \sup_s \|L_s\|_r$,

$$\|V\|_r \le \Big\|\sum_{s \text{ odd}} Q_s M_{l_s} p_{l_s}\Big\|_r + \Big\|\sum_{s \text{ even}} Q_s M_{l_s} p_{l_s}\Big\|_r \le 2 \sup_s \|M_s\|_r,$$

$UQ = U$ and

$$U D_1 V = \sum_s \imath_{l_s} L_{l_s} p_{m_{l_s}} D_1 Q_s M_{l_s} p_{l_s} = \sum_s \imath_{l_s} p_{l_s}.$$

Set $P := \sum_s \imath_{l_s} p_{l_s}$. Let L be a sublattice of $P(X)$, isometrically Riesz isomorphic to X and complemented by a norm-1 positive projection W (it is easy to produce one such sublattice). Let $\phi : X \to L$ be an isometric Riesz isomorphism and let $\imath : L \to X$ be the natural embedding. Then $\mathrm{id}_X = \phi^{-1}WP\imath\phi$. Since $UTV = UD_1V + UAV = P + UAV$, one sees that $\|\phi^{-1}WUTV\imath\phi - \mathrm{id}_X\|_r \leq \|U\|_r\|A\|_r\|V\|_r < 1$. That $\mathrm{id}_X \in \mathcal{A}_T$ follows readily from this.

• $p = 1$. In this case, $\mathcal{E}$ is an L^1-space. Since $\mathcal{T}$ is not semicompact, by [SW, Proposition 1.1], $\mathrm{sol}(\mathcal{T}(\mathcal{X}_{[1]}))$ (:= solid hull of $\mathcal{T}(\mathcal{X}_{[1]})$) must contain a disjoint sequence, (ξ_i) say, with $\inf_i \|\xi_i\| =: \delta > 0$. Let $(x_i) \subset \mathcal{X}_{[1]}$ be such that $|\mathcal{T}x_i| \geq |\xi_i|$ $(i \in \mathbb{N})$ and let q_i be the principal band projection on $\mathcal{E}$ generated by $|\xi_i|$ $(i \in \mathbb{N})$. For each $i \in \mathbb{N}$, choose $g_i \in \mathcal{E}'_{[1]} \cap \mathcal{E}'_+$ so that $g_i \circ q_i = g_i$ and $g_i(|\xi_i|) = \|\xi_i\|$, and choose $f_i \in \mathcal{E}'$ so that $|f_i| \leq g_i$ and $f_i(\mathcal{T}x_i) = g_i(|\mathcal{T}x_i|)$. One then finds that $f_i(\mathcal{T}x_i) = g_i(q_i|\mathcal{T}x_i|) \geq g_i(|\xi_i|) = \|\xi_i\| \geq \delta$ $(i \in \mathbb{N})$. Also note that (f_i) is a disjoint sequence. Thus, by Lemma 3.2, there is a strictly increasing sequence $(j_n) \subset \mathbb{N}$, and for every $n \in \mathbb{N}$, a sequence $(\xi_{j_n,i})_{i=1}^n \subset X_{[1]}$ and a disjoint sequence $(\phi_{j_n,i})_{i=1}^n \subset (E'_{j_n})_{[1]}$ such that $\phi_{j_n,i}(p_{j_n}T(\xi_{j_n,i})) > \delta/2$ $(1 \leq i \leq n)$. For each $n \in \mathbb{N}$, define $R_n := \sum_i \phi_{j_n,i} \otimes e_{i,n}$ and $S_n := \sum_i e^*_{i,n} \otimes \xi_{j_n,i}$. From this point onwards, the argument continues exactly as in the case $1 < p < \infty$.

In view of the above and Lemma 3.3, $\mathcal{L}^r(X)$ has a unique maximal ideal given by $\{T \in \mathcal{B} : ST \in \mathcal{B}\ (S \in \mathcal{L}^r(X))\}$. To conclude this part of the proof, it will suffice to notice that when $q = 0$, $\mathcal{A}$ is also a left ideal (and therefore, $\mathcal{M}_X^{\mathrm{oa}} = \mathcal{A}$). Indeed, when $q = 0$, $\imath : X \to X_\infty$ is just the canonical embedding $\kappa_X : X \to X''$, and so, given $T \in \mathcal{A}$, for every $S \in \mathcal{L}^r(X)$, $\imath ST = \kappa_X ST = S''\kappa_X T \in \mathcal{SK}^r(X, X_\infty)$.

(II) Now let $X := (\bigoplus_i \ell_p^i)_q$ with $1 \leq q < p \leq \infty$. The argument is very similar to that from the previous part, and so we omit a few details. This time set

$$\mathcal{A} := \{T \in \mathcal{L}^r(X) : \imath T' \in \mathcal{SK}^r(X', X'_\infty)\},$$
$$\mathcal{B} := \mathrm{sp}\,\{T \in \mathcal{L}^r(X)_+ : \rho((\imath'_i T')_{\mathcal{U}}) = 0\ (\mathcal{U} \in \mathfrak{U})\}.$$

Recall $\imath$ stands now for the canonical embedding of X' into $X'_\infty = (\bigoplus_i \ell_{p'}^i)_\infty$, where p' stands for the conjugate exponent of p.

For all $R, S \in \mathcal{A}$ and $T \in \mathcal{L}^r(X)$, we have $\imath S'T', \imath R' + \imath S' \in \mathcal{SK}^r(X', X'_\infty)$, so $\mathcal{A}$ is a left algebra ideal. To see $\mathcal{A}$ is an order ideal, just note $|T| \leq S \in \mathcal{A}$ $\Rightarrow \imath T'_\pm \leq \imath S' \in \mathcal{SK}^r(X', X'_\infty) \Rightarrow T \in \mathcal{A}$. Also, $\mathrm{id}_X \notin \mathcal{A}$, for if $y \in \prod_n E'_n$ is such that $\sup\{\|(|\imath(x)| - y)_+\| : x \in X'_{[1]}\} \leq \varepsilon < 1$, then $\|y_n\|_{p'} \geq (1-\varepsilon)n^{1/p'}$ $(n \in \mathbb{N})$. Thus, $\mathcal{A}$ is a proper order and left algebra ideal of $\mathcal{L}^r(X)$. Next, one verifies $\mathcal{A} = \mathcal{B}$ (the argument here is almost the same as before, simply replace $p_i S$ by $\imath'_i S'$ and X by X' in the corresponding paragraphs of (I)).

As in part (I), we show next that $\mathcal{B}$ contains all proper ideals of $\mathcal{L}^r(X)$. We shall do this, once again, by showing that if $T \in \mathcal{L}^r(X)_+ \setminus \mathcal{B}$ then $\mathcal{A}_T = \mathcal{L}^r(X)$. So let $T \in \mathcal{L}^r(X)_+ \setminus \mathcal{B}$ and let $\mathcal{V}$ be a free ultrafilter on $\mathbb{N}$ such that $(\imath_i' T')_{\mathcal{V}}$ fails to be semicompact. To shorten the argument, we handle all possible values of q (i.e., also $q = 1$) at once. We consider separately the cases $1 < p < \infty$ and $p = \infty$:

• $1 < p < \infty$. As in part (I), this time with $\mathcal{T} := (\imath_i' T')_{\mathcal{V}} : (X')_{\mathcal{V}} \to (E_i')_{\mathcal{V}}$, one obtains a strictly increasing sequence $(j_n) \subset \mathbb{N}$, and for every $n \in \mathbb{N}$, disjoint sequences $(\xi_{j_n,i})_{i=1}^n \subset X'_{[1]}$ and $(\phi_{j_n,i})_{i=1}^n \subset (E_{j_n})_{[1]}$ such that

$$\xi_{j_n,i}(T\imath_{j_n}(\phi_{j_n,i})) = \phi_{j_n,i}(\imath_{j_n}' T'(\xi_{j_n,i})) > \delta/2 \quad (1 \le i \le n), \tag{3.3}$$

where $0 < \delta < \rho(\mathcal{T})$. (The proof that X' satisfies an upper p'-estimate is basically the same as that of X satisfying an upper p-estimate in part (I); the case $q' = \infty$ is treated exactly as the case $q = 0$.)

Define

$$R_n := \sum_{i=1}^n \xi_{j_n,i} \otimes e_{i,n} \quad \text{and} \quad S_n := \sum_{i=1}^n e_{i,n}^* \otimes \phi_{j_n,i} \qquad (n \in \mathbb{N}),$$

where, as before, $(e_{i,n})_{i=1}^n$ stands for the unit vector basis of E_n, and $(e_{i,n}^*)_{i=1}^n$ for the corresponding basis of biorthogonal functionals in E_n'. Note that $\|R_n'\| = \|\sum_i e_{i,n} \otimes \xi_{j_n,i} : E_n' \to X'\|$ and $\|S_n'\| = \|\sum_i \phi_{j_n,i} \otimes e_{i,n}^* : E_{j_n}' \to E_n'\|$, so arguing as in the first part of the proof, one finds that $\|R_n\|_r, \|S_n\|_r \le 1$ $(n \in \mathbb{N})$. Also note that

$$e_{i,n}^*(R_n T \imath_{j_n} S_n(e_{i,n})) = \xi_{j_n,i}(T\imath_{j_n}(\phi_{j_n,i})) \quad (1 \le i \le n,\, n \in \mathbb{N}). \tag{3.4}$$

Now, for every $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \{\pm 1\}$, one has

$$\Big\|\sum_{i=1}^n \alpha_i T \imath_{j_n} S_n e_{i,n}\Big\| \le \|T\| \Big(\sum_i |\alpha_i|^p\Big)^{1/p} = \|T\| n^{1/p},$$

and so, given $\varepsilon > 0$, the number of vectors in $(T\imath_{j_n} S_n e_{i,n})_{i=1}^n$ with the same entry outside $(-\varepsilon, \varepsilon)$ cannot exceed $\lceil \varepsilon^{-1} n^{1/p} \|T\| \rceil$. In turn, for $m \in \mathbb{N}$ fixed and every $n > m\lceil \varepsilon^{-1} n^{1/p} \|T\| \rceil$, $(T\imath_{j_n} S_n e_{i,n})_{i=1}^n$ must contain at least $n - m\lceil \varepsilon^{-1} n^{1/p} \|T\| \rceil$ vectors with all their first m entries in $(-\varepsilon, \varepsilon)$. There is thus a strictly increasing sequence (n_k) such that, for every $k \in \mathbb{N}$, the set $J_k := \{i : \|P_k T \imath_{j_{n_k}} S_{n_k} e_{i,n_k}\| < 2^{-k}\}$ contains at least k elements.

For each k, choose $\{j_1, \dots, j_k\} \subset J_k$, and let $\sigma_k : E_k \to E_{n_k}$ and $\tau_k : E_{n_k} \to E_k$ be the linear maps defined by $\sigma_k(e_{i,k}) = e_{j_i,n_k}$ and $\tau_k'(e_{i,k}^*) = e_{j_i,n_k}^*$ $(1 \le i \le k)$. Set $S := \sum_k \imath_{j_{n_k}} S_{n_k} \sigma_k p_k$ and $T_1 := TS$. It follows from our choice of (n_k) that, for every $i \in \mathbb{N}$, if $j, N \in \mathbb{N}$ are such that $j > 1 + \dots + N$ and $N > i$, then

$$
\begin{aligned}
\|x_i^* \circ T_1(\mathrm{id} - P_j)\| = \|x_i^* \circ (TS(\mathrm{id} - P_j))\| &\le \sum_{k>N} \|x_i^* T \imath_{j_{n_k}} S_{n_k} \sigma_k p_k (\mathrm{id} - P_j)\| \\
&\le \sum_{k>N} \|x_i^* P_k T \imath_{j_{n_k}} S_{n_k} \sigma_k p_k\| \le \sum_{k>N} \|P_k T \imath_{j_{n_k}} S_{n_k} \sigma_k p_k\| \\
&= \sum_{k>N} \Big\| \sum_{i=1}^{k} e_{i,k}^* \otimes (P_k T \imath_{j_{n_k}} S_{n_k} e_{j_i,n_k}) \Big\| \le \sum_{k>N} k 2^{-k},
\end{aligned}
$$

and hence that $\lim_j \|x_i^* \circ T_1(\mathrm{id} - P_j)\| = 0$. Furthermore, letting $\widetilde{R_k} := \tau_k R_{n_k}$ and combining (3.3) and (3.4), one sees that

$$
e_{i,k}^*(\widetilde{R}_k T_1 \imath_k(e_{i,k})) = e_{j_i,n_k}^*(R_{n_k} T \imath_{j_{n_k}} S_{n_k}(e_{j_i,n_k})) > \delta/2 \quad (1 \le i \le k).
$$

From this point onwards, one can argue along the same lines as in part (I). First, by Lemma 3.4, there are bounded sequences of operators $(A_k : X \to E_k)_{k\in\mathbb{N}}$ and $(B_k : E_k \to E_{n_k})_{k\in\mathbb{N}}$ such that $\mathrm{id}_{E_k} = A_k T_1 \imath_{n_k} B_k$ $(k \in \mathbb{N})$. Then, by Lemma 2.1, there is a subsequence (n_{k_j}) of (n_k) and block diagonal regular operators D_1, D_2 such that $T_1 - D_1 - D_2 =: A \in \mathcal{A}^r(X)$ and $D_2(\sum_j \imath_{n_{k_j}} p_{n_{k_j}}) = 0$. Assume $\|A\|_r \le (4 \sup_k \|A_k\|_r \sup_k \|B_k\|_r)^{-1}$. Let π_j and γ_j be maps as in part (I), and define $L_j := \pi_j A_{k_j}$ and $M_j := B_{k_j}(A_{k_j} D_1 \imath_{n_{k_j}} B_{k_j})^{-1} \gamma_j$ $(j \in \mathbb{N})$. One readily checks that $\mathrm{id}_{E_j} = L_j D_1 \imath_{n_{k_j}} M_j$, that $\|L_j\|_r \le \|A_{k_j}\|_r$ and $\|M_j\|_r \le 2\|B_{k_j}\|_r$ $(j \in \mathbb{N})$. Set $m_j := n_{k_j}$ $(j \in \mathbb{N})$ and construct, inductively, strictly increasing sequences (l_k) and (i_k) such that $P_{l_{k-1}} \imath_{m_{i_k}} = 0$, $P_{l_k} D_1 = D_1 P_{l_k}$ and $P_{l_k} \imath_{m_{i_k}} = \imath_{m_{i_k}}$ $(k \in \mathbb{N})$, where $l_0 = 0$ and $P_0 = 0$. Define

$$
U := \sum_k \imath_{i_k} L_{i_k} Q_k \quad \text{and} \quad V := \sum_k \imath_{m_{i_k}} M_{i_k} p_{i_k}.
$$

Then $\|U\|_r \le 2 \sup_k \|L_k\|_r$, $\|V\|_r \le \sup_k \|M_k\|_r$ and $UTV = \sum_k \imath_{i_k} p_{i_k} + UAV$. The rest of the argument remains the same as in part (I).

• $p = \infty$: Again, arguing as in part (I), with $\mathcal{T} := (\imath_i' T')_{\mathcal{V}} : (X')_{\mathcal{V}} \to (E_i')_{\mathcal{V}}$, one obtains a strictly increasing sequence $(j_n) \subset \mathbb{N}$, and for every $n \in \mathbb{N}$, a disjoint sequence $(\phi_{j_n,i})_{i=1}^n \subset (E_{j_n})_{[1]}$ and a sequence $(\xi_{j_n,i})_{i=1}^n \subset X'_{[1]}$ such that $\xi_{j_n,i}(T \imath_{j_n}(\phi_{j_n,i})) = \phi_{j_n,i}(\imath_{j_n}' T'(\xi_{j_n,i})) > \delta/2$ $(1 \le i \le n)$, with $0 < \delta < \rho(\mathcal{T})$. Let $R_n := \sum_{i=1}^n \xi_{j_n,i} \otimes e_{i,n}$ and $S_n := \sum_{i=1}^n e_{i,n}^* \otimes \phi_{j_n,i}$ $(n \in \mathbb{N})$. From this point onwards, one proceeds exactly as above.

To see $\mathcal{M}_X^{\mathrm{oa}} = \mathcal{L} = \{T \in \mathcal{L}^r(X) : T' \in \mathcal{SK}^r(X')\}$ if $q = 1$, simply note that, in this situation, $X'_\infty = X'$ and $\imath = \mathrm{id}_{X'}$, so for every $T \in \mathcal{A}$ and $S \in \mathcal{L}^r(X)$, $\imath(TS)' = S'T' \in \mathcal{SK}^r(X', X'_\infty)$ (because $\mathcal{SK}^r(X', X'_\infty)$ is closed under composition with regular maps from the left), and therefore $\mathcal{A}$ is an ideal. ∎

REMARK 3.5. Let $\mathbf{e} := (e_i)$ be a normalized 1-unconditional Schauder basis for some Banach space E, and let $X = (\bigoplus_i \ell_p^i)_{\mathbf{e}} := \{(x_i) \in \prod_i \ell_p^i : \sum_i \|x_i\|_p e_i$ converges$\}$. Endowed with the norm $\|(x_i)\| := \|\sum_i \|x_i\|_p e_i\|_E$ ($(x_i) \in X$) and the order induced by the cone $X \cap \prod_i (\ell_p^i)_+$, the latter becomes a Banach lattice. It is easy to see that if $\mathbf{e}$ is subsymmetric shrinking, $p < \infty$ and X satisfies an upper p-estimate (e.g., if E is p-convex) (resp. $\mathbf{e}$ is subsymmetric, $1 < p$ and X satisfies a lower p-estimate (e.g., if E is p-concave)), then almost the same argument as above implies that $\mathcal{L}^r(X)$ has a unique maximal ideal. Since the modifications needed are fairly straightforward, we leave the details to the reader.

It should be noticed that the use of Lemma 3.4, in the proof of Theorem 3.1, could be easily avoided. Indeed, if X is as in part (I) and $T \in \mathcal{L}^r(X)_+ \setminus \mathcal{B}$, then from Lemma 2.1 and the fact that $\mathcal{A}^r(X) \subseteq \mathcal{B}$ it follows that there is a positive block diagonal operator $D \in \mathcal{A}_T \setminus \mathcal{B}$. Arguing exactly as in the proof of Theorem 3.1, one constructs (j_n), (R_n) and (S_n) such that $e_{i,n}^*(R_n p_{j_n} D S_n(e_{i,n})) \geq 2^{-1}\delta$ $(1 \leq i \leq n,\ n \in \mathbb{N})$. One then defines (l_k) and (n_k) so that $p_{j_{n_k}} P_{l_{k-1}} = 0$, $P_{l_k} D = D P_{l_k}$ and $p_{j_{n_k}} P_{l_k} = p_{j_{n_k}}$ $(k \in \mathbb{N})$, with $l_0 = 0$ and $P_0 = 0$. The maps

$$U := \sum_k \imath_{n_k} R_{n_k} p_{j_{n_k}} \quad \text{and} \quad V := \sum_k (P_{l_k} - P_{l_{k-1}}) S_{n_k} p_{n_k}$$

are both regular and satisfy $UDV \geq 2^{-1}\delta \sum_k \imath_{n_k} p_{n_k}$. Thus, $\sum_k \imath_{n_k} p_{n_k}$ belongs to the ideal generated by T, and in turn so does id_X. If X is as in part (II) and $T \in \mathcal{L}^r(X)_+ \setminus \mathcal{B}$, one first constructs $T_1 \in \mathcal{L}^r(X)_+ \setminus \mathcal{B}$ as in the proof given, and then continues as we just explained.

On the other hand, unlike the argument of the previous paragraph, the argument of the proof of Theorem 3.1 can be almost immediately adapted to the algebraic situation to give the following analogue of Theorem 3.1 (below, $\mathcal{SK}(X,Y)$ stands for the vector space of all semicompact operators from X to Y).

THEOREM 3.6. *Let* $X = (\bigoplus_i \ell_p^i)_q$ *with* $p \in [1,\infty]$, $q \in [1,\infty) \cup \{0\}$ *and* $p \neq q$. *Let* (p_i), $(\imath_i)$ *and* $\imath$ *be as in Theorem* 3.1.

(I) *Suppose that* $1 \leq p < \infty$ *and* $p < q < \infty$ *or* $q = 0$, *and let* $\mathcal{R} = \{T \in \mathcal{L}^r(X) : \imath T \in \mathcal{SK}(X, X_\infty)\}$. *Then*

$$\mathcal{M}_X^a := \{T \in \mathcal{R} : ST \in \mathcal{R}\ (S \in \mathcal{L}^r(X))\}$$

is the only maximal algebra ideal of $\mathcal{L}^r(X)$.

(II) *Suppose that* $1 \leq q < p \leq \infty$ *and let* $\mathcal{L} = \{T \in \mathcal{L}^r(X) : \imath T' \in \mathcal{SK}(X', X'_\infty)\}$. *Then*

$$\mathcal{M}_X^a := \{T \in \mathcal{L} : TS \in \mathcal{L}\ (S \in \mathcal{L}^r(X))\}$$

is the only maximal algebra ideal of $\mathcal{L}^r(X)$.

Proof. Almost the same argument as in the proof of Theorem 3.1 works in the present situation, so we only need to indicate the modifications needed and the reasons for them.

(I) Let $X = (\bigoplus_i \ell_p^i)_q$ with $1 \le p < \infty$ and either $p < q < \infty$ or $q = 0$. Set $\mathcal{A} := \{T \in \mathcal{L}^r(X) : \imath T \in \mathcal{SK}(X, X_\infty)\}$ and $\mathcal{B} := \{T \in \mathcal{L}^r(X) : \rho((p_i T)_{\mathcal{U}}) = 0 (\mathcal{U} \in \mathfrak{U})\}$, where, as before, $\mathfrak{U}$ stands for the set of all free ultrafilters on $\mathbb{N}$. This time one only needs to check that $\mathcal{A}$ is a proper right algebra ideal of $\mathcal{L}^r(X)$. The proof that $\mathcal{A} = \mathcal{B}$ is basically the same (the argument given for a positive T works equally well without the positivity assumption). That $T \in \mathcal{L}^r(X) \setminus \mathcal{B}$ implies $\mathcal{A}_T = \mathcal{L}^r(X)$ is proved in exactly the same way as if T were positive (since $\mathcal{B}$ need not be an order ideal, it is not enough to consider the case $T \ge 0$, as in the proof of Theorem 3.1). Lastly, the desired conclusion follows on applying the analogue of Lemma 3.3 in the merely ring-theoretical situation.

(II) Let $X = (\bigoplus_i \ell_p^i)_q$ with $1 \le q < p \le \infty$. Set $\mathcal{A} := \{T \in \mathcal{L}^r(X) : \imath T' \in \mathcal{SK}(X', X'_\infty)\}$ and $\mathcal{B} := \{T \in \mathcal{L}^r(X) : \rho((\imath_i' T')_{\mathcal{U}}) = 0\ (\mathcal{U} \in \mathfrak{U})\}$. One readily sees that $\mathcal{A}$ is a proper left algebra ideal of $\mathcal{L}^r(X)$. The rest remains almost the same as in part (II) of the proof of Theorem 3.1 (the remarks made above apply here as well). Once again, one relies on the ring-theoretical analogue of Lemma 3.3 to finish the proof. ∎

Regarding the relation between maximal (order and algebra) ideals and maximal algebra ideals of the algebras $\mathcal{L}^r((\bigoplus_i \ell_p^i)_q)$, our next result provides a partial answer.

PROPOSITION 3.7. *Let X be as in Theorem* 3.1. *If* $1 < p < \infty$ *then* $\mathcal{M}_X^{\mathrm{oa}} \subsetneq \mathcal{M}_X^a$.

Proof. Let $1 < p < \infty$ and let $T \in \mathcal{L}^r(\ell_p)$ be a compact operator such that $|T| \notin \mathcal{A}^r(\ell_p)$. By Lemma 2.1, we can assume T is block diagonal. Let (Q_i) be a sequence of disjoint band projections, commuting with T, and such that $\lim_i \|Q_i T Q_i\| = 0$ and $\inf_i \|Q_i T Q_i\|_r =: \delta > 0$. Let $n_0 := 0$, and for each $k \in \mathbb{N}$, let $n_k := 1 + \cdots + k$, let $\widetilde{Q}_k := \sum_{n_{k-1} < i \le n_k} Q_i$, let $F_k := \widetilde{Q}_k(\ell_p)$, let $l_k := \operatorname{rk} \widetilde{Q}_k$, and lastly, let $\phi_k : E_{l_k} \to F_k$ be the linear map defined by $\phi_k(e_{i,l_k}) := e_{i+m_k}$ $(1 \le i \le l_k)$, where $m_k := \sum_{i<k} l_i$ and e_j $(j \in \mathbb{N})$ stands for the jth unit vector in ℓ_p. Define $S := \sum_k \imath_{l_k} \phi_k^{-1} (\widetilde{Q}_k T \widetilde{Q}_k)|_{F_k}^{F_k} \phi_k p_{l_k}$ (as usual, convergence of the series to be understood in the strong operator sense). Then $S \in \mathcal{L}^r(X)$ is compact and it is easy to produce regular operators A and B such that $\mathrm{id}_X = A|S|B$ (e.g., let $(\lambda_i) \subset (\ell_p')_{[1]}$ and $(\xi_i) \subset (\ell_p)_{[1]}$ be such that $\lambda_i Q_i = \lambda_i$, $Q_i \xi_i = \xi_i$ and $\lambda_i(Q_i |T| Q_i \xi_i) = \delta$ $(i \in \mathbb{N})$; then one can take $A := \delta^{-1} \sum_k \imath_k (\sum_{i=1}^k \lambda_{i+n_{k-1}} \otimes e_{i,k}) \phi_k p_{l_k}$ and $B := \sum_k \imath_{l_k} \phi_k^{-1} (\sum_{i=1}^k e_{i,k}^* \otimes \xi_{i+n_{k-1}}) p_k)$. Since S is compact, $\mathcal{A}_S$ is a proper algebra ideal of $\mathcal{L}^r(X)$, and hence, $\mathcal{A}_S \subseteq \mathcal{M}_X^a$. On the other hand, since

$\mathcal{A}_{|S|} = \mathcal{L}^r(X)$, $\mathcal{A}_S$ cannot be contained in $\mathcal{M}_X^{\mathrm{oa}}$ (otherwise, we would have $|S| \in \mathcal{M}_X^a$). Thus, $\mathcal{M}_X^{\mathrm{oa}} \subsetneq \mathcal{M}_X^a$. ∎

We do not know what happens when $p = 1$ or $p = \infty$.

There is yet another possible description of the maximal algebra ideal of $\mathcal{L}^r((\bigoplus_i \ell_p^i)_q)$, in line with that of [KL] and [Le] for the corresponding algebra $\mathcal{B}((\bigoplus_i \ell_p^i)_q)$, which we believe is worth mentioning. First, mimicking the Banach space situation, let us say that $T \in \mathcal{L}^r(X)$ *r-factors through* $L \in \mathcal{L}^r(E,F)$ if there are regular operators $R : F \to X$ and $S : X \to E$ such that $RLS = T$. The last result of the section can then be stated as follows:

PROPOSITION 3.8. *Let X be as in Theorem* 3.1. *Then* $\mathcal{M}_X^a = \{T \in \mathcal{L}^r(X) :$ id_X *does not r-factor through* $T\}$.

Proof. As in the Banach space situation, it will suffice to show that for every $T \in \mathcal{L}^r(X)$, either $T \notin \mathcal{M}_X^r$ or $\mathrm{id} - T \notin \mathcal{M}_X^r$ (see the proof of [DJ, Proposition 5.1]—the same proof works for $\mathcal{L}^r(X)$). Now, let $T \in \mathcal{L}^r(X)$ be arbitrary, let

$$a_k := |\{1 \le i \le k : |e_{i,k}^*(Te_{i,k})| \ge 2^{-1}\}|,$$
$$b_k := |\{1 \le i \le k : |e_{i,k}^*((\mathrm{id} - T)e_{i,k})| \ge 2^{-1}\}| \quad (k \in \mathbb{N}),$$

where $|\cdot|$ stands for the size of the underlying set. It is clear that either $\sup\{a_i : i \in \mathbb{N}\} = \infty$ or $\sup\{b_i : i \in \mathbb{N}\} = \infty$, so to show id_X r-factors through either T or $\mathrm{id} - T$, it will suffice to show that if $S \in \mathcal{L}^r(X)$ satisfies $\inf\{|e_{i,k}^*(Se_{i,k})| : 1 \le i \le k,\ k \in \mathbb{N}\} =: \delta > 0$, then id_X r-factors through S. Furthermore, we can clearly restrict attention to the case where $e_{i,k}^*(Se_{i,k}) > 0$ $(1 \le i \le k,\ k \in \mathbb{N})$.

So let $S \in \mathcal{L}^r(X)$ be such that $e_{i,k}^*(Se_{i,k}) \ge \delta > 0$ $(1 \le i \le k,\ k \in \mathbb{N})$, and for each $n \in \mathbb{N}$, set $j_n := n$, $R_n := p_n$ and $S_n := \imath_n$. Then, arguing exactly as in part (I) of the proof of Theorem 3.1 if the unit vector basis of X is shrinking, i.e., if $q \ne 1$, and as in part (II) of that proof if $q = 1$, one constructs the required factorization of id_X through S. ∎

Theorem 3.6 and Proposition 3.8 also hold in the complex case. The only difference between the proofs in the real and complex cases is that, in the complex case, one needs to use the complex versions of Lemmas 3.3 and 3.4 (the proofs of these lemmas in the complex case are completely analogous to those in the real case—the modifications needed are minor and fairly straightforward). The proof of Proposition 3.7 is clearly independent of the underlying field.

4. Further results about the lattices of closed ideals of $\mathcal{L}^r((\bigoplus_i \ell_p^i)_0)$ and $\mathcal{L}^r((\bigoplus_i \ell_p^i)_1)$ for $1 < p < \infty$. Let $X = (\bigoplus_i \ell_p^i)_0$ (resp. $X = (\bigoplus_i \ell_p^i)_1$) and $E = c_0$ (resp. $E = \ell_1$). We shall say that $T \in \mathcal{L}^r(X)$ *r-factors through*

E if it r-factors through id_E (see the paragraph preceding Proposition 3.8), i.e., if there are regular operators $R : E \to X$ and $S : X \to E$ such that $RS = T$. We let

$$\begin{aligned} \mathit{\Gamma}_{E,r}(X) &:= \operatorname{sp}\{T \in \mathcal{L}^r(X) : T \text{ } r\text{-factors through } E\}, \\ \mathit{\Gamma}^o_{E,r}(X) &:= \{T \in \mathcal{L}^r(X) : \pm T \le S \text{ for some } S \in \mathit{\Gamma}_{E,r}(X)\}. \end{aligned}$$

It is easy to see that $\mathit{\Gamma}_{E,r}(X)$ is an algebra ideal and that $\mathit{\Gamma}^o_{E,r}(X)$ is the smallest (order and algebra) ideal that contains $\mathit{\Gamma}_{E,r}(X)$. (The above, of course, makes sense for any pair of Banach lattices X, E, but we shall not make use of such generality here.) Furthermore, one can verify, exactly as in the Banach space situation, that $\mathit{\Gamma}_{E,r}(X) = \{T \in \mathcal{L}^r(X) : T \text{ } r\text{-factors through } E\}$.

The first result of the section concerns the ideals $\mathit{\Gamma}^o_{c_0,r}((\bigoplus_i \ell_p^i)_0)$ and $\mathit{\Gamma}^o_{\ell_1,r}((\bigoplus_i \ell_p^i)_1)$. Recall that an operator $T \in \mathcal{L}^r(X)$ is said to be *majorizing* if there is a constant $m \ge 0$ such that for every finite sequence $(x_i)_{i=1}^n \subset X$,

$$\Big\|\sup_i |Tx_i|\Big\| \le m \sup_i \|x_i\|,$$

and *cone absolutely summing* if there is a constant $l \ge 0$ such that for every finite sequence $(x_i)_{i=1}^n \subset X_+$,

$$\sum_i \|Tx_i\| \le l \Big\|\sum_i x_i\Big\|$$

(here we have used [Sc, Chapter IV, Propositions 3.3(d) and 3.4(d)]). It is fairly straightforward to verify that the collections

$$\begin{aligned} \mathcal{M}(X) &:= \{T \in \mathcal{L}^r(X) : T \text{ is majorizing}\}, \\ \mathit{\Pi}^+(X) &:= \{T \in \mathcal{L}^r(X) : T \text{ is cone absolutely summing}\}, \end{aligned}$$

are both order and algebra ideals of $\mathcal{L}^r(X)$.

Proposition 4.1.

(i) *If* $X := (\bigoplus_i \ell_p^i)_0$, $1 \le p < \infty$, *then* $\mathit{\Gamma}_{c_0,r}(X) = \mathit{\Gamma}^o_{c_0,r}(X) = \mathcal{M}(X)$.

(ii) *If* $X := (\bigoplus_i \ell_p^i)_1$, $1 < p \le \infty$, *then* $\mathit{\Gamma}_{\ell_1,r}(X) = \mathit{\Gamma}^o_{\ell_1,r}(X) = \mathit{\Pi}^+(X)$.

Proof. (i) Let $X = (\bigoplus_i \ell_p^i)_0$, $1 \le p < \infty$. Since $\mathcal{M}(X)$ is an order ideal, it will suffice to show that $\mathcal{M}(X) = \mathit{\Gamma}_{c_0,r}(X)$.

Suppose first $T \in \mathcal{M}(X)$. Let (P_i) be the sequence of natural projections with respect to the unit vector basis (x_i) of X, and for each $k \in \mathbb{N}$, choose n_k so that $\|(\mathrm{id}_X - P_{n_k})Tx_k\| \le 2^{-k}$. Let $N := \sum_k x_k^* \otimes ((\mathrm{id} - P_{n_k})Tx_k)$, where x_k^* ($k \in \mathbb{N}$) stands for the kth biorthogonal functional with respect to (x_i), and set $T_1 := T - N$. Note that $N \in \mathit{\Gamma}_{c_0,r}(X)$, for N can be written as the composite of the natural embedding $\imath : X \to c_0$ with $R : c_0 \to X$, $(\alpha_k) \mapsto \sum_k \alpha_k(\mathrm{id} - P_{n_k})Tx_k$. Also note that since $\mathcal{M}(X)$ is an order ideal and $|T_1| \le |T|$, $T_1 \in \mathcal{M}(X)$. Hence, by [Sc, Proposition 3.4(c′)], there exists an AM-space Z, an operator $V \in \mathcal{B}(X,Z)$ and an injective Riesz homomorphism $U : Z \to X$ such that $U(Z)$ is an order ideal of X and $UV = T_1$.

We claim that we can further assume $Z = c_0$. Indeed, since Z is Riesz isomorphic to $U(Z)$ and the latter is an ideal of X, the atomicity of X implies that of Z. Let $\{z_j : j \in J\}$ be the set of all normalized atoms of Z. Since each $T_1 x_i$ ($= UVx_i$) is finitely supported in X and U is an injective Riesz homomorphism, we have $Vx_i \in \mathrm{sp}\,\{z_j : j \in J\}$, and so we can replace Z with $[z_j]_{j\in J}$, and U with the corresponding restriction. The claim follows readily from this.

So suppose $Z = c_0$. We show next that V is regular. For this, let $\varepsilon > 0$ and $x \in X_+$ be arbitrary, and let $\imath : c_0 \to \ell_\infty$ be the natural inclusion. Choose $0 \le \eta \le x$ of finite support so that $\big\||\imath \circ V|(x-\eta)\big\|_\infty < \varepsilon/2$, and then choose n_0 such that $n \ge n_0 \Rightarrow e_n^*(|\imath\circ V|(\eta)) < \varepsilon/2$, where $e_n^* : \ell_\infty \to \mathbb{R}$, $(\alpha_i) \mapsto \alpha_n$. Then, for every $n \ge n_0$ and $\xi \in X$, with $|\xi| \le x$,

$$e_n^*(\imath(|V\xi|)) \le e_n^*(|\imath\circ V|(x)) \le e_n^*(|\imath\circ V|(\eta)) + \big\||\imath\circ V|(x-\eta)\big\|_\infty < \varepsilon.$$

It follows that $\sup\{|V\xi| : |\xi| \le x\} \in c_0$ $(x \in X_+)$, and hence $|V|$ exists. Thus, $T_1 \in \mathit{\Gamma}_{c_0,r}(X)$, and since $N \in \mathit{\Gamma}_{c_0,r}(X)$ and $\mathit{\Gamma}_{c_0,r}(X)$ is a linear space, $T \in \mathit{\Gamma}_{c_0,r}(X)$.

In the opposite direction, let $T \in \mathit{\Gamma}_{c_0,r}(X)$ be arbitrary, and let $V : X \to c_0$ and $U : c_0 \to X$ be regular operators such that $UV = T$. To show $T \in \mathcal{M}(X)$, it will suffice (by [Sc, Proposition 3.4(b)]) to show that there exists $f = (f_i) \in (\bigoplus_i \ell_p^i)_\infty$ such that $T(X_{[1]}) \subseteq [-f, f]$. The latter follows readily on noting that, for every $i \in \mathbb{N}$, $p_iT = (p_iU)V = (p_iU)''(\kappa_{c_0}V)$, and so, for every $x \in X_{[1]}$,

$$|p_iTx| \le |(p_iU)''|\,|\kappa_{c_0}Vx| \le \|V\|_r|(p_iU)''|(e) =: f_i,$$

where e stands for the constant sequence of 1s in ℓ_∞. Clearly, $\|f_i\| \le \|U\|_r\|V\|_r$ $(i \in \mathbb{N})$, whence the desired result.

(ii) Let $X = (\bigoplus_i \ell_p^i)_1$, $1 < p \le \infty$. As in the previous case, it will suffice to show that $\mathit{\Gamma}_{\ell_1,r}(X) = \mathit{\Pi}^+(X)$.

Suppose first $T \in \mathit{\Gamma}_{\ell_1,r}(X)$, and let $V \in \mathcal{L}^r(X,\ell_1)$ and $U \in \mathcal{L}^r(\ell_1, X)$ be such that $T = UV$. Then, for every sequence $x_1, \ldots, x_n \in X_+$,

$$\begin{aligned}\sum_i \|Tx_i\| \le \|U\|_r \sum_i \big\||V|(x_i)\big\| &= \|U\|_r \Big\|\sum_i |V|(x_i)\Big\| \\ &\le \|U\|_r\|V\|_r\Big\|\sum_i x_i\Big\|\end{aligned}$$

(the equality holds because for any $\xi_1, \ldots, \xi_n \in \ell_1$, $\big\|\sum_i |\xi_i|\big\| = \sum_i \|\xi_i\|$), and so, $T \in \mathit{\Pi}^+(X)$.

Next, let $T \in \mathit{\Pi}^+(X)$ be arbitrary. By [Sc, Proposition 3.3(c′)], there is an AL-space Z, a lattice homomorphism $V : X \to Z$, with dense range, and a linear map $U : Z \to X$ such that $T = UV$. In this situation, Z must be isometrically Riesz isomorphic to a vector sublattice of ℓ_1. To see this, first note that since V is a Riesz homomorphism, (Vx_i) must be a disjoint sequence in Z. But $[Vx_i : i \in \mathbb{N}] = Z$ and Z is an AL-space, so each Vx_i different from zero must be an atom in Z. We can thus assume $Z = \ell_1$. That U is regular is now obvious, and so, $T \in \Gamma_{\ell_1,r}(X)$. ∎

We now come to the main result of the section. For comparison, recall first from [LLR] and [LSZ] that the only proper non-trivial closed ideals of $\mathcal{B}((\bigoplus_i \ell_2^i)_0)$ and $\mathcal{B}((\bigoplus_i \ell_2^i)_1)$ are the corresponding ideals of approximable operators and the norm-closures of the ideals $\{T \in \mathcal{B}((\bigoplus_i \ell_p^i)_0) : T \text{ factors through } c_0\}$ and $\{T \in \mathcal{B}((\bigoplus_i \ell_p^i)_1) : T \text{ factors through } \ell_1\}$, respectively. It will be seen next that the analogues of these results for the algebras $\mathcal{L}^r((\bigoplus_i \ell_2^i)_0)$ and $\mathcal{L}^r((\bigoplus_i \ell_2^i)_1)$ are no longer true.

THEOREM 4.2.

(i) *Let* $X := (\bigoplus_i \ell_p^i)_0$, $1 < p < \infty$. *Then*

$$\mathcal{A}^r(X) \subsetneq \overline{\Gamma^o_{c_0,r}(X)} \subsetneq \mathcal{M}_X^{\mathrm{oa}},$$

and any non-trivial proper closed ideal $\mathcal{I}$ *of* $\mathcal{L}^r(X)$, *different from* $\mathcal{A}^r(X)$, *must satisfy*

$$\overline{\Gamma^o_{c_0,r}(X)} \subseteq \mathcal{I} \subseteq \mathcal{M}_X^{\mathrm{oa}}.$$

In particular, the lattice of closed ideals of $\mathcal{L}^r(X)$ *contains at least three distinct non-trivial proper closed ideals.*

(ii) *Let* $X := (\bigoplus_i \ell_p^i)_1$, $1 < p < \infty$. *Then*

$$\mathcal{A}^r(X) \subsetneq \overline{\Gamma^o_{\ell_1,r}(X)} \subsetneq \mathcal{M}_X^{\mathrm{oa}},$$

and any non-trivial proper closed ideal $\mathcal{I}$ *of* $\mathcal{L}^r(X)$, *different from* $\mathcal{A}^r(X)$, *must satisfy*

$$\overline{\Gamma^o_{\ell_1,r}(X)} \subseteq \mathcal{I} \subseteq \mathcal{M}_X^{\mathrm{oa}}.$$

In particular, the lattice of closed ideals of $\mathcal{L}^r(X)$ *contains at least three distinct non-trivial proper closed ideals.*

REMARK 4.3. The algebra ideal $\mathcal{A}^r(X)$ was introduced in Section 2. It is clearly the smallest closed non-trivial algebra ideal of $\mathcal{L}^r(X)$. When X is atomic and order continuous (which is the case for all Banach lattices considered in the note), $\mathcal{A}^r(X)$ is also an order ideal [Wi1, Theorem 4.1].

Our proof of Theorem 4.2 will rely on the following result concerning semicompact operators on $L^p[0,1]$ $(1 < p < \infty)$.

THEOREM 4.4. *For every* $\tau \in (0,1)$, *there exists* $S_\tau \in \mathcal{SK}^r(L^p[0,1])_+$ *which coincides with* $\tau\, \mathrm{id}_{L^p[0,1]}$ *on the linear span of the Rademacher functions.*

For $1 < p < 2$, the above result was first proved by Tradacete [Tr, Lemma 1 and Example 2]. We are grateful to the anonymous referee for bringing this to our attention. The elegant argument that we provide below was communicated to us by Anton Schep. The operator in question is the so-called *convolution by a bias coin operator* (in [Tr, Example 1] a modified version of the same operator is used). Note that an immediate consequence of the above is that the ideal $\mathcal{K}^r(L^p[0,1])$, defined as the span of the positive compact operators on $L^p[0,1]$, is properly contained in the ideal $\mathcal{SK}^r(L^p[0,1])$ ($1 < p < \infty$). (Of course, this is trivially true for $p = \infty$, and for $p = 1$ it follows readily from the obvious fact that the map $\ell_1 \to [r_i]_{i\in\mathbb{N}}$, $(\alpha_i) \mapsto \sum_i \alpha_i r_i$, is semicompact, where r_i stands for the ith Rademacher function.)

Proof of Theorem 4.4. Fix $1 < p < \infty$, let D^∞ stand for the *Cantor group* (i.e., the set $\{\pm 1\}^{\mathbb{N}}$ with pointwise product as group operation) and let μ be the *Haar measure* on D^∞ (i.e., $\mu := \prod_n \mu_n$, where $\mu_n(\{1\}) = \mu_n(\{-1\}) = 1/2$ ($n \in \mathbb{N}$)). Let $0 < \tau < 1$, let $\nu_\tau := \prod_n \nu_{\tau,n}$, where $\nu_{\tau,n}(\{1\}) = (1+\tau)/2$ and $\nu_{\tau,n}(\{-1\}) = (1-\tau)/2$ ($n \in \mathbb{N}$), and for every $f \in L^p(D^\infty, \mu)$, let $T_\tau(f) := \nu_\tau * f$, where

$$(\nu_\tau * f)(x) = \int_{D^\infty} f(xy)\, d\nu_\tau(y) \qquad (x \in D^\infty).$$

By [Bo, Chapitre III, Théorème 3], ν_τ is an improving measure, and so there exist $\delta > 0$ and $C > 0$ such that

$$T_\tau(f) \in L^{p+\delta}(D^\infty, \mu) \quad \text{and} \quad \|T_\tau(f)\|_{p+\delta} \le C\|f\|_p \qquad (f \in L^p(D^\infty, \mu)).$$

Furthermore, for each $i \in \mathbb{N}$, if $\widetilde{r}_i : D^\infty \to \mathbb{R}$, $(s_n) \mapsto s_i$, then an easy calculation shows that $T_\tau(\widetilde{r}_i) = \tau \widetilde{r}_i$. Let $\imath : L^{p+\delta}(D^\infty, \mu) \to L^p(D^\infty, \mu)$ be the natural inclusion map. By [MN, Proposition 3.6.20], $\imath$ is semicompact, so $\imath \circ T_\tau : L^p(D^\infty, \mu) \to L^p(D^\infty, \mu)$ is semicompact too and coincides with $\tau\, \mathrm{id}_{L^p(D^\infty,\mu)}$ on $[\widetilde{r}_i]_{i\in\mathbb{N}}$.

To finish, let $\varphi : D^\infty \to [0,1]$, $(s_n) \mapsto \sum_n (1-s_n)/2^{n+1}$, and let (r_i) be the sequence of Rademacher functions on $[0,1]$. Then $\Phi : L^p[0,1] \to L^p(D^\infty, \mu)$, $f \mapsto f \circ \varphi$, is an isometric Riesz isomorphism such that $\Phi(r_i) = r_i \circ \varphi = \widetilde{r}_i$ ($i \in \mathbb{N}$). The map $S_\tau := \Phi^{-1} \circ T_\tau \circ \Phi$ has all the required properties. ∎

Proof of Theorem 4.2. (i) Let $X := (\bigoplus_i \ell_p^i)_0$, $1 < p < \infty$. Fix $\tau \in (0,1)$ and let $S_\tau \in \mathcal{SK}^r(L^p[0,1])$ be as in Theorem 4.4. For each $k \in \mathbb{N}$, let $\chi_{i,k}$ ($1 \le i \le 2^k$) be the characteristic function of $[(i-1)2^{-k}, i2^{-k}]$ and let $E_k := \mathrm{sp}\,\{\chi_{i,k} : 1 \le i \le 2^k\}$ (here, we are thinking of E_k as a subspace of $L^p[0,1]$, and so a function stands for the class of all functions that differ from it on a set of Lebesgue measure zero). Further, let $P_k : L^p[0,1] \to E_k$ be the corresponding average projection, i.e.,

$$P_k(f) := \sum_{i=1}^{2^k}\Big(2^k \int_{[0,1]} f(t)\chi_{i,k}(t)\,d\lambda(t)\Big)\chi_{i,k} \quad (f \in L^p[0,1]),$$

let $\psi_k : \ell_p^{2^k} \to L^p[0,1]$, $(\alpha_i) \mapsto \sum_i \alpha_i \|\chi_{i,k}\|_p^{-1}\chi_{i,k}$, and let $\phi_k := (\psi_k|^{E_k})^{-1} : E_k \to \ell_p^{2^k}$. Define $T : X \to X$ by

$$T((x_k)) := \sum_k \imath_{2^k}\phi_k P_k S_\tau \psi_k x_{2^k} \quad ((x_k) \in X),$$

where, as before, $\imath_n$ ($n \in \mathbb{N}$) stands for the nth coordinate embedding into X.

It is easy to see that T is well-defined and regular. Furthermore, given $f \in L^p[0,1]$, for every $k \in \mathbb{N}$,

$$\begin{aligned}\big(|\phi_k P_k S_\tau(\psi_k x)| - \phi_k P_k f\big)_+ &\le \phi_k\big((P_k|S_\tau(\psi_k x)| - P_k f)_+\big)\\ &\le \phi_k P_k\big((|S_\tau(\psi_k x)| - f)_+\big) \quad (x \in \ell_p^{2^k}),\end{aligned}$$

and so if f is such that $\|(|S_\tau\xi| - f)_+\| < \varepsilon$ ($\xi \in L^p[0,1]_{[1]}$), we must have

$$\sup_k \|(|\phi_k P_k S_\tau(\psi_k x_{2^k})| - \phi_k P_k f)_+\| \le \varepsilon \quad ((x_k) \in X_{[1]}).$$

This shows that $\imath T : X \to (\bigoplus_i \ell_p^i)_\infty$ (where $\imath : X \to (\bigoplus_i \ell_p^i)_\infty$ is the natural embedding) is semicompact, and therefore $T \in \mathcal{M}_X^{\mathrm{oa}}$ (by Theorem 3.1).

We show next that $T \notin \overline{\Gamma^o_{c_0,r}(X)}$. Let (r_i) stand for the sequence of Rademacher functions in $L^p[0,1]$. Note that, by our definition of T,

$$T\imath_{2^k}\phi_k(r_i) = \tau\imath_{2^k}\phi_k(r_i) \quad (1 \le i \le k,\ k \in \mathbb{N}).$$

Let $Q_p : L^p[0,1] \to [r_i]_{i\in\mathbb{N}}$ be a bounded projection, and for each $k \in \mathbb{N}$, let $\rho_k : [r_i]_{i\in\mathbb{N}} \to [r_i]_{i=1}^k$ be the natural projection onto $[r_i]_{i=1}^k$, let $q_k := \rho_k \circ Q_p \circ \psi_k \circ p_{2^k}$ and let $h_k : [r_i]_{i=1}^k \to X$ be the restriction of $\imath_{2^k} \circ \phi_k$ to $[r_i]_{i=1}^k$. Note that

$$\|q_k\| \le \|\rho_k Q_p\| \le \Big(\sup_i \|\rho_i\|\Big)\|Q_p\| \quad (k \in \mathbb{N}).$$

Suppose towards a contradiction that there is $L \in \Gamma^o_{c_0,r}(X)$ $(= \Gamma_{c_0,r}(X)$ by Proposition 4.1) such that

$$\|T - L\|_r \le \frac{\tau}{2(\sup_i \|\rho_i\|)\|Q_p\|},$$

and let $R : c_0 \to X$ and $S : X \to c_0$ be regular maps such that $RS = L$. (Recall that an operator $L : Y \to Z$, between normed spaces Y and Z, is p-summing ($1 \le p < \infty$) if there is $k \ge 0$ with $\sum_i \|Ly_i\|^p \le k^p \sup_{y'\in Y'_{[1]}} \sum_i |y'(y_i)|^p$ for every finite sequence $(y_i) \subset Y$; the infimum of such k's, denoted $\pi_p(L)$, defines a norm in the space of p-summing operators from Y to Z.) Every operator from c_0 to ℓ_2 is 2-summing (see for instance [Pi, Theorem 5.4]), and so there is a constant C such that $\pi_2(q_kR) \le C\|q_kR\|$ ($k \in \mathbb{N}$). It follows from the above, combined with the fact that $\pi_2(\mathrm{id}_E) = \sqrt{\dim E}$ for every finite-dimensional normed space E, that

$$\begin{aligned}\tau\sqrt{k} &= \tau\pi_2(\mathrm{id}_{[r_i]_{i=1}^k}) = \pi_2(q_kTh_k)\\ &\le \pi_2(q_kRSh_k) + \pi_2(q_kTh_k - q_kRSh_k)\\ &\le \pi_2(q_kR)\|Sh_k\| + \|q_kTh_k - q_kRSh_k\|\pi_2(\mathrm{id}_{[r_i]_{i=1}^k})\\ &\le C\|q_kR\|\,\|S\| + \|q_k\|\,\|T-L\|_r\pi_2(\mathrm{id}_{[r_i]_{i=1}^k})\\ &\le C\|q_k\|\,\|R\|\,\|S\| + \frac{\tau}{2}\sqrt{k},\end{aligned}$$

i.e., $\tau\sqrt{k} \le 2C(\sup_i \|\rho_i\|)\|Q_p\|\,\|R\|\,\|S\|$ ($k \in \mathbb{N}$), which is clearly impossible. Thus, $T \notin \overline{\mathit{\Gamma}^o_{c_0,r}(X)}$. Furthermore, since $\mathcal{M}^{\mathrm{oa}}_X$ is the only maximal ideal of $\mathcal{L}^r(X)$ (by Theorem 3.1), we must have $\overline{\mathit{\Gamma}^o_{c_0,r}(X)} \subsetneq \mathcal{M}^{\mathrm{oa}}_X$.

To finish this part of the proof, it only remains to show that if $\mathcal{I}$ is a nontrivial proper closed ideal of $\mathcal{L}^r(X)$, different from $\mathcal{A}^r(X)$, then necessarily $\overline{\mathit{\Gamma}^o_{c_0,r}(X)} \subseteq \mathcal{I}$. To this end, let $T \in \mathcal{L}^r(X) \setminus \mathcal{A}^r(X)$, let

$$\delta := \inf\{\|T-S\|_r : S \in \mathcal{F}(X)\},$$

let $\mathcal{I}_T$ be the closed (order and algebra) ideal of $\mathcal{L}^r(X)$ generated by T, and let (Q_k) be the sequence of natural projections with respect to the unit vector basis of X. (At this point, one could appeal to Lemma 2.1 to simplify this part of the proof, however, this would not work for $q=1$, so for the sake of a more unified argument, we shall do things differently.) We construct sequences (m_k) and (n_k) in $\mathbb{N}$ as follows: first choose n_1 so that $\|TQ_{n_1}\|_r \ge 2^{-1}\delta$ and then choose $m_1 \ge n_1$ so that $\|(\mathrm{id}-Q_{m_1})TQ_{n_1}\|_r < 4^{-1}\delta$; in general, if $m_1,\dots,m_{k-1}$ have been chosen for some $k>1$, choose $n_k > m_{k-1}$ so that

$$\|(\mathrm{id}-Q_{m_{k-1}})T(\mathrm{id}-Q_{m_{k-1}})Q_{n_k}\|_r \ge 2^{-1}\delta$$

and then choose $m_k \ge n_k$ so that

$$\|(\mathrm{id}-Q_{m_k})T(\mathrm{id}-Q_{m_{k-1}})Q_{n_k}\|_r < 4^{-1}\delta.$$

Let $\mathcal{Q}_k := Q_{m_k} - Q_{m_{k-1}}$ ($k \in \mathbb{N}$), with $m_0 := 0$ and $Q_0 := 0$. Then, for every $k \in \mathbb{N}$, one has

$$\begin{aligned}\|\mathcal{Q}_kT\mathcal{Q}_k\|_r &\ge \|(Q_{m_k} - Q_{m_{k-1}})T(Q_{n_k} - Q_{m_{k-1}})\|_r\\ &\ge \|(\mathrm{id}-Q_{m_{k-1}})T(Q_{n_k} - Q_{m_{k-1}})\|_r\\ &\quad - \|(\mathrm{id}-Q_{m_k})T(Q_{n_k} - Q_{m_{k-1}})\|_r > \delta/4.\end{aligned}$$

Let $D := \sup_k \mathcal{Q}_kT\mathcal{Q}_k$, and let $(x_k) \subset X_{[1]} \cap X_+$ and $(\lambda_k) \subset X'_{[1]} \cap X'_+$ be sequences such that $\mathcal{Q}_kx_k = x_k$, $\mathcal{Q}'_k\lambda_k = \lambda_k$ and $\lambda_k(|D|(x_k)) \ge \delta$ ($k \in \mathbb{N}$). Let (e_k) be the unit vector basis of c_0 and let (e^*_k) be the corresponding sequence of biorthogonal functionals. By passing to subsequences if needed, one can assume that (x_k) and (λ_k) are equivalent to (e_k) and (e^*_k), respectively.

Set $d_k := \lambda_k(|D|(x_k))$ ($k \in \mathbb{N}$), and define $A : X \to c_0$, $x \mapsto \sum_k d_k^{-1}\lambda_k(x)e_k$, and $B : c_0 \to X$, $(a_k) \mapsto \sum_k a_k x_k$. The operators A and B are both regular and satisfy $A|D|B = \mathrm{id}_{c_0}$. Since $|D| \in \mathcal{I}_T$, we have $\Gamma_{c_0,r}(X) \subseteq \mathcal{I}_T$, and in turn $\overline{\Gamma^o_{c_0,r}(X)} \subseteq \mathcal{I}_T$. The desired conclusion follows readily from this.

(ii) Let $X := (\bigoplus_i \ell_p^i)_1$, $1 < p < \infty$. The proof in this case is very similar, so we shall only outline the differences. Define $T : X \to X$ exactly as before. The adjoint of T is then given by

$$T'((f_k)) := \sum_k \psi'_k S'_\tau P'_k \phi'_k \imath'_{2^k}(f_{2^k}) \quad ((f_k) \in X').$$

Since S_τ is semicompact, so is S'_τ (see [Za, Theorem 127.1]), and hence for every $\varepsilon > 0$ there exists $f \in L^{p'}[0,1]$ such that $\|(|S'_\tau(\xi)| - f)_+\| < \varepsilon$ ($\xi \in L^{p'}[0,1]_{[1]}$), where p' denotes the conjugate index to p. It follows that

$$\begin{aligned}(|\psi'_k S'_\tau(\xi)| - \psi'_k f)_+ &\le (\psi'_k |S'_\tau(\xi)| - \psi'_k f)_+ \\ &\le \psi'_k\big((|S'_\tau(\xi)| - f)_+\big) \quad (\xi \in L^{p'}[0,1]_{[1]}),\end{aligned}$$

and hence that $\sup_k \|(|\psi'_k S'_\tau(P'_k \phi'_k \imath'_{2^k} f_{2^k})| - \psi'_k f)_+\| \le \varepsilon$ ($(f_k) \in X'_{[1]}$). The latter implies $T' \in \mathcal{SK}^r(X')$, and once again, by Theorem 3.1, $T \in \mathcal{M}_X^{\mathrm{oa}}$.

To show $T \notin \overline{\Gamma_{\ell_1,r}(X)}$, suppose towards a contradiction that there exists $L \in \Gamma^o_{\ell_1,r}(X)$ ($= \Gamma_{\ell_1,r}(X)$, by Proposition 4.1) such that

$$\|T - L\|_r \le \frac{\tau}{2(\sup_i \|\rho_i\|)\|Q_p\|}.$$

Let $R : \ell_1 \to X$ and $S : X \to \ell_1$ be regular operators such that $RS = L$, and define q_k and h_k ($k \in \mathbb{N}$) as before. Every operator from ℓ_1 to ℓ_2 is 2-summing (indeed, every operator from ℓ_1 to ℓ_2 is 1-summing (see for instance [Pi, Theorem 5.12]), and the 1-summing norm always dominates the 2-summing one [Pi, Corollary 1.6]), so $\pi_2(q_k R) \le C\|q_k R\|$ ($k \in \mathbb{N}$) for some constant C. Now one can argue exactly as in the previous part of the proof to achieve a contradiction. That $\overline{\Gamma^o_{\ell_1,r}(X)} \subset \mathcal{M}_X^{\mathrm{oa}}$ follows again from Theorem 3.1.

The proof that any non-trivial proper closed ideal of $\mathcal{L}^r(X)$, different from $\mathcal{A}^r(X)$, must contain $\overline{\Gamma^o_{\ell_1,r}(X)}$, is completely analogous to that in the case where $q = 0$. ∎

We do not know whether the three ideals listed in Theorem 4.2 are the only non-trivial proper closed ideals of the algebras $\mathcal{L}^r((\bigoplus_i \ell_2^i)_0)$ and $\mathcal{L}^r((\bigoplus_i \ell_2^i)_1)$. Our methods also fall short of shedding any light onto what happens in the cases $\mathcal{L}^r((\bigoplus_i \ell_1^i)_0)$ and $\mathcal{L}^r((\bigoplus_i \ell_\infty^i)_1)$. In view of the above results, we suspect further progress on these questions, or more generally, on the problem of the structure of the lattice of closed ideals of the algebras $\mathcal{L}^r((\bigoplus_i \ell_p^i)_0)$ ($1 \le p < \infty$) and $\mathcal{L}^r((\bigoplus_i \ell_p^i)_1)$ ($1 < p \le \infty$), may be linked to the question of the ideal structure of $\mathcal{SK}^r(L^p[0,1])$ ($1 < p < \infty$), of which, at present, not much seems to be known.

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