

On the genesis of BBP formulas

by

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1. Introduction. More than 20 years ago, D. H. Bailey, P. Borwein and S. Plouffe [5] presented an efficient algorithm to compute deep binary or hexadecimal digits of π without the need to compute the previous ones. Their algorithm is based on a series representation for π given by a formula discovered by S. Plouffe,

$$(1) \quad \pi = \sum_{k=0}^{+\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

Formulas of similar form for other transcendental constants were known long time ago, like the classical formula for $\log 2$, known to J. Bernoulli,

$$\log 2 = \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{1}{k}.$$

The reader can find in [4] an illustration of how to extract binary digits from this type of formulas.

Many new formulas of this type, named BBP formulas, have been found for π and other higher transcendental constants in the last decades (see [1], [26]). Plouffe's formula, and others for π , can be derived using integral periods (as in [5]), or more directly using polylogarithm ladder relations at precise algebraic values (as in [14]). The polylogarithm ladder relations can be viewed as generalizations of Machin–Störmer relations (see [24] and [25]) for rational values of the arctangent function, after taking appropriate Taylor series expansions. In particular, we can recover in that way Bellard's formula (see Bellard's webpage [9]), which seems to be the most efficient one to compute deep binary digits of π (see Remark 1.2),

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$$(2) \quad \pi = \frac{1}{2^6} \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{10k}} \left(-\frac{2^5}{4k+1} - \frac{1}{4k+3} + \frac{2^8}{10k+1} - \frac{2^6}{10k+3} - \frac{2^2}{10k+5} - \frac{2^2}{10k+7} + \frac{1}{10k+9} \right).$$

Many of these formulas are rearrangements of each other, or related by null BBP formulas that represent 0. The origin of null BBP formulas is somewhat mysterious. Most of them have been found by extensive computer search over parameter space using the PSLQ algorithm to detect integer relations. So their true origin and nature remain somewhat mysterious. As the authors of [5] explain:

We found the identity by a combination of inspired guessing and extensive searching using the PSLQ integer relation algorithm.

And in [2]:

This formula was found using months of PSLQ computations after corresponding but simpler n -th digit formulas were identified for several other constants, including $\log(2)$. This is likely the first instance in history that a significant new formula for π was discovered by a computer.

To compute *all* digits of π up to a certain order, there are more efficient formulas given by rapidly convergent series of a modular nature, initiated by S. Ramanujan [23], which are at the origin of Chudnovsky's algorithm based on Chudnovsky's formula (see [16])

$$\frac{1}{\pi} = 12 \sum_{k=0}^{+\infty} \frac{(-1)^k (6k)! (545140134k + 13591409)}{(3k)! (k!)^3 (640320)^{3k+3/2}}.$$

Other methods of algorithmic nature include the Borwein quartic algorithm for π (see [12]) that approximately quadruples the number of correct digits with each iteration, and the Borwein nonic algorithm for π that approximately increases the number of correct digits nine-fold.

A general BBP formula as defined in [6] for the constant α is a series of the form

$$\alpha = Q(d, b, m, \mathbf{A}) = \sum_{k=0}^{+\infty} \frac{1}{b^k} \sum_{l=1}^m \frac{a_l}{(km+l)^d},$$

where b, d, m , are integers, $b \geq 2$, and $\mathbf{A} = (a_1, \dots, a_m)$ is an integer vector. The integer $d \geq 1$ is the degree of the formula. The classical BBP formula (1) and Bellard formula (2) are of degree 1. In this article we study formulas of degree 1. It would be interesting to extend the present results to get higher degree formulas. The integer b is called the base of the BBP formula, and digits in base b can be computed efficiently. Particular attention has been given to base $b = 2^n$ formulas, as they are useful in computing binary digits. They are called *binary* BBP formulas. While there are both base 2 and base 3

BBP formulas for some constants like π^2 (see [13]), there are no non-binary BBP formulas for π (see [13]). However, this does not rule out the possibility of some completely different scheme, not based on BBP formulas, to directly compute non-binary digits of π .

More generally, we can define BBP-like formulas to be of the general form

$$(3) \quad \alpha = Q(r_0, r_1, d, b, m, \mathbf{A}) = r_0 + r_1 \sum_{k=0}^{+\infty} \frac{1}{b^k} \sum_{l=1}^m \frac{a_l}{(km+l)^d},$$

where r_0 and r_1 are rational numbers. Such formulas have potentially similar computational applications.

But the interest of these formulas is also theoretical. A *normal number* in base $b \geq 2$ is an irrational number α whose expansion in base b contains any string of n consecutive digits with frequency b^{-n} . These numbers were introduced in 1909 by É. Borel who proved that Lebesgue almost every number is normal in any base $b \geq 2$ ([10], and the survey [22]). This result is a direct application of Birkhoff's Ergodic Theorem to the dynamical system given by the transformation $T : \mathbb{T} \rightarrow \mathbb{T}$, $T(x) = bx$ modulo 1, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The transformation T preserves the Lebesgue measure which is an ergodic invariant measure. It is not difficult to construct explicit normal numbers, and numbers that are not normal, but there is no known example of "natural" transcendental constant that is normal in every base. It is conjectured that this holds for π and other natural transcendental constants, but this remains an open question. It is not even known if a given digit appears infinitely often in the base 10 expansion for π . We note the recent results of [7, 8] where the normality of a certain class of constants has been proved, yet not including π .

An approach to prove normality in base b for any transcendental constant which admits a BBP formula in base b is proposed in [6]. The criterion, named "Hypothesis A", seems related to Furstenberg's "multiplication by 2 and 3" conjecture (see [17]). Only a very particular class of period-like numbers have BBP formulas (for instance π^2 , as mentioned before). It is also natural to investigate the class of numbers with a BBP or BBP-like representation.

The main goal of this article is to present a general procedure to generate the most basic BBP and BBP-like formulas of degree 1 that correspond to the simplest transcendental numbers $\log p$ and π . With this new procedure we derive the classical formulas, like the Bailey–Borwein–Plouffe or Bellard formulas, and understand better their origin, in particular the origin of null formulas, and the relation of BBP formulas for $\log 2$ and π that correspond to taking the real or imaginary parts of the same complex formula. We also better understand the redundancy of rearrangements in these formulas, and the method provides a tool to search for more formulas with a more

conceptual approach. Although we do not find new BBP formulas, we recover the most important ones and we believe that the method presented can be further developed to discover new ones. We plan to carry this out in the future.

The procedure to generate BBP formulas is elementary and is motivated by considering the bases for first order asymptotics at infinity of the Euler Gamma function and higher Barnes Gamma functions and the transalgebraic considerations that play an important role in [21] (see also [20]). To construct these asymptotic bases, we consider the family of iterated integrals of $1/s$ defined by $I_0(s) = 1/s$, and for $n \geq 0$,

$$I_{n+1}(s) = \int_1^s I_n(u) \, du = \cdots = \int_1^s \int_1^{u_n} \cdots \int_1^{u_0} \frac{1}{u_0} \, du_0 \dots du_{n-1} \, du_n.$$

It is elementary to check by induction that

$$I_n(s) = A_n(s) \log s + B_n(s),$$

where $A_n, B_n \in \mathbb{Q}[s]$ are polynomials with rational coefficients, with $\deg A_n = \deg B_n = n - 1$, and

$$A_n(s) = \frac{s^{n-1}}{(n-1)!}.$$

We have

THEOREM 1.1. *If $s \in \mathbb{C}$ and $|s - 1| < 1$, or $|s - 1| = 1$ and $n \geq 2$, then*

$$I_n(s) = \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n}}{n! \binom{j+n}{n}} = \frac{s^{n-1}}{(n-1)!} \log s + B_n(s),$$

$$\log s = \frac{(1-s)^n}{ns^{n-1}} \sum_{j=0}^{+\infty} \frac{(1-s)^j}{\binom{j+n}{j}} - \frac{(n-1)!}{s^{n-1}} B_n(s).$$

Since $B_n(s)$ has rational coefficients, we can take $s = 1 - 1/b$ and we get a BBP-like formula for $\log s$. Taking suitable complex values for s , and separating real and imaginary parts, we also obtain BBP and BBP-like formulas for π . We prove that formulas for different values of n provide non-obvious rearrangements of the summations, which in part explains the rich “rearrangement algebra” of BBP formulas.

We recover many formulas with this procedure. For instance, all the formulas for $\log 2$ appearing in Wikipedia [27] are given in (6)–(16). We also get the following classical formulas:

$$\begin{aligned} \log 2 &= \frac{5}{6} - \sum_{k=1}^{+\infty} \frac{1}{2^k} \left(\frac{1}{k} - \frac{3}{k+1} + \frac{3}{k+2} - \frac{1}{k+3} \right), \\ \log 2 &= \frac{2}{3} + \sum_{k=1}^{+\infty} \frac{1}{16^k} \left(\frac{2}{8k} + \frac{1}{8k+2} + \frac{1/2}{8k+4} + \frac{1/4}{8k+6} \right), \\ \pi &= \frac{8}{3} + 4 \sum_{k=1}^{+\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3} \right), \\ \pi &= \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{2}{8k+1} + \frac{2}{8k+2} + \frac{1}{8k+3} - \frac{1/2}{8k+5} - \frac{1/2}{8k+6} - \frac{1/4}{8k+7} \right). \end{aligned}$$

Also combining our formulas we can get some null formulas, for example the following one appearing in [5]:

$$(4) \quad 0 = \sum_{k=0}^{+\infty} \frac{1}{16^k} \left(\frac{-8}{8k+1} + \frac{8}{8k+2} + \frac{4}{8k+3} + \frac{8}{8k+4} + \frac{2}{8k+5} + \frac{2}{8k+6} - \frac{1}{8k+7} \right).$$

This gives some explanations of the mysteries mentioned before. For example, formulas for $\log 2$ and π are related by taking the real and imaginary parts of formulas for complex values for s , for example for $s = (1+i)/2$. Null formulas can appear when comparing our formulas for different complex values of s and taking real or imaginary parts. For example for $s = 1/2$ and $s = (1+i)/2$ we get the previous null formula. It is natural to ask if all null BBP formulas of degree 1 can be obtained by combining formulas from Theorem 1.1 for different values of s .

Clearly, we also recover the classical BBP formula (1) and Bellard formula (2).

REMARK 1.2. We can measure the *efficiency* of a BBP-like formula (3) for computing a number α as $\bar{m}/\log b$, where \bar{m} is the number of non-zero coefficients in $\mathbf{A} = (a_1, \dots, a_m)$, as this measures the number of non-zero summands for going to the next step in the digit computation. Binary BBP formulas, when $b = 2$, are of special relevance, since they allow computing α in binary form. In that case, we can take the logarithm in base 2. The efficiency of (1) is 1, whereas the efficiency of (2) is $7/10$, a 43% faster.

The techniques of this article extend to other bases of iterated functions that we will discuss in future articles. We hope that our approach can be useful in finding more efficient BBP-formulas for π by more powerful algebraic computer search algorithms.

2. Laplace–Hadamard regularization of polar parts. The Laplace–Hadamard regularization is related to [20] and [21].

For each $n \geq 0$ we define $P_0 = 0$, and for $n \geq 1$,

$$P_n(s, t) = \sum_{k=0}^{n-1} \frac{(1-s)^k}{k!} t^k.$$

We also define the iterated primitives of $1/s$ by $I_0(s) = 1/s$, and for $n \geq 0$,

$$I_{n+1}(s) = \int_1^s I_n(u) du = \cdots = \int_1^s \int_1^{u_n} \cdots \int_1^{u_0} \frac{1}{u_0} du_0 \cdots du_{n-1} du_n.$$

We call the integrals $I_n(s)$ the *Laplace–Hadamard regularization* or the *Laplace–Hadamard transform* of $1/t^n$. The functions $I_n(s)$ are holomorphic in $\mathbb{C} -]-\infty, 0]$ and have an isolated singularity at 0 with non-trivial monodromy when $n \geq 1$. We have a single integral expression for $I_n(s)$ as a Laplace–Hadamard regularization:

PROPOSITION 2.1. *If $n \geq 0$ and $\Re s > 0$, or $\Re s = 0$ and $n \geq 2$, then*

$$I_n(s) = (-1)^n \int_0^{+\infty} \frac{1}{t^n} (e^{-st} - P_n(s, t)e^{-t}) dt.$$

Proof. For $n = 0$ we have

$$\int_0^{+\infty} e^{-st} dt = \frac{1}{s},$$

and by induction we get the result by integrating over u between 1 and s ,

$$I_{n+1}(s) = \int_1^s \int_0^{s+\infty} \frac{1}{t^n} (e^{-ut} - P_n(u, t)e^{-t}) dt du,$$

and using

$$\int_1^s e^{-ut} du = -\frac{1}{t}(e^{-st} - e^{-t}), \quad \int_1^s P_n(u, t) du = -\frac{1}{t}(P_{n+1}(s, t) - 1). \blacksquare$$

Note that $P_n(s, t) \rightarrow e^{(1-s)t}$ as $n \rightarrow +\infty$ uniformly on compact sets, and $P_n(s, t)$ is the n th order jet of $e^{(1-s)t}$ at $t = 0$. So for $t \rightarrow 0$ we have

$$e^{-st} - P_n(s, t)e^{-t} = O(t^n).$$

For $n = 0$ we get the elementary integral

$$I_0(s) = \int_0^{+\infty} e^{-st} dt = \frac{1}{s}.$$

For $n = 1$ we get the old Frullani integral [11, p. 98]

$$I_1(s) = - \int_0^{+\infty} \frac{1}{t} (e^{-st} - e^{-t}) dt = \log s.$$

We have

$$\begin{aligned} I_1(s) &= \log s, \\ I_2(s) &= s \log s - (s - 1), \\ I_3(s) &= \frac{1}{2} s^2 \log s - \frac{1}{4} (s - 1)(3s - 1), \\ I_4(s) &= \frac{1}{6} s^3 \log s - \frac{1}{36} (s - 1)(11s^2 - 7s + 2), \\ I_5(s) &= \frac{1}{24} s^4 \log s - \frac{1}{288} (s - 1)(25s^3 - 23s^2 + 13s - 3). \end{aligned}$$

A simple induction shows

PROPOSITION 2.2. *We have*

$$I_n(s) = A_n(s) \log s + B_n(s),$$

where $A_n, B_n \in \mathbb{Q}[s]$ are polynomials, with $\deg A_n = \deg B_n = n - 1$, and

$$A_n(s) = \frac{s^{n-1}}{(n-1)!}. \blacksquare$$

Regarding the polynomials B_n , the relation $I'_{n+1}(s) = I_n(s)$ shows that

$$(5) \quad B'_{n+1}(s) = B_n(s) - \frac{s^{n-1}}{n!}.$$

This equation together with the condition $B_{n+1}(1) = 0$ determines B_{n+1} uniquely from B_n .

We have a formula for B_n (see [19], where $I_n(s) = f_{n-1}(x)$ with $x = s - 1$, and [18]):

PROPOSITION 2.3. *For $n \geq 0$,*

$$B_{n+1}(s) = -\frac{1}{n!} \sum_{k=1}^n \binom{n}{k} (H_n - H_{n-k})(s-1)^k,$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ are the harmonic numbers.

Proof. The formula holds for $n = 0$, and it satisfies $B_{n+1}(1) = 0$ and the recurrence relation

$$\begin{aligned} B'_{n+1}(s) &= -\frac{1}{n!} \sum_{k=1}^n k \binom{n}{k} (H_n - H_{n-k})(s-1)^{k-1} \\ &= -\frac{1}{(n-1)!} \sum_{k=1}^n \binom{n-1}{k-1} (H_n - H_{n-k})(s-1)^{k-1} \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{(n-1)!} \sum_{k=1}^{n-1} \binom{n-1}{k-1} (H_{n-1} - H_{n-1-(k-1)}) (s-1)^{k-1} \\
&\quad - \frac{1}{(n-1)!} \sum_{k=1}^n \frac{1}{n} (H_n - H_{n-1}) (s-1)^{k-1} \\
&= B_n(s) - \frac{1}{n!} \sum_{k=1}^n \binom{n-1}{k-1} (s-1)^{k-1} \\
&= B_n(s) - \frac{1}{n!} ((s-1) + 1)^{n-1} = B_n(s) - \frac{s^{n-1}}{n!}. \blacksquare
\end{aligned}$$

Now we prove

LEMMA 2.4. For $n \geq 1$,

$$I_{n+1}(0) = B_{n+1}(0) = \frac{(-1)^{n+1}}{n \cdot n!}.$$

We first establish a useful integral representation for harmonic numbers:

LEMMA 2.5.

$$H_n = \int_0^{+\infty} \frac{1 - e^{-(n+1)t}}{1 - e^{-t}} dt.$$

Proof. We have

$$H_n = \sum_{k=1}^n \frac{1}{k} = \sum_{k=1}^n \int_0^{+\infty} e^{-kt} dt = \int_0^{+\infty} \frac{1 - e^{-(n+1)t}}{1 - e^{-t}} dt. \blacksquare$$

LEMMA 2.6.

$$\sum_{k=0}^n \binom{n}{k} (-1)^k H_{n-k} = \frac{(-1)^n}{n}.$$

Proof. We have

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} (-1)^k H_{n-k} \\
&= \int_0^{+\infty} \frac{(\sum_{k=0}^n \binom{n}{k} (-1)^k) - (\sum_{k=0}^n \binom{n}{k} (-1)^k e^{-(n-k)t}) e^{-t}}{1 - e^{-t}} dt \\
&= \int_0^{+\infty} \frac{(1-1)^n - (-1 + e^{-t})^n e^{-t}}{1 - e^{-t}} dt \\
&= (-1)^{n+1} \int_0^{+\infty} (1 - e^{-t})^{n-1} e^{-t} dt = (-1)^n \int_0^1 x^{n-1} dx = \frac{(-1)^n}{n}. \blacksquare
\end{aligned}$$

Proof of Lemma 2.4. We have

$$\begin{aligned} B_{n+1}(0) &= -\frac{1}{n!} \sum_{k=1}^n \binom{n}{k} (H_n - H_{n-k}) (-1)^{k-1} \\ &= -\frac{1}{n!} \left(H_n + \sum_{k=1}^n \binom{n}{k} (-1)^k H_{n-k} \right) \\ &= -\frac{1}{n!} \left(H_n + \left(\frac{(-1)^n}{n} - H_n \right) \right) = \frac{(-1)^{n+1}}{n \cdot n!}. \blacksquare \end{aligned}$$

COROLLARY 2.7. For $n \geq 2$,

$$B'_{n+1}(0) = \frac{(-1)^n}{(n-1)(n-1)!}.$$

Proof. From (5) we have $B'_{n+1}(0) = B_n(0)$, and the result follows from Lemma 2.4. \blacksquare

This is related to the following identity for harmonic numbers:

LEMMA 2.8. For $n \geq 2$, we have

$$\sum_{k=0}^n k \binom{n}{k} (-1)^k H_{n-k} = (-1)^{n-1} \frac{n}{n-1}.$$

Proof. For $n \geq 2$, we have

$$\sum_{k=0}^n k \binom{n}{k} (-1)^k a^{n-k} = x \frac{d}{dx} (a+x)^n \Big|_{x=-1} = -n(a-1)^{n-1},$$

therefore

$$\begin{aligned} \sum_{k=0}^n k \binom{n}{k} (-1)^k H_{n-k} &= \int_0^{+\infty} \frac{-n(1-t)^{n-1} + n(e^{-t}-1)^{n-1}e^{-t}}{1-e^{-t}} dt \\ &= (-1)^{n-1} n \int_0^{+\infty} (1-e^{-t})^{n-2} e^{-t} dt \\ &= (-1)^{n-1} n \int_0^1 x^{n-2} dx = (-1)^{n-1} \frac{n}{n-1}. \blacksquare \end{aligned}$$

3. Egyptian formulas for rational numbers. We start with the simplest case: an Egyptian formula for rationals. The following is an “infinite Egyptian fraction decomposition” for $1/n$.

PROPOSITION 3.1 (Infinite Egyptian fraction decomposition). For $n \geq 2$,

$$\frac{1}{n} = \sum_{j=1}^{+\infty} \frac{1}{\binom{j+n+1}{n+1}}.$$

Proof. Notice that from Proposition 2.1 we have

$$I_n(s) = (-1)^n \int_0^{+\infty} \frac{1}{t^n} (e^{-st} - P_n(s, t)e^{-t}) dt$$

with

$$P_n(s, t) = \sum_{k=0}^{n-1} \frac{(1-s)^k t^k}{k!},$$

hence

$$P_n(0, t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}.$$

So for $n \geq 2$, we can develop and exchange the integral and the summation:

$$\begin{aligned} (-1)^n I_n(0) &= \int_0^{+\infty} \frac{e^{-t}}{t^n} \left(e^t - \sum_{k=0}^{n-1} \frac{t^k}{k!} \right) dt = \int_0^{+\infty} e^{-t} \sum_{j=0}^{+\infty} \frac{t^j}{(j+n)!} dt \\ &= \sum_{j=0}^{+\infty} \frac{j!}{(j+n)!} = \frac{1}{n!} \sum_{j=0}^{+\infty} \frac{1}{\binom{j+n}{n}}. \end{aligned}$$

Now from Lemma 2.4 we have

$$I_n(0) = B_n(0) = \frac{(-1)^n}{(n-1)(n-1)!},$$

thus

$$\frac{n}{n-1} = \sum_{j=0}^{\infty} \frac{1}{\binom{j+n}{n}},$$

and the result follows. ■

As one referee has pointed out, Proposition 3.1 also follows by a telescoping sum over

$$\frac{n}{\binom{j+n+1}{n+1}} = \frac{j+n+1}{\binom{j+n+1}{n+1}} - \frac{j+n+2}{\binom{j+n+2}{n+1}},$$

which is found using Gosper's algorithm. We show here that this formula results from our general approach.

4. BBP-like formulas for $\log s$. In general we have

PROPOSITION 4.1. *If $|s-1| < 1$, or $|s-1| = 1$ and $n \geq 2$, then*

$$I_n(s) = \frac{(-1)^n}{n!} \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n}}{\binom{j+n}{n}}.$$

Proof. The condition $\Re s > 0$ ensures the convergence of the integrals and the assumption that $|s - 1| < 1$, or $|s - 1| = 1$ and $n \geq 2$, ensures the convergence of the series:

$$\begin{aligned}
 (-1)^n I_n(s) &= \int_0^{+\infty} \frac{1}{t^n} (e^{-st} - P_n(s, t)e^{-t}) dt = \int_0^{+\infty} \frac{e^{-t}}{t^n} (e^{(1-s)t} - P_n(s, t)) dt \\
 &= \int_0^{+\infty} \frac{e^{-t}}{t^n} \left(e^{(1-s)t} - \sum_{k=0}^{n-1} \frac{(1-s)^k t^k}{k!} \right) dt \\
 &= \int_0^{+\infty} \frac{e^{-t}}{t^n} \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n} t^{j+n}}{(j+n)!} dt = \int_0^{+\infty} e^{-t} \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n} t^j}{(j+n)!} dt \\
 &= \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n} j!}{(j+n)!} = \frac{1}{n!} \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n}}{\binom{j+n}{n}}. \blacksquare
 \end{aligned}$$

REMARK 4.2. The formula in Proposition 4.1 also holds for $|s - 1| = 1$ and $s \neq 0$, but the convergence of the sum is only conditional. This can be checked by continuity of both sides, letting $|s - 1| \rightarrow 1$.

Now, we have

$$I_n(s) = \frac{s^{n-1}}{(n-1)!} \log s + B_n(s),$$

and since $B_n \in \mathbb{Q}[s]$ we get

$$I_n(s) = (-1)^n \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n}}{n! \binom{j+n}{n}} = \frac{s^{n-1}}{(n-1)!} \log s + B_n(s).$$

In particular, for $s \in \mathbb{Q}$ we have

$$\sum_{j=0}^{+\infty} \frac{(1-s)^{j+n}}{n! \binom{j+n}{n}} \in \mathbb{Q} \oplus \mathbb{Q} \log s.$$

THEOREM 4.3. *If $|s - 1| < 1$, or $|s - 1| = 1$ and $n \geq 2$, then*

$$\log s = -\frac{(n-1)!}{s^{n-1}} B_n(s) + (-1)^n \frac{(n-1)!}{s^{n-1}} \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n}}{n! \binom{j+n}{n}}. \blacksquare$$

We get a group of formulas for $\log 2$ by specializing at $s = 2$. We have

$$\log 2 = -\frac{(n-1)!}{2^{n-1}} B_n(2) + \frac{(n-1)!}{2^{n-1}} \sum_{j=0}^{+\infty} \frac{(-1)^j}{(j+1)(j+2)\dots(j+n)}.$$

Using the values $B_2(2) = -1$, $B_3(2) = -\frac{1}{2}$, $B_4(2) = -\frac{8}{9}$, $B_5(2) = -\frac{131}{240}$, $B_6(2) = -\frac{661}{3600}$, we get

$$(6) \quad \log 2 = \frac{1}{2} + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j+1)},$$

$$(7) \quad \log 2 = \frac{5}{8} + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j+1)(j+2)},$$

$$(8) \quad \log 2 = \frac{2}{3} + \frac{3}{4} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j+1)(j+2)(j+3)},$$

$$(9) \quad \log 2 = \frac{131}{192} + \frac{3}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j+1)(j+2)(j+3)(j+4)},$$

$$(10) \quad \log 2 = \frac{661}{960} + \frac{15}{4} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j+1)(j+2)(j+3)(j+4)(j+5)}.$$

Specializing at $s = 1/2$, we get the following formula for $n \geq 1$:

$$\log 2 = -\frac{(n-1)!}{2^{n-1}} B_n(1/2) + (-1)^n (n-1)! \sum_{j=0}^{+\infty} \frac{1}{2^{j+1}(j+1)(j+2)\dots(j+n)}.$$

Using the values $B_2(1/2) = -2$, $B_3(1/2) = -1$, $B_4(1/2) = -\frac{40}{36}$, $B_5(1/2) = \frac{7}{18}$, $B_6(1/2) = -\frac{47}{225}$, we get the formulas

$$(11) \quad \log 2 = \sum_{j=1}^{\infty} \frac{1}{2^j j},$$

$$(12) \quad \log 2 = 1 - \sum_{j=1}^{\infty} \frac{1}{2^j j(j+1)},$$

$$(13) \quad \log 2 = \frac{1}{2} + 2 \sum_{j=1}^{\infty} \frac{1}{2^j j(j+1)(j+2)},$$

$$(14) \quad \log 2 = \frac{5}{6} - 6 \sum_{j=1}^{\infty} \frac{1}{2^j j(j+1)(j+2)(j+3)},$$

$$(15) \quad \log 2 = \frac{7}{12} + 24 \sum_{j=1}^{\infty} \frac{1}{2^j j(j+1)(j+2)(j+3)(j+4)},$$

$$(16) \quad \log 2 = \frac{47}{60} - 120 \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2^j j(j+1)(j+2)(j+3)(j+4)(j+5)}.$$

All these formulas appear in [27].

It is customary to write the formulas above by splitting the denominators into simple fractions. For instance, the fourth formula can be written as

$$\log 2 = \frac{5}{6} - \sum_{j=1}^{+\infty} \frac{1}{2^j} \left(\frac{1}{j} - \frac{3}{j+1} + \frac{3}{j+2} - \frac{1}{j+3} \right).$$

If we group for $j = 4k, 4k + 1, 4k + 2, 4k + 3$, we get

$$\begin{aligned} \log 2 &= \frac{5}{6} - \sum_{k=1}^{+\infty} \frac{1}{2^{4k}} \left(\frac{1}{4k} - \frac{3}{4k+1} + \frac{3}{4k+2} - \frac{1}{4k+3} \right) \\ &\quad - \sum_{k=1}^{+\infty} \frac{1}{2^{4k}} \left(\frac{1/2}{4k+1} - \frac{3/2}{4k+2} + \frac{3/2}{4k+3} - \frac{1/2}{4k+4} \right) \\ &\quad - \sum_{k=1}^{+\infty} \frac{1}{2^{4k}} \left(\frac{1/4}{4k+2} - \frac{3/4}{4k+3} + \frac{3/4}{4k+4} - \frac{1/4}{4k+5} \right) \\ &\quad - \sum_{k=1}^{+\infty} \frac{1}{2^{4k}} \left(\frac{1/8}{4k+3} - \frac{3/8}{4k+4} + \frac{3/8}{4k+5} - \frac{1/8}{4k+6} \right) \\ &= \frac{2}{3} + \sum_{k=1}^{+\infty} \frac{1}{2^{4k}} \left(\frac{1}{4k} + \frac{1/2}{4k+1} + \frac{1/4}{4k+2} + \frac{1/8}{4k+3} \right). \end{aligned}$$

We rewrite it in more classical form:

$$(17) \quad \log 2 = \frac{2}{3} + \frac{1}{4} \sum_{k=1}^{+\infty} \frac{1}{16^k} \left(\frac{8}{8k} + \frac{4}{8k+2} + \frac{2}{8k+4} + \frac{1}{8k+6} \right).$$

We can obtain many more binary BBP-like formulas. Specializing at $s = 3/2$ we get the formula for $n \geq 1$,

$$\begin{aligned} \log(3/2) &= -(2/3)^{n-1} (n-1)! B_n(3/2) \\ &\quad + 2(n-1)! 3^{n-1} \sum_{j=0}^{+\infty} \frac{(-1)^j}{2^j (j+1)(j+2) \cdots (j+n)}. \end{aligned}$$

For instance, $n = 4$ gives

$$\log(3/2) = \frac{65}{162} + \frac{1}{216} \sum_{j=0}^{+\infty} \frac{(-1)^j}{2^j \binom{j+4}{j}}.$$

As before, the sum can also be written as

$$\log(3/2) = \frac{65}{162} + \frac{1}{27} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{2^k} \left(\frac{1}{k} - \frac{3}{k+1} + \frac{3}{k+2} - \frac{1}{k+3} \right).$$

In general, binary BBP formulas can be obtained from Theorem 4.3 by taking $s = 1 \pm \frac{1}{2^N}$,

$$\begin{aligned} \log \frac{2^N \pm 1}{2^N} &= \frac{-(n-1)!}{(1 \pm 2^{-N})^{n-1}} B_n(1 \pm 2^{-N}) \\ &+ \frac{(-1)^n (n-1)!}{(1 \pm 2^{-N})^{n-1}} \sum_{j=0}^{+\infty} \frac{1}{2^{N(j+n)} (j+1)(j+2) \dots (j+n)}. \end{aligned}$$

Formulas of this sort were also obtained by Chamberland [15].

The numbers 2 and $2^N \pm 1$, $N \geq 1$, generate a multiplicative subgroup of \mathbb{Q}^* , and for the elements k in that subgroup, we have binary BBP formulas for $\log k$. The first prime that is not in this subgroup is $k = 23$. Note that $2^{11} - 1 = 23 \cdot 89$, but these two primes appear always together in the factor decomposition of $2^N - 1$ when N is a multiple of 11, and do not appear for other values of N . Also they do not appear at all in $2^N + 1$, for any natural number N . This can be checked as follows: first $2^{11} \equiv 1 \pmod{23}$, so the order of 2 in \mathbb{Z}_{23} is 11. In particular it cannot be that $2^N \equiv -1 \pmod{23}$, since otherwise $2^{2N} \equiv 1 \pmod{23}$, and hence $2N \mid 11$, so $N \mid 11$ and thus $2^N \equiv 1 \pmod{23}$. On the other hand, if $2^N \equiv 1 \pmod{23}$ then N is a multiple of 11, and then $23 \cdot 89 \mid (2^{11} - 1) \mid (2^N - 1)$.

5. BBP-like formulas for π . We may use Theorem 4.3 for a complex value of s ; then we can get BBP-formulas for $\log k$ and also for π by separating the real and imaginary parts. For $n = 1$ (using Remark 4.2), we have

$$\log s = (s-1) \sum_{j=0}^{+\infty} \frac{(1-s)^j}{\binom{j+1}{1}} = - \sum_{j=1}^{+\infty} \frac{(1-s)^j}{j},$$

which is the classical series for $\log s$. Let $s = 1 + i$. We have $0 < \Re(1+i) = 1 < 2$ and

$$\log(1+i) = \log \sqrt{2} + i \frac{\pi}{4} = \frac{1}{2} \log 2 + i \frac{\pi}{4},$$

and

$$\log(1+i) = i \sum_{j=0}^{+\infty} \frac{i^j}{\binom{j+1}{1}}.$$

Separating the real and imaginary parts and $j = 2k$ or $j = 2k+1$ we get two BBP formulas, one for $\log 2$ and the other for π :

$$\log 2 = 2 \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{\binom{2k+2}{1}} = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{k+1}, \quad \pi = 4 \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1}.$$

This last formula is just the first Machin formula for π , related to $\pi/4 = \arctan 1$.

For general $n \geq 2$, we take $s = 1 + i$, and we have

$$\begin{aligned} \log(1+i) &= \frac{1}{2} \log 2 + i \frac{\pi}{4} \\ &= -\frac{(n-1)!}{(1+i)^{n-1}} B_n(1+i) \\ &\quad + (-1)^n \frac{(1-i)^{n-1} (n-1)!}{2^{n-1}} \sum_{j=0}^{\infty} \frac{(-i)^{j+n}}{(j+1)(j+2)\dots(j+n)}. \end{aligned}$$

Let

$$c_n = -\Im \left(\frac{(n-1)!}{(1+i)^{n-1}} B_n(1+i) \right),$$

so that

$$\pi = 4c_n + 4(-1)^n \frac{(n-1)!}{2^{n-1}} \sum_{a=0}^{n-1} \binom{n-1}{a} \sum_{j \equiv n+a+1(2)} \frac{(-1)^{(j+n+a+1)/2}}{(j+1)(j+2)\dots(j+n)}.$$

With this machinery at hand, we recover a number of known formulas.

PROPOSITION 5.1 (Leibniz). *We have*

$$\pi = \frac{8}{3} + 4 \sum_{k=1}^{+\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3} \right).$$

Proof. We apply the above to $n = 2$, where we have $B_2(1+i) = -i$ and $c_2 = -\Im(-i/(1+i)) = 1/2$, thus

$$\begin{aligned} \pi &= 2 + 2 \left(\sum_{j=0}^{+\infty} \frac{(-1)^j}{(2j+2)(2j+3)} + \sum_{j=0}^{+\infty} \frac{(-1)^j}{(2j+1)(2j+2)} \right) \\ &= 2 + 2 \sum_{j=0}^{+\infty} (-1)^j \left(\frac{1}{2j+2} - \frac{1}{2j+3} + \frac{1}{2j+1} - \frac{1}{2j+2} \right) \\ &= 2 + 2 \sum_{j=0}^{+\infty} (-1)^j \left(\frac{1}{2j+1} - \frac{1}{2j+3} \right) \\ &= 2 + 2 \sum_{k=0}^{+\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3} - \frac{1}{4k+3} + \frac{1}{4k+5} \right) \\ &= \frac{8}{3} + 4 \sum_{k=1}^{+\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3} \right). \quad \blacksquare \end{aligned}$$

The original BBP formula from [5] reads as follows:

THEOREM 5.2 (Bailey–Borwein–Plouffe). *We have*

$$\pi = \sum_{k=0}^{+\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

Proof. Take $s = (1+i)/2$, so $\log s = -\frac{1}{2} \log 2 + i\frac{\pi}{4}$. Using the formula for $n = 1$, we have

$$\log \frac{1+i}{2} = - \sum_{j=1}^{\infty} \frac{(1-s)^j}{j} = - \sum_{j=1}^{\infty} \frac{(1-i)^j}{j2^j}.$$

Taking the imaginary part, and grouping terms for $j = 8k+r$, $r = 1, \dots, 8$, we get

$$\frac{\pi}{4} = - \sum_{k=0}^{\infty} \frac{1}{16^{k+1}} \left(\frac{-8}{8k+1} - \frac{8}{8k+2} - \frac{4}{8k+3} + \frac{2}{8k+5} + \frac{2}{8k+6} + \frac{1}{8k+7} \right),$$

so

$$(18) \quad \pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{2}{8k+1} + \frac{2}{8k+2} + \frac{1}{8k+3} - \frac{1/2}{8k+5} - \frac{1/2}{8k+6} - \frac{1/4}{8k+7} \right).$$

Similarly, by taking the real part, we get

$$\begin{aligned} & -\frac{1}{2} \log 2 \\ &= -\frac{71}{210} - \sum_{k=1}^{\infty} \frac{1}{16^{k+1}} \left(\frac{16}{8k} + \frac{8}{8k+1} - \frac{4}{8k+3} - \frac{4}{8k+4} - \frac{2}{8k+5} + \frac{1}{8k+7} \right), \end{aligned}$$

so

$$(19) \quad \log 2 = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{2}{8k} + \frac{1}{8k+1} - \frac{1/2}{8k+3} - \frac{1/2}{8k+4} - \frac{1/4}{8k+5} + \frac{1/8}{8k+7} \right).$$

Subtracting (19) and our previous formula (17), we get the null formula

$$(20) \quad 0 = \sum_{k=0}^{+\infty} \frac{1}{16^k} \left(\frac{1}{8k+1} - \frac{1}{8k+2} - \frac{1/2}{8k+3} - \frac{1}{8k+4} - \frac{1/4}{8k+5} - \frac{1/4}{8k+6} + \frac{1/8}{8k+7} \right)$$

(note that the term $k = 0$ gives exactly $1/105 = 71/105 - 2/3$). Adding (18) to twice this formula, we get

$$\pi = \sum_{k=0}^{+\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \blacksquare$$

In the proof we have proved and used the following null BBP formula that appears in [5]:

PROPOSITION 5.3. *We have*

$$(21) \quad \sum_{k=0}^{+\infty} \frac{1}{16^k} \left(\frac{-8}{8k+1} + \frac{8}{8k+2} + \frac{4}{8k+3} + \frac{8}{8k+4} \right. \\ \left. + \frac{2}{8k+5} + \frac{2}{8k+6} - \frac{1}{8k+7} \right) = 0.$$

Null BBP formulas are interesting and useful for rewriting BBP formulas. They are obtained by comparing BBP formulas for the same number at different values of s .

PROPOSITION 5.4. *We have*

$$\sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left(\frac{16}{6k+1} - \frac{24}{6k+2} - \frac{8}{6k+3} - \frac{6}{6k+4} + \frac{1}{6k+5} \right) = 0.$$

Proof. We use the formulas

$$\begin{aligned} \log \frac{3}{2} &= - \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{(-1)^k}{k} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left(\frac{1/2}{6k+1} + \frac{-1/4}{6k+2} + \frac{1/8}{6k+3} - \frac{1/16}{6k+4} + \frac{1/32}{6k+5} - \frac{1/64}{6k+6} \right), \\ \log \frac{3}{4} &= - \sum_{k=1}^{\infty} \frac{1}{4^k} \frac{1}{k} = - \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left(\frac{1/4}{3k+1} + \frac{1/16}{3k+2} + \frac{1/64}{3k+3} \right) \\ &= - \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left(\frac{1/2}{6k+2} + \frac{1/8}{6k+4} + \frac{1/32}{6k+6} \right), \\ \log \frac{9}{8} &= - \sum_{k=1}^{\infty} \frac{1}{8^k} \frac{(-1)^k}{k} = \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left(\frac{1/8}{2k+1} - \frac{1/64}{2k+2} \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left(\frac{3/8}{6k+3} - \frac{3/64}{6k+6} \right). \end{aligned}$$

Adding the first two and subtracting the third, we get

$$\sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left(\frac{1/2}{6k+1} - \frac{3/4}{6k+2} - \frac{1/4}{6k+3} - \frac{3/16}{6k+4} + \frac{1/32}{6k+5} \right) = 0,$$

and multiplying by 32 we get the result. ■

Finally, we include a proof of Bellard's formula.

THEOREM 5.5 (F. Bellard). *We have*

$$\pi = \frac{1}{2^6} \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{10k}} \left(-\frac{2^5}{4n+1} - \frac{1}{4n+3} + \frac{2^8}{10n+1} - \frac{2^6}{10n+3} - \frac{2^2}{10n+5} - \frac{2^2}{10n+7} + \frac{1}{10n+9} \right).$$

Proof. We use the factorization

$$1+i = \left(\frac{2+i}{2} \right)^2 \left(\frac{7+i}{8} \right)^{-1},$$

and taking the imaginary parts yields

$$\frac{\pi}{4} = 2\Im \log(1+i/2) - \Im \log((7+i)/8).$$

For $s = (7+i)/8$ and $n = 1$, we get

$$\begin{aligned} (22) \quad \Im \log((7+i)/8) &= -\Im \sum_{j=1}^{\infty} \frac{(1-i)^j}{j8^j} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{20k}} \left(\frac{1/8}{8k+1} + \frac{2/8^2}{8k+2} + \frac{2/8^3}{8k+3} - \frac{4/8^5}{8k+5} - \frac{8/8^6}{8k+6} - \frac{8/8^7}{8k+7} \right) \\ &= \frac{1}{256} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{10l}} \left(\frac{32}{4l+1} + \frac{8}{4l+2} + \frac{1}{4l+3} \right), \end{aligned}$$

writing $j = 8k + r$, $r = 1, \dots, 8$, and then $2k = l$.

Now take $s = 1 + i/2$ and $n = 1$ to get

$$\begin{aligned} (23) \quad \Im \log(1+i/2) &= -\Im \sum_{j=1}^{\infty} \frac{(-i)^j}{j2^j} = \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} \frac{(-1)^k}{2k+1} \\ &= \frac{1}{256} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{10l}} \left(\frac{128}{10l+1} - \frac{32}{10l+3} + \frac{8}{10l+5} - \frac{2}{10l+7} + \frac{1/2}{10l+9} \right). \end{aligned}$$

Taking twice (23) minus (22), and using $2 \frac{8}{10l+5} - \frac{8}{4l+2} = -\frac{4}{10l+5}$, we get the result. ■

6. On the classical BBP form. As defined in [6] the classical BBP form is

$$Q(b, d, m, A) = \sum_{k=0}^{+\infty} \frac{1}{b^k} \sum_{l=1}^m \frac{a_l}{(km+l)^d},$$

where b, d, m are integers and $A = (a_1, \dots, a_m)$ is a vector of integers. The degree is d and the base is b . Let us check that with our formula from Theorem 4.3 we get BBP formulas of degree 1.

LEMMA 6.1. *We have*

$$\frac{1}{\binom{j+n}{n}} = \sum_{l=1}^n \frac{c_l}{j+l},$$

where for $l = 1, \dots, n$, c_l is the integer given by

$$c_l = (-1)^{l-1} n \binom{n-1}{l-1}.$$

Proof. As usual, multiply by $j+l$ and set $j = -l$ to get

$$\begin{aligned} c_l &= \frac{n!}{(n-l)(n-l-1)\cdots 2 \cdot 1 \cdot (-1)(-2)\cdots(-(l-1))} \\ &= (-1)^{l+1} l \binom{n}{l} = (-1)^{l-1} n \binom{n-1}{l-1}. \blacksquare \end{aligned}$$

We have a general reorganization lemma that shows that any sum of BBP form with more than m fractions can be reorganized into one with m terms.

LEMMA 6.2. *We have*

$$\sum_{j=0}^{+\infty} b^{-j} \left(\sum_{i=1}^n \frac{c_i}{(mj+i)^d} \right) = \sum_{k=0}^{+\infty} b^{-k} \left(\sum_{l=1}^m \frac{a_l}{(km+l)^d} \right),$$

with

$$a_l = \sum_i c_i b^{(i-l)/m},$$

where the sum extends over $l+1 \leq i \leq n$ such that $mj+i = mk+l$.

Proof. For $k = 0, 1, \dots$ and $l = 1, \dots, m$, group the fractions of the sum modulo m with $mj+i = mk+l$. \blacksquare

These two lemmas prove that the BBP formulas that we get from Theorem 1.1 are of type $Q(1, b, 1, (a_1))$.

We can apply this reorganization to the summation in the formula from Theorem 1.1 and get (regrouping the terms with $j = k-l+1$ in the third equality)

$$\begin{aligned} (24) \quad \sum_{j=0}^{+\infty} \frac{(1-s)^{j+n}}{\binom{j+n}{n}} &= \sum_{j=0}^{+\infty} \sum_{l=1}^n \frac{c_l}{j+l} (1-s)^{j+n} \\ &= \sum_{k=0}^{+\infty} \sum_{l=1}^n \frac{c_l}{k+1} (1-s)^{k+1+n-l} - \sum_{\substack{0 \leq k \leq l-2 \\ l \leq n}} \frac{c_l}{k+1} (1-s)^{k+1+n-l} \\ &= - \sum_{\substack{0 \leq k \leq l-2 \\ l \leq n}} \frac{c_l (1-s)^{k+n+1-l}}{k+1} + \sum_{k=0}^{+\infty} \frac{a_k}{k+1} (1-s)^{k+1} \end{aligned}$$

with $a_k = \sum_{l=1}^n c_l(1-s)^{n-l}$. But

$$\begin{aligned} a_k &= \sum_{l=1}^n c_l(1-s)^{n-l} = \sum_{l=1}^n (-1)^{l-1} n \binom{n-1}{l-1} (1-s)^{n-l} \\ &= (-1)^{n-1} n (1 - (1-s))^{n-1} = (-1)^{n-1} n s^{n-1}. \end{aligned}$$

Hence, we recognize $\log s$ in the last sum of (24), so the formula in Theorem 1.1 for $n \geq 2$ is a rearrangement of the formula for $n = 1$ that is the classical Taylor formula for $\log s$,

$$(25) \quad \log s = - \sum_{k=0}^{\infty} \frac{(1-s)^{k+1}}{k+1}.$$

We can use this rearrangement to recover the formula for the polynomials B_n directly:

$$\begin{aligned} \sum_{m \geq 1} \frac{(1-s)^{n+m}}{m(m+1) \cdots (m+n)} &= \frac{(-1)^n}{n!} \sum_{m \geq 1} (1-s)^{m+n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{m+k} \\ &= \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (1-s)^{n-k} \left(\log(s) - \sum_{i=1}^k \frac{(1-s)^i}{i} \right) \end{aligned}$$

but

$$\begin{aligned} A_n(s) &= \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (1-s)^{n-k} = (1 - (1-s))^n = \frac{(-1)^n}{n!} s^n, \\ B_n(s) &= \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (1-s)^{n-k} \sum_{i=1}^k \frac{(1-s)^i}{i}, \end{aligned}$$

which after some rearrangement gives the expression for $B_n(s)$.

Formally, there is no extra content in the formulas for the same parameter s but different integers $n \geq 2$. However, these rearrangements are computationally useful, and they are not easy to produce. The iterated integrals $I_n(s)$ or Proposition 2.2 gives a systematic method to find a family of such resummations. The expression in terms of combinatorial coefficients in the denominator that arise by the iterated integrals in this type of sums can sometimes present some advantages. Of course this immediately reminds (even if it is a formula of higher degree) the famous Apéry sum for $\zeta(3)$, a starting point of his proof of the irrationality of this number. Recall that Apéry's formula is

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Appendix. Location of the zeros of B_n . The application of the formula in Theorem 4.3 to roots of B_n , in particular to real roots, gives BBP-like formulas of a special form. We study the location of the roots of B_n and the number of real roots.

To understand the polynomials $B_n(s)$, we introduce the polynomials $C_n(x)$ of degree $n - 2$, for $n \geq 2$, defined by

$$(26) \quad B_n(s) = -\frac{1}{(n-1)!}(s-1)C_n(s-1),$$

so that by Proposition 2.3,

$$(27) \quad C_n(x) = \sum_{k=0}^{n-2} \binom{n-1}{k+1} (H_{n-1} - H_{n-k-2}) x^k.$$

We list the first such polynomials:

$$B_1(s) = 0,$$

$$B_2(s) = -(s-1),$$

$$B_3(s) = -\frac{1}{4}(s-1)(3s-1),$$

$$B_4(s) = -\frac{1}{6}(s-1)\left(1 + \frac{5}{2}(s-1) + \frac{11}{6}(s-1)^2\right),$$

$$B_5(s) = -\frac{1}{20}(s-1)\left(1 + \frac{7}{2}(s-1) + \frac{13}{3}(s-1)^2 + \frac{25}{12}(s-1)^3\right),$$

$$B_6(s) = -\frac{1}{120}(s-1)\left(1 + \frac{9}{2}(s-1) + \frac{47}{6}(s-1)^2 + \frac{77}{12}(s-1)^3 + \frac{137}{60}(s-1)^4\right),$$

$$B_7(s) = -\frac{1}{740}(s-1)$$

$$\times \left(1 + \frac{11}{2}(s-1) + \frac{37}{3}(s-1)^2 + \frac{57}{4}(s-1)^3 + \frac{87}{10}(s-1)^4 + \frac{49}{20}(s-1)^5\right),$$

and accordingly,

$$C_2(x) = x,$$

$$C_3(x) = \frac{1}{2}(3x+2),$$

$$C_4(x) = \frac{1}{6}(6+15x+11x^2),$$

$$C_5(x) = \frac{1}{12}(12+42x+52x^2+25x^3),$$

$$C_6(x) = \frac{1}{60}(60+180x+470x^2+389x^3+137x^4),$$

$$C_7(x) = \frac{1}{60}(60+330x+740x^2+859x^3+522x^4+147x^5).$$

We want to locate the zeros of $C_n(x)$.

LEMMA 6.3. *We have*

$$C_n(0) = 1, \quad C_n(-1) = (n-1)!B_n(0) = \frac{(-1)^n}{n-1}.$$

Proof. The value at $x = -1$ follows from Lemma 2.4, and the value at $x = 0$ by (27). ■

Let

$$D_n(x) = x C_n(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} (H_{n-1} - H_{n-k-1}) x^k.$$

The zeros of D_n are those of C_n and an extra zero at $x = 0$. Now we have two interesting equalities:

$$\begin{aligned}
 (28) \quad D'_n(x) &= \sum_{k=1}^{n-1} k \binom{n-1}{k} (H_{n-1} - H_{n-k-1}) x^{k-1} \\
 &= \sum_{k=1}^{n-1} (n-1) \binom{n-2}{k-1} (H_{n-1} - H_{n-k-1}) x^{k-1} \\
 &= \sum_{k=0}^{n-2} (n-1) \binom{n-2}{k} \left(\frac{1}{n-1} + H_{n-2} - H_{n-k-2} \right) x^k \\
 &= \sum_{k=0}^{n-2} (n-1) \binom{n-2}{k} (H_{n-2} - H_{(n-1)-k-1}) x^k + \sum_{k=0}^{n-2} \binom{n-2}{k} x^k \\
 &= (n-1) D_{n-1}(x) + (1+x)^{n-2},
 \end{aligned}$$

and

$$\begin{aligned}
 (29) \quad (1+x) D'_n - (n-1) D_n &= (n-1)(1+x) D_{n-1} - (n-1) D_n + (1+x)^{n-1} \\
 &= \sum_{k=0}^{n-2} (n-1) \binom{n-2}{k} (H_{n-2} - H_{n-k-2}) x^k \\
 &\quad + \sum_{k=0}^{n-2} (n-1) \binom{n-2}{k} (H_{n-2} - H_{n-k-2}) x^{k+1} \\
 &\quad - \sum_{k=0}^{n-1} (n-1) \binom{n-1}{k} (H_{n-1} - H_{n-k-1}) x^k + (1+x)^{n-1} \\
 &= \sum_{k=0}^{n-2} (n-1) \binom{n-2}{k} \left(H_{n-2} - H_{n-k-1} + \frac{1}{n-k-1} \right) x^k \\
 &\quad + \sum_{k=1}^{n-1} (n-1) \binom{n-2}{k-1} (H_{n-2} - H_{n-k-1}) x^k \\
 &\quad - \sum_{k=0}^{n-1} (n-1) \binom{n-1}{k} \left(\frac{1}{n-1} + H_{n-2} - H_{n-k-1} \right) x^k + (1+x)^{n-1} \\
 &= \sum_{k=0}^{n-2} \frac{n-1}{n-k-1} \binom{n-2}{k} x^k - \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + (1+x)^{n-1}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{n-2} \binom{n-1}{k} x^k - \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + (1+x)^{n-1} \\
&= (1+x)^{n-1} - x^{n-1} = Q_n(x).
\end{aligned}$$

Using these equalities, we can prove the following:

PROPOSITION 6.4. *For $n \geq 2$ even, the polynomial $C_n(x)$ has no real roots. For $n \geq 3$ odd, $C_n(x)$ has only one real root and it lies in the interval $] -1, 0[$.*

Proof. We want to prove by induction that:

- For n even, $x = 0$ is the only (simple) zero of $D_n(x)$. And $D_n(x) < 0$ for $x < 0$ and $D_n(x) > 0$ for $x > 0$.
- For n odd, D_n has two zeros, at some $x_0 \in] -1, 0[$ and at $x = 0$. And $D_n(x) > 0$ for $x \in] -\infty, x_0[\cup] 0, \infty[$ and $D_n(x) < 0$ for $x \in] x_0, 0[$.

Let n be even. We want to prove that $D_n(x)$ has only a zero at $x = 0$. Note that $D_n(0) = 0$ and $D'_n(0) = 1$, so D_n is increasing at $x = 0$. For n even we have $Q_n(x) > 0$ everywhere.

- If $x \leq -1$ then $D_{n-1}(x) < 0$ by induction hypothesis. By (28) we have $D'_n(x) > 0$, so D_n is increasing there. By Lemma 6.3, $D_n(-1) < 0$, so there are no zeros in $] -\infty, -1]$.
- If $x > 0$ then $D_{n-1}(x) > 0$ by induction hypothesis. By (28) we have $D'_n(x) > 0$, so D_n is increasing there. As $D_n(0) = 0$, there are no zeros in $] 0, \infty[$.
- If $x \in] -1, 0[$ then $Q_n(x) > 0$. If $D_n(x) = 0$ then (29) says that $(1+x)D'_n(x) > 0$. So D_n is increasing at every zero. As $x = 0$ is a zero, this implies that there is only one zero of D_n .

Now let n be odd. We want to prove that $D_n(x)$ has a zero at some $x_0 \in] -1, 0[$ and at $x = 0$, it is positive on $] -\infty, x_0[\cup] 0, \infty[$ and negative at $] x_0, 0[$. Note that $D_n(0) = 0$ and $D'_n(0) = 1$, so D_n is increasing at $x = 0$. Note that for n odd we have $Q_n(x) > 0$ for $x > -1/2$, and $Q_n(x) < 0$ for $x < -1/2$.

- If $x \leq -1$ then $D_{n-1}(x) < 0$ by induction hypothesis. By (28) we have $D'_n(x) < 0$, so D_n is decreasing there. By Lemma 6.3, $D_n(-1) > 0$, so there are no zeros in $] -\infty, -1]$.
- If $x > 0$ then $D_{n-1}(x) > 0$ by induction hypothesis. By (28) we have $D'_n(x) > 0$, so D_n is increasing there. As $D_n(0) = 0$, there are no zeros in $] 0, \infty[$.
- If $x \in] -1/2, 0[$ then (29) says that $(1+x)D'_n(x) > (n-1)D_n(x)$. So if there is a zero, D_n is increasing. As the last zero before $x = 0$ cannot be increasing, this last zero has to be $x_0 \leq -1/2$.

- For $x = -1/2$, if it was a zero of D_n , then it is also a zero of D'_n because of (29). Then we write $x = -1/2 + h$, and develop (29) to see that $D'_n(-1/2 + h) > 0$ for $h > 0$ small. But this implies that there must be another zero of D_n in $] -1/2, 0[$ with decreasing slope, which contradicts the previous item.
- If $x \in] -1, -1/2[$ then (29) says that $(1 + x)D'_n(x) < (n - 1)D_n(x)$. So if there is a zero, D_n is decreasing. There must be at least one zero, but there cannot be two zeros, since there cannot be two decreasing consecutive zeros. ■

It is of interest to locate the complex zeros of $C_n(x)$. The polynomial C_4 has a pair of conjugate complex roots $x \approx -0.68182 \pm 0.28386i$. The polynomial C_5 has one real root $x_0 \approx -0.61852$ and a pair of conjugate complex roots $x \approx -0.73074 \pm 0.49200i$. The polynomial C_6 has two pairs of conjugate complex roots: $x \approx -0.18154 \pm 0.39220i$, $x \approx -1.2382 \pm 0.9009i$. We may expect that all roots of $C_n(x)$ have $\Re x \in] -\infty, 0[$.

To locate the complex roots of $C_n(x)$, we rewrite the differential equation (29) as

$$((1 + x)^{-(n-1)}D_n(x))' = (1 + x)^{-n}Q_n(x).$$

Take $f_n(x) = (1 + x)^{-(n-1)}D_n(x)$, hence $df_n = (1 + x)^{-n}((1 + x)^{n-1} - x^{n-1})dx$. We make the change of variables $w = \frac{x}{1+x}$ to get $df_n = \frac{1-w^{n-1}}{1-w}dw = (1 + w + \dots + w^{n-2})dw$, and integrating

$$f_n = w + \frac{1}{2}w^2 + \dots + \frac{1}{n-1}w^{n-1},$$

where we have used the fact that for $w = 0$, we have $x = 0$ and hence $f_n = 0$. Note that $f_n(w)$ is the truncation of the series $-\log(1 - w)$, which is convergent on $|w| < 1$.

PROPOSITION 6.5. *The polynomial $f_n(w)$ has no roots in $|w| \leq 1$ except $w = 0$.*

Proof. We will look at the polynomial

$$Q(w) = f_n(w)(1 - w)/w = 1 - \sum_{k=1}^{n-2} \frac{1}{k(k+1)}w^k - \frac{1}{n-1}w^{n-1},$$

for which we want to check that the only root in the disc $|w| \leq 1$ is $w = 1$. For $|w| \leq 1$, we have

$$\left| \sum_{k=2}^{n-2} \frac{1}{k(k+1)}w^k + \frac{1}{n-1}w^{n-1} \right| \leq \sum_{k=2}^{n-2} \frac{1}{k(k+1)} + \frac{1}{n-1} = \frac{1}{2}.$$

Then if $Q(w) = 0$, we have

$$\left|1 - \frac{1}{2}w\right| \leq \frac{1}{2},$$

which implies $|w - 2| \leq 1$. Combined with $|w| \leq 1$, this yields $w = 1$. ■

Undoing the change of variables $w = \frac{x}{1+x}$, we find that all roots of $C_n(x)$ are in $\Re x < -1/2$. Therefore, with (26) we see that the roots of $B_n(s)$ are $s = 1$ and the others lie in $\Re s < 1/2$.

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