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BUMP CONDITIONS AND TWO-WEIGHT INEQUALITIES FOR COMMUTATORS OF FRACTIONAL INTEGRALS

BY

YONGMING WEN (Zhangzhou) and HUOXIONG WU (Xiamen)

Abstract. This paper gives new two-weight bump conditions for sparse operators related to iterated commutators of fractional integrals. As applications, two-weight bounds for iterated commutators of fractional integrals under more general bump conditions are obtained. Necessary two-weight bump conditions including the converse of Bloom type estimates for iterated commutators of fractional integrals are also studied.

1. Introduction and main results. Let $0 < \alpha < n$, $m \in \mathbb{Z}^+$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The fractional integral operator I_α and its higher order commutator $I_\alpha^{b,m}$ are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \qquad I_\alpha^{b,m} f(x) = \int_{\mathbb{R}^n} (b(x)-b(y))^m \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

In this paper, we consider two-weight estimates for $I_\alpha^{b,m}$,

$$\Big(\int_{\mathbb{R}^n} |I_\alpha^{b,m} f(x)|^q \mu(x)\,dx\Big)^{1/q} \le C\Big(\int_{\mathbb{R}^n} |f(x)|^p \nu(x)\,dx\Big)^{1/p},$$

where (μ,ν) is a pair of weights. First, we provide a brief review of some previously known results.

Let $1 < p < n/\alpha$ and $1/p - 1/q = \alpha/n$. It is well known that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Given a function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we say that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|\,dx < \infty,$$

where $b_Q = |Q|^{-1}\int_Q b(x)\,dx$. In 1982, Chanillo [4] proved that if $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $I_\alpha^{b,1}$ is bounded from $L^p(\mathbb{R}^n)$

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to $L^q(\mathbb{R}^n)$. By a *weight* ω, we mean a nonnegative locally integrable function on $\mathbb{R}^n$. We say that $\omega \in A_{p,q}$ if

$$[\omega]_{A_{p,q}} = \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)^q\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\right)^{q/p'} < \infty, \qquad 1 < p < q < \infty.$$

Muckenhoupt and Wheeden [21] proved that I_α is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$, where $0 < \alpha < n$, $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$ and $\omega \in A_{p,q}$. Under the same conditions as in [21] with $b \in \mathrm{BMO}(\mathbb{R}^n)$, Segovia and Torrea [29] obtained the weighted $L^p \to L^q$ boundedness for commutators of fractional integral operators.

Though the forms of two-weight inequalities for singular integral operators and related operators are the generalization of one-weight inequalities, two weight estimates for operators are more difficult. For instance, it is well known that the A_p condition

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} < \infty$$

is a sufficient condition for singular integral operators and related operators to be bounded on $L^p(\omega)$. However, in general, the A_p condition for a pair (μ, ν) of weights,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \mu(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \nu(x)^{1-p'}\,dx\right)^{p-1} < \infty, \tag{1.1}$$

is necessary but never sufficient for operators to be bounded from $L^p(\nu)$ to $L^p(\mu)$ (see [6]). In the case of the Hardy–Littlewood maximal operator M, to solve this problem, Sawyer [28] first introduced the following testing condition: M is bounded from $L^p(\nu)$ to $L^p(\mu)$ if and only if

$$\int_Q M(\nu^{1-p'}\chi_Q)(x)^p \mu(x)\,dx \le C\int_Q \nu(x)^{1-p'}\,dx < \infty.$$

However, this condition can be hard to verify since it involves the operator M itself. This drawback prompted researchers to search for some simpler sufficient conditions, which are close to (1.1) in some sense. Neugebauer [23] first proved that for some $r > 1$, if a pair (μ, ν) of weights satisfies the following power bump condition:

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \mu(x)^r\,dx\right)^{1/r}\left(\frac{1}{|Q|}\int_Q \nu(x)^{r(1-p')}\,dx\right)^{(p-1)/r} < \infty,$$

then

$$\int_{\mathbb{R}^n} (Mf(x))^p \mu(x)\,dx \le C\int_{\mathbb{R}^n} |f(x)|^p \nu(x)\,dx.$$

To continue our overview of works seeking for appropriate bump conditions which are sufficient for two-weight inequalities for singular integral operators and related operators, we recall some facts about Orlicz spaces.

We say $A : [0,\infty) \to [0,\infty)$ is a *Young function* if it is increasing, convex, $A(0) = 0$ and $A(t)/t \to \infty$ as $t \to \infty$. Given a Young function A, the associated complementary function $\bar{A}$ is defined by

$$\bar{A}(t) = \sup_{s>0}\{st - A(s)\}.$$

Let $1 < p < \infty$ and A be a Young function. We say that $A \in B_p$ if

$$\int_1^\infty \frac{A(t)}{t^p}\,\frac{dt}{t} < \infty.$$

Given a Young function A, the *Orlicz average* of a function f on a cube Q is defined by

$$\|f\|_{A,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q A\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

In 1995, Pérez [26] improved Neugebauer's result by eliminating the power bump on the weight μ and replacing the power bump on ν by the "Orlicz bump". More precisely, he proved that if a pair (μ,ν) of weights satisfies

$$\sup_Q \|\mu^{1/p}\|_{p,Q}\|\nu^{-1/p}\|_{\Phi,Q} < \infty, \qquad 1 < p < \infty,$$

and $\bar{\Phi} \in B_p$, then $M : L^p(\nu) \to L^p(\mu)$. For the Calderón–Zygmund operator T, Cruz-Uribe and Pérez [12] conjectured that if both terms in (1.1) are bumped, then $T : L^p(\nu) \to L^p(\mu)$. This conjecture was partially solved in [22] and completely solved by Lerner [17]. Lerner proved that if a pair of weights (μ,ν) satisfies

$$\sup_Q \|\mu^{1/p}\|_{\Psi,Q}\|\nu^{-1/p}\|_{\Phi,Q} < \infty, \qquad 1 < p < \infty, \tag{1.2}$$

and $\bar{\Phi} \in B_p$, $\bar{\Psi} \in B_{p'}$, then $T : L^p(\nu) \to L^p(\mu)$. The separated bump conjecture, which arises from the work of Cruz-Uribe et al. [11], asserts that $T : L^p(\nu) \to L^p(\mu)$ provided that, in place of (1.2),

$$\sup_Q \|\mu^{1/p}\|_{p,Q}\|\nu^{-1/p}\|_{\Phi,Q} < \infty \quad \text{and} \quad \|\mu^{1/p}\|_{\Psi,Q}\|\nu^{-1/p}\|_{p',Q} < \infty.$$

In [13], Cruz-Uribe et al. proved this conjecture for $\Phi(t) = t^{p'}[\log(e+t)]^{p'-1+\delta}$ and $\Psi(t) = t^p[\log(e+t)]^{p-1+\delta}$ for some $\delta > 0$. This conjecture is still open, and we refer the readers to [16, 19, 20] for more recent relevant works. Analogously to the case of singular integral operators, Pérez [25] gave the following sufficient condition:

$$\sup_Q |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{A,Q}\|\nu^{-1/p}\|_{B,Q}<\infty, \qquad \bar{A}\in B_p,\ \bar{B}\in B_{q'},$$

for $I_\alpha : L^p(\nu)\to L^q(\mu)$. The conditions $\bar{A}\in B_p$, $\bar{B}\in B_{q'}$ were improved to $\bar{A}\in B_{p,q}$, $\bar{B}\in B_{q',p'}$ in [8]. Here, we say that $A\in B_{p,q}$ if

$$\int_1^\infty \frac{A(t)^{q/p}}{t^q}\frac{dt}{t}<\infty.$$

Recently, Rahm [27] used "entropy bumps" and "direct comparison bumps" to get the two-weight boundedness for fractional sparse operators.

On the other hand, Cruz-Uribe and Moen [7] showed that if $b\in\mathrm{BMO}(\mathbb{R}^n)$ and a pair (μ,ν) of weights satisfies

$$\sup_Q \|\mu^{1/p}\|_{L^p(\log L)^{2p-1+\delta},Q}\|\nu^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\delta},Q}<\infty,$$

then the commutator of the Calderón–Zygmund operator T_b is bounded from $L^p(\nu)$ to $L^p(\mu)$. This result was recently improved by Lerner et al. [19], who provided a wider class of weights (μ,ν):

$$\sup_Q \|\mu^{1/p}\|_{L^p(\log L)^{(m+1)p-1+\delta}}\|\nu^{-1/p}\|_{B,Q} + \sup_Q \|\mu^{1/p}\|_{A,Q}\|\nu^{-1/p}\|_{L^p(\log L)^{(m+1)p-1+\delta},Q}<\infty,$$

for which $\|T_b^m\|_{L^p(\nu)\to L^p(\mu)}<\infty$, where $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $\bar{A}\in B_{p'}, \bar{B}\in B_p$. Very recently, Cruz-Uribe et al. [9] generalized the work of [19] by assuming the Young functions satisfy $\bar{A},\bar{C}\in B_{p'}$, $\bar{B},\bar{D}\in B_p$ and (μ,ν) satisfies

$$\sup_Q \|\mu^{1/p}\|_{A,Q}\|(b-b_Q)^m\nu^{-1/p}\|_{B,Q} + \sup_Q \|(b-b_Q)^m\mu^{1/p}\|_{C,Q}\|\nu^{-1/p}\|_{B,Q}<\infty.$$

We also refer the readers to [15] for a result in the matrix setting. For commutators of fractional integral operators, Cruz-Uribe [5] showed that if a pair of weights (μ,ν) satisfies

$$\sup_Q |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{L^q(\log L)^{2q-1+\delta},Q}\|\nu^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\delta},Q}<\infty, \tag{1.3}$$

then $I_\alpha^{b,1}$ is bounded from $L^p(\nu)$ to $L^p(\mu)$. Recently, Cardenas and Isralowitz [3] established a two-weight inequality for $I_\alpha^{b,1}$ in the matrix setting.

Inspired by [3, 9, 19], in this paper, we mainly consider two-weight inequalities for $I_\alpha^{b,m}$. Our first main result can be formulated as follows.

THEOREM 1.1. *Let* $1 < p \le q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$, $b \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ *and let* $\mathcal{S}$ *be a sparse family.*

(i) *Suppose that* A, B, C, D *are Young functions which satisfy* $\bar{A}, \bar{C} \in B_{q'}$ *and* $\bar{B}, \bar{D} \in B_{p,q}$. *If a pair* (μ, ν) *of weights satisfies*

$$\sup_{Q\in\mathcal{S}} |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{A,Q}\|(b-b_Q)^m\nu^{-1/p}\|_{B,Q} + \sup_{Q\in\mathcal{S}} |Q|^{\alpha/n+1/q-1/p}\|(b-b_Q)^m\mu^{1/q}\|_{C,Q}\|\nu^{-1/p}\|_{D,Q} < \infty,$$

then

$$\|T^{\mathcal{S},b,m}_\alpha f\|_{L^q(\mu)} + \|(T^{\mathcal{S},b,m}_\alpha)^* f\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\nu)}. \tag{1.4}$$

(ii) *Conversely, if* (1.4) *holds, then*

$$\sup_{Q\in\mathcal{S}} |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{q,Q}\|(b-b_Q)^m\nu^{-1/p}\|_{p',Q} + \sup_{Q\in\mathcal{S}} |Q|^{\alpha/n+1/q-1/p}\|(b-b_Q)^m\mu^{1/q}\|_{q,Q}\|\nu^{-1/p}\|_{p',Q} < \infty.$$

Here

$$T^{b,m}_{\mathcal{S},\alpha} f(x) = \sum_{Q\in\mathcal{S}} |Q|^{\alpha/n}\Big(|Q|^{-1}\int_Q |b(x)-b_Q|^m|f(x)|\,dx\Big)\chi_Q(x),$$

$$(T^{b,m}_{\mathcal{S},\alpha})^* f(x) = \sum_{Q\in\mathcal{S}} |Q|^{\alpha/n}|b(x)-b_Q|^m\Big(|Q|^{-1}\int_Q |f(x)|\,dx\Big)\chi_Q(x).$$

As an application, the following two-weight bump condition for iterated commutators $I^{b,m}_\alpha$ follows from Theorem 1.1 via sparse domination.

THEOREM 1.2. *Let* $1 < p \le q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$, $b \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ *and let* $I^{b,m}_\alpha$ *be commutators of fractional integral operators. Suppose that* A, B, C, D *are Young functions which satisfy* $\bar{A}, \bar{C} \in B_{q'}$ *and* $\bar{B}, \bar{D} \in B_{p,q}$. *If a pair* (μ, ν) *of weights satisfies*

$$\sup_{Q} |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{A,Q}\|(b-b_Q)^m\nu^{-1/p}\|_{B,Q} + \sup_{Q} |Q|^{\alpha/n+1/q-1/p}\|(b-b_Q)^m\mu^{1/q}\|_{C,Q}\|\nu^{-1/p}\|_{D,Q} < \infty,$$

then $\|I^{b,m}_\alpha f\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\nu)}$.

Furthermore, as a consequence of Theorem 1.2, we can obtain the more traditional bump conditions by assuming that the multiplier b lies in an oscillation class related to $\mathrm{BMO}(\mathbb{R}^n)$.

THEOREM 1.3. *Let* $1 < p \le q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$ *and let* $I^{b,m}_\alpha$ *be commutators of fractional integral operators. Assume that* A, B, C, D, X, Y

are Young functions which satisfy $\bar{A}, \bar{C} \in B_{q'}$*,* $\bar{B}, \bar{D} \in B_{p,q}$ *and*

$$X^{-1}(t) \lesssim \frac{B^{-1}(t)}{\Phi^{-1}(t)^m} \quad \text{and} \quad Y^{-1}(t) \lesssim \frac{C^{-1}(t)}{\Phi^{-1}(t)^m}$$

for large t*. If* $b \in \operatorname{Osc}(\Phi)$ *and a pair* (μ,ν) *of weights satisfies*

$$\sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{A,Q} \|\nu^{-1/p}\|_{X,Q} + \sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{Y,Q} \|\nu^{-1/p}\|_{D,Q} < \infty,$$

then $\|I_\alpha^{b,m} f\|_{L^q(\mu)} \lesssim \|b\|^m_{\operatorname{Osc}(\Phi)} \|f\|_{L^p(\nu)}$*, where* $\operatorname{Osc}(\Phi)$ *is the space of functions* $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with*

$$\|b\|_{\operatorname{Osc}(\Phi)} = \sup_Q \|b - b_Q\|_{\Phi,Q} < \infty.$$

When $b \in \mathrm{BMO}(\mathbb{R}^n)$, we may take $\Phi(t) = \exp t - 1$ in Theorem 1.3. Then we have the following result.

Corollary 1.4. *Let* $1 < p \le q < \infty$*,* $0 < \alpha < n$*,* $m \in \mathbb{Z}^+$ *and let* $I_\alpha^{b,m}$ *be commutators of fractional integral operators. Suppose that* A, D *are Young functions which satisfy* $\bar{A} \in B_{q'}$*,* $\bar{D} \in B_{p,q}$*. If* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and a pair* (μ,ν) *of weights satisfies*

$$\sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{A,Q} \|\nu^{-1/p}\|_{L^{p'}(\log L)^{(m+1)p'-1+\delta},Q} + \sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{L^q(\log L)^{(m+1)q-1+\delta},Q} \|\nu^{-1/p}\|_{D,Q} < \infty,$$

then $\|I_\alpha^{b,m} f\|_{L^q(\mu)} \lesssim \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\nu)}$*.*

In particular, if we take $A(t) = t^q[\log(e+t)]^{q-1+\delta}$ and $D(t) = t^{p'}[\log(e+t)]^{p'-1+\delta}$, then we can get the following two-weight bump conditions for $I_\alpha^{b,m}$, more general than (1.3).

Corollary 1.5. *Let* $1 < p \le q < \infty$*,* $0 < \alpha < n$*,* $m \in \mathbb{Z}^+$ *and let* $I_\alpha^{b,m}$ *be commutators of fractional integral operators. If* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and a pair* (μ,ν) *of weights satisfies*

$$\sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{L^q(\log L)^{q-1+\delta},Q} \|\nu^{-1/p}\|_{L^{p'}(\log L)^{(m+1)p'-1+\delta},Q} + \sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{L^q(\log L)^{(m+1)q-1+\delta},Q} \|\nu^{-1/p}\|_{L^{p'}(\log L)^{p'-1+\delta},Q} < \infty \tag{1.5}$$

for some $\delta > 0$*, then* $\|I_\alpha^{b,m} f\|_{L^q(\mu)} \lesssim \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\nu)}$*.*

Remark 1.6. It is clear that the bump condition in (1.5) for $m = 1$ is more general than the one in (1.3). Therefore, Corollary 1.5 is an essential improvement of the corresponding result in [5].

Next, we turn to the necessity of bump conditions for the two-weight boundedness of $I_\alpha^{b,m}$, which is addressed in the following theorem.

THEOREM 1.7. *Let $1 < p \le q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$ and let $I_\alpha^{b,m}$ be commutators of fractional integral operators. Suppose that μ is a doubling weight and for any $b \in \mathrm{BMO}(\mathbb{R}^n)$,*

$$\|I_\alpha^{b,m} f\|_{L^{q,\infty}(\mu)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^m \|f\|_{L^p(\nu)}.$$

Then

$$\sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{L^q,Q} \|\nu^{-1/p}\|_{L^{p'}(\log L)^{mp'},Q} < \infty.$$

Finally, we consider the inverse result related to the Bloom type estimate for $I_\alpha^{b,m}$. We first provide some background. Let η be a weight. We say that $b \in \mathrm{BMO}_\eta$ if

$$\|b\|_{\mathrm{BMO}_\eta} := \sup_Q \frac{1}{\eta(Q)} \int_Q |b(x) - b_Q|\, dx < \infty.$$

Bloom [2] first characterized BMO_η via the two-weight estimate of the commutator of the Hilbert transform H. For the commutator of a fractional integral operator, Accomazzo et al. [1] proved that if $\lambda, \mu \in A_{p,q}$ and $\eta = (\mu\lambda^{-1})^{1/m}$, then

$$b \in \mathrm{BMO}_\eta \implies \|I_\alpha^{b,m}\|_{L^q(\lambda^q)} \lesssim \|b\|_{\mathrm{BMO}_\eta}^m \|f\|_{L^p(\mu^p)}$$

and

$$\|I_\alpha^{b,m}\|_{L^q(\lambda^q)} \lesssim \|f\|_{L^p(\mu^p)} \implies b \in \mathrm{BMO}_\eta.$$

The corresponding result for $m = 1$ was obtained by Holmes et al. [14]. Our next theorem can be regarded as the converse of the above Bloom type estimate for $I_\alpha^{b,m}$.

THEOREM 1.8. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$, $m \in \mathbb{Z}^+$, $\lambda, \mu \in A_{p,q}$ and let $I_\alpha^{b,m}$ be commutators of fractional integral operators. If η is an arbitrary weight which satisfies*

$$b \in \mathrm{BMO}_\eta \implies \|I_\alpha^{b,m}\|_{L^q(\lambda^q)} \lesssim \|b\|_{\mathrm{BMO}_\eta}^m \|f\|_{L^p(\mu^p)} \tag{1.6}$$

and

$$\|I_\alpha^{b,m}\|_{L^q(\lambda^q)} \lesssim \|f\|_{L^p(\mu^p)} \implies b \in \mathrm{BMO}_\eta, \tag{1.7}$$

then $\eta \sim (\mu\lambda^{-1})^{1/m}$ almost everywhere.

We organize the rest of the paper as follows. Section 2 is devoted to the proofs of Theorems 1.1–1.3 and Corollaries 1.4 and 1.5. In Section 3, we provide the proof of Theorem 1.7, and Theorem 1.8 is settled in Section 4.

We end this section by making some conventions. We denote $f \lesssim g$, $f \sim g$ if $f \le Cg$ and $f \lesssim g \lesssim f$, respectively. For any ball $B := B(x_0, r) \subset \mathbb{R}^n$, x_0 and r are the center and the radius of B, respectively, and f_B means the mean value of f over B, while χ_B represents the characteristic function of B. For any cube $Q \subset \mathbb{R}^n$, the diameter of Q is denoted by diam Q. $C_c^\infty(\mathbb{R}^n)$ is the space of all smooth functions with compact support.

2. Two-weight boundedness for $I_\alpha^{b,m}$. In this section, we will prove Theorems 1.1–1.3 and Corollaries 1.4–1.5. We begin by recalling some notations, definitions and facts related to sparse families (see [18, 24] for more details). Given a cube $Q \subset \mathbb{R}^n$, let $\mathcal{D}(Q)$ be the set of cubes obtained by repeatedly subdividing Q and its descendants into 2^n congruent subcubes.

Definition 2.1. A collection $\mathcal{D}$ of cubes is called a *dyadic lattice* if it satisfies the following properties:

(1) if $Q \in \mathcal{D}$, then every child of Q is also in $\mathcal{D}$;
(2) for any two cubes $Q_1, Q_2 \in \mathcal{D}$, there is a common ancestor $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$;
(3) for any compact set $K \subset \mathbb{R}^n$, there is a cube $Q \in \mathcal{D}$ such that $K \subset Q$.

Definition 2.2. A subset $\mathcal{S} \subset \mathcal{D}$ is called an *η-sparse family* with $\eta \in (0,1)$ if for every cube $Q \in \mathcal{S}$ there is a measurable subset $E_Q \subset Q$ such that $\eta|Q| \leq |E_Q|$, and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are mutually disjoint.

In [1], Accomazzo et al. proved the following sparse domination result for commutators of fractional integral operators.

Lemma 2.3 ([1]). *Let $0 < \alpha < n$ and $m \in \mathbb{Z}^+$. For every $f \in C_c^\infty(\mathbb{R}^n)$ and $b \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$, there exist a family $\{\mathcal{D}_j\}_{j=1}^{3^n}$ of dyadic lattices and a family $\{\mathcal{S}_j\}_{j=1}^{3^n}$ of sparse families such that $\mathcal{S}_j \subset \mathcal{D}_j$ for each j, and*

$$|I_\alpha^{b,m} f(x)| \lesssim \sum_{j=1}^{3^n} \sum_{Q\in\mathcal{S}_j} \sum_{k=0}^{m} |b(x) - b_Q|^{m-k} |Q|^{\alpha/n} \chi_Q(x) \times \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^k |f(x)|\, dx\right).$$

Based on Lemma 2.3 and adapting some ideas in [9], we can prove the following lemma.

Lemma 2.4. *Let $0 < \alpha < n$, $m \in \mathbb{Z}^+$, $b \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ and let $I_\alpha^{b,m}$ be commutators of fractional integral operators. Then for $f \in C_c^\infty(\mathbb{R}^n)$, there exist 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$, $j = 1, \ldots, 3^n$, such that*

$$|I_\alpha^{b,m} f(x)| \lesssim \sum_{j=1}^{3^n} \big(T_{\mathcal{S}_j,\alpha}^{b,m} f(x) + (T_{\mathcal{S}_j,\alpha}^{b,m})^* f(x)\big),$$

where $T_{\mathcal{S},\alpha}^{b,m}$ and $(T_{\mathcal{S},\alpha}^{b,m})^*$ are defined in Theorem 1.1.

Proof. Fix a sparse family $\mathcal{S}$, and let $Q \in \mathcal{S}$ and $x \in Q$. Then

$$\begin{aligned}
&|Q|^{\alpha/n}\sum_{k=0}^{m}|b(x)-b_Q|^{m-k}\frac{1}{|Q|}\int_Q |b(y)-b_Q|^k|f(y)|\,dy \\
&\le \frac{1}{|Q|}\int_Q \Big(\sum_{k=0}^{m}\max\{|b(x)-b_Q|,|b(y)-b_Q|\}^m\Big)|f(y)|\,dy|Q|^{\alpha/n} \\
&= (m+1)\frac{1}{|Q|}\int_Q \max\{|b(x)-b_Q|^m,|b(y)-b_Q|^m\}|f(y)|\,dy|Q|^{\alpha/n} \\
&\lesssim |b(x)-b_Q|^m\frac{1}{|Q|}\int_Q|f(y)|\,dy|Q|^{\alpha/n}+\frac{1}{|Q|}\int_Q|b(y)-b_Q|^m|f(y)|\,dy|Q|^{\alpha/n}.
\end{aligned}$$

This, together with Lemma 2.3, leads to the desired conclusion. ∎

The proof of Theorem 1.1 reduces to the following two propositions.

PROPOSITION 2.5. *Let $0<\alpha<n, m\in\mathbb{Z}^+$, $b\in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ and let $\mathcal{S}$ be a sparse family. Assume that $1<p\le q<\infty$ and A,B are Young functions that satisfy $\bar{A}\in B_{q'}$, $\bar{B}\in B_{p,q}$. If (μ,ν) is a pair of weights that satisfies*

$$\sup_{Q\in\mathcal{S}}|Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{A,Q}\|(b-b_Q)^m\nu^{-1/p}\|_{B,Q}<\infty,$$

then

$$\|T^{b,m}_{\mathcal{S},\alpha}f\|_{L^q(\mu)}\le C\|f\|_{L^p(\nu)}. \tag{2.1}$$

Conversely, if $T^{b,m}_{\mathcal{S},\alpha}$ satisfies (2.1), then the pair (μ,ν) of weights satisfies

$$\sup_{Q\in\mathcal{S}}|Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{q,Q}\|(b-b_Q)^m\nu^{-1/p}\|_{p',Q}<\infty.$$

Proof. By duality, there exists a nonnegative measurable function $g\in L^{q'}(\mu)$ with $\|g\|_{L^{q'}(\mu)}=1$ such that

$$\begin{aligned}
\|T^{b,m}_{\mathcal{S},\alpha}\|_{L^q(\mu)} &= \int_{\mathbb{R}^n} T^{b,m}_{\mathcal{S},\alpha}f(x)g(x)\mu(x)\,dx \\
&\le \sum_{Q\in\mathcal{S}}|Q|^{\alpha/n+1}\bigg(\frac{1}{|Q|}\int_Q|b(x)-b_Q|^m|f(x)|\,dx\bigg) \\
&\quad\times\bigg(\frac{1}{|Q|}\int_Q|g(x)|\mu(x)\,dx\bigg)\bigg(\frac{1}{|Q|}\int_Q|g(x)|\mu(x)\,dx\bigg).
\end{aligned} \tag{2.2}$$

Let $1/p-1/q=\beta/n$. It was proved in [8] that

$$M_{\beta,\bar{B}}:L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n).$$

From this, by (2.2), the generalized Hölder inequality and our assumptions yield

$$
\begin{aligned}
\|T_{\mathcal{S},\alpha}^{b,m}\|_{L^q(\mu)} &\le \sum_{Q\in\mathcal{S}} |Q|^{1+\alpha/n+1/q-1/p+\beta/n}\|(b-b_Q)^m\nu^{-1/p}\|_{B,Q}\|f\nu^{1/p}\|_{\bar{B},Q} \\
&\quad\times \|\mu^{1/q}\|_{A,Q}\|g\mu^{1/q'}\|_{\bar{A},Q} \\
&\lesssim |E_Q|\,|Q|^{\beta/n}\|f\nu^{1/p}\|_{\bar{B},Q}\|g\mu^{1/q'}\|_{\bar{A},Q} \\
&\le \int_{\mathbb{R}^n} M_{\bar{A}}(g\mu^{1/q'})(x)M_{\beta,\bar{B}}(f\nu^{1/p})(x)\,dx \\
&\le \|M_{\beta,\bar{B}}(f\nu^{1/p})\|_{L^q}\|M_{\bar{A}}(g\mu^{1/q'})\|_{L^{q'}} \lesssim \|f\|_{L^p(\nu)}.
\end{aligned}
$$

Next, we turn to necessity. Fix $Q\in\mathcal{S}$, and let

$$
f=|b-b_Q|^{m(p'-1)}\nu^{-p'/p}\chi_Q.
$$

For $x\in Q$, it is easy to see that

$$
T_{\mathcal{S},\alpha}^{b,m}f(x)\ge |Q|^{\alpha/n-1}\int_Q |b(x)-b_Q|^{mp'}\nu(x)^{-p'/p}\,dx,
$$

which implies that

$$
\begin{aligned}
&\Big(\int_Q T_{\mathcal{S},\alpha}^{b,m}f(x)^q\mu(x)\,dx\Big)^{1/q} \\
&\qquad\ge |Q|^{\alpha/n-1}\int_Q |b(x)-b_Q|^{mp'}\nu(x)^{-p'/p}\,dx\Big(\int_Q \mu(x)\,dx\Big)^{1/q}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Big(\int_Q T_{\mathcal{S},\alpha}^{b,m}f(x)^q\mu(x)\,dx\Big)^{1/q} &\le C\Big(\int_{\mathbb{R}^n} |f(x)|^p\nu(x)\,dx\Big)^{1/p} \\
&= C\Big(\int_Q |b(x)-b_Q|^{mp'}\nu(x)^{-p'/p}\,dx\Big)^{1/p}.
\end{aligned}
$$

Hence, we conclude that

$$
\begin{aligned}
&|Q|^{\alpha/n-1}\int_Q |b(x)-b_Q|^{mp'}\nu(x)^{-p'/p}\,dx\Big(\int_Q \mu(x)\,dx\Big)^{1/q} \\
&\quad\le C\Big(\int_Q |b(x)-b_Q|^{mp'}\nu(x)^{-p'/p}\,dx\Big)^{1/p}.
\end{aligned}
$$

The desired result follows by rearranging the above terms. ∎

Arguing similarly, we can obtain the following proposition; we leave the details to the reader.

PROPOSITION 2.6. *Let $0<\alpha<n, m\in\mathbb{Z}^+$, $b\in L^m_{\rm loc}(\mathbb{R}^n)$ and let $\mathcal{S}$ be a sparse family. Assume that $1<p\le q<\infty$ and C,D are Young functions* *that satisfy* $\bar{C} \in B_{q'}, \bar{D} \in B_{p,q}$. *If* (μ,ν) *is a pair of weights that satisfies*

$$\sup_{Q\in\mathcal{S}} |Q|^{\alpha/n+1/q-1/p}\|(b-b_Q)^m\mu^{1/q}\|_{C,Q}\|\nu^{-1/p}\|_{D,Q} < \infty,$$

then

$$\|(T^{b,m}_{\mathcal{S},\alpha})^* f\|_{L^q(\mu)} \le C\|f\|_{L^p(\nu)}. \tag{2.3}$$

Conversely, if $(T^{b,m}_{\mathcal{S},\alpha})^*$ *satisfies* (2.3), *then the pair* (μ,ν) *of weights satisfies*

$$\sup_{Q\in\mathcal{S}} |Q|^{\alpha/n+1/q-1/p}\|(b-b_Q)^m\mu^{1/q}\|_{q,Q}\|\nu^{-1/p}\|_{p',Q} < \infty.$$

Proofs of Theorems 1.1 and 1.2. Theorem 1.1 follows from Propositions 2.5 and 2.6, and Theorem 1.2 follows from Lemma 2.4 and Theorem 1.1. ∎

Next, we prove Theorem 1.3. We first recall the following lemma.

LEMMA 2.7 ([6, p. 99]). *Let* A, B *be continuous and strictly increasing functions on* $[0,\infty)$ *and* C *be a Young function that satisfies* $A^{-1}(t)B^{-1}(t) \lesssim C^{-1}(t)$ *for* t *large. Then*

$$\|fg\|_{C,Q} \lesssim \|f\|_{A,Q}\|g\|_{B,Q}.$$

Proof of Theorem 1.3. Denote $\Phi_m(t) = \Phi(t^{1/m})$. Since B, X, Φ satisfy

$$\Phi^{-1}(t)^m X^{-1}(t) \lesssim B^{-1}(t)$$

for t large, by Lemma 2.7 we have

$$\begin{aligned}\|(b-b_Q)^m\nu^{-1/p}\|_{B,Q} &\lesssim \|(b-b_Q)^m\|_{\Phi_m,Q}\|\nu^{-1/p}\|_{X,Q}\\ &= \|(b-b_Q)\|^m_{\Phi,Q}\|\nu^{-1/p}\|_{X,Q}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_Q |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{A,Q}\|(b-b_Q)^m\nu^{-1/p}\|_{B,Q}\\ &\qquad\lesssim \|b\|^m_{\mathrm{Osc}(\Phi)}\sup_Q |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{A,Q}\|\nu^{-1/p}\|_{X,Q} < \infty.\end{aligned}$$

By Proposition 2.5, we get

$$\|T^{b,m}_{\mathcal{S},\alpha} f\|_{L^q(\mu)} \le C\|f\|_{L^p(\nu)}.$$

Similarly, we have

$$\begin{aligned}&\sup_Q |Q|^{\alpha/n+1/q-1/p}\|(b-b_Q)^m\mu^{1/q}\|_{C,Q}\|\nu^{-1/p}\|_{D,Q}\\ &\qquad\lesssim \|b\|^m_{\mathrm{Osc}(\Phi)}\sup_Q |Q|^{\alpha/n+1/q-1/p}\|\mu^{1/q}\|_{Y,Q}\|\nu^{-1/p}\|_{D,Q} < \infty.\end{aligned}$$

This, together with Proposition 2.6, yields

$$\|(T^{b,m}_{\mathcal{S},\alpha})^* f\|_{L^q(\mu)} \le C\|f\|_{L^p(\nu)}.$$

Summing up the estimates above with Lemma 2.4, we obtain the conclusion of Theorem 1.3. ■

To prove Corollary 1.4, we need to recall the following fact. For $\varphi(t) = t^p(\log(e+t))^q$ with $p>1$ and $q \in \mathbb{R}$, Cruz-Uribe et al. [10] showed that

$$\varphi^{-1}(t) \sim \frac{t^{1/p}}{(\log(e+t))^{q/p}}, \qquad \bar{\varphi}(t) = \frac{t^{p'}}{(\log(e+t))^{p'q/p}}. \tag{2.4}$$

Proof of Corollary 1.4. We need only choose some Young functions that satisfy the conditions of Theorem 1.3. For some $\delta > 0$, choose

$$X(t) = t^{p'}[\log(e+t)]^{(m+1)p'-1+\delta}, \qquad Y(t) = t^q[\log(e+t)]^{(m+1)q-1+\delta},$$
$$B(t) = t^{p'}[\log(e+t)]^{p'-1+\delta}, \qquad C(t) = t^q[\log(e+t)]^{q-1+\delta}, \qquad \Phi(t) = e^t - 1.$$

It is not hard to check that

$$\bar{B}(t) \sim \frac{t^p}{[\log(e+t)]^{1+p\delta/p'}} \in B_p \subset B_{p,q}, \qquad \bar{C}(t) \sim \frac{t^{q'}}{[\log(e+t)]^{1+q'\delta/q}} \in B_{q'}$$

and $\Phi^{-1}(t) = \log(e+t)$. By (2.4), we have

$$X^{-1}(t) \sim \frac{t^{1/p'}}{[\log(e+t)]^{m+1/p+\delta/p'}}, \qquad Y^{-1}(t) \sim \frac{t^{1/q}}{[\log(e+t)]^{m+1/q'+\delta/q}},$$
$$B^{-1}(t) \sim \frac{t^{1/p'}}{[\log(e+t)]^{1/p+\delta/p'}}, \qquad C^{-1}(t) \sim \frac{t^{1/q}}{[\log(e+t)]^{1/q'+\delta/q}}.$$

Then

$$X^{-1}(t)\Phi^{-1}(t)^m \sim \frac{t^{1/p'}}{[\log(e+t)]^{m+1/p+\delta/p'}}[\log(e+t)]^m \sim B^{-1}(t),$$
$$Y^{-1}(t)\Phi^{-1}(t)^m \sim \frac{t^{1/q}}{[\log(e+t)]^{m+1/q'+\delta/q}}[\log(e+t)]^m \sim C^{-1}(t).$$

Finally, by the John–Nirenberg inequality and $t \lesssim \Phi(t)$, we get $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sim \|b\|_{\mathrm{Osc}(\Phi)}$. Thus, Theorem 1.3 implies Corollary 1.4. ■

Proof of Corollary 1.5. Choose

$$A(t) = t^q[\log(e+t)]^{q-1+\delta}, \qquad D(t) = t^{p'}[\log(e+t)]^{p'-1+\delta}$$

in Corollary 1.4. Then

$$\bar{A}(t) \sim \frac{t^{q'}}{[\log(e+t)]^{1+q'\delta/q}} \in B_{q'}, \qquad \bar{D}(t) \sim \frac{t^p}{[\log(e+t)]^{1+p\delta/p'}} \in B_p \subset B_{p,q}.$$

Hence, Corollary 1.5 directly follows from Corollary 1.4. ■

3. Necessity of two-weight inequalities for $I_\alpha^{b,m}$. In this section, we give the proof of Theorem 1.7. We need the following two lemmas.

LEMMA 3.1. *Let* $K_\alpha(x,y)=\frac{1}{|x-y|^{n-\alpha}}$. *Then for each* $A\geq 4$ *and each ball* $B:=B(y_0,r)$, *there exists a disjoint ball* $\tilde{B}:=B(x_0,r)$ *with* $\operatorname{dist}(B,\tilde{B})\sim Ar$ *such that* $|K_\alpha(x_0,y_0)|=\frac{1}{A^{n-\alpha}r^{n-\alpha}}$, *and for any* $y\in B$ *and* $x\in\tilde{B}$,

$$|K_\alpha(x,y)-K_\alpha(x_0,y_0)|\lesssim\frac{\epsilon_A}{A^{n-\alpha}r^{n-\alpha}},$$

where $\epsilon_A\to 0$ *as* $A\to\infty$.

Proof. Fix a ball $B=B(y_0,r)$ and $A\geq 4$, and take $x_0=y_0+Ar\theta_0$, where $\theta_0\in\mathbb{S}^{n-1}$. Let $\tilde{B}:=B(x_0,r)$. It is easy to see that $\operatorname{dist}(B,\tilde{B})\sim Ar$ and $K_\alpha(x_0,y_0)=\frac{1}{|x_0-y_0|^{n-\alpha}}=\frac{1}{A^{n-\alpha}r^{n-\alpha}}$. For any $y\in B$ and $x\in\tilde{B}$, by the mean value theorem, we have

$$\begin{aligned}|K_\alpha(x,y)-K_\alpha(x_0,y_0)|&\leq|K_\alpha(x,y)-K_\alpha(x_0,y)|+|K_\alpha(x_0,y)-K_\alpha(x_0,y_0)|\\&\lesssim\frac{|x-x_0|}{|x_0-y|^{n-\alpha+1}}\leq\frac{1/A}{(Ar)^{n-\alpha}}=:\frac{\epsilon_A}{(Ar)^{n-\alpha}}.\ \blacksquare\end{aligned}$$

LEMMA 3.2 ([19]). *Assume that* $f\in\mathrm{BMO}(\mathbb{R}^n)$, *and let* Q *be a cube such that* $f_Q=0$. *Then there exists a function* φ *such that* $\varphi=f$ *on* Q, $\varphi=0$ *on* $\mathbb{R}^n\backslash 2Q$ *and* $\|\varphi\|_{\mathrm{BMO}(\mathbb{R}^n)}\lesssim\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

Proof of Theorem 1.7. For any cube $Q\subset\mathbb{R}^n$, we define

$$g(x)=\log^+\left(\frac{M(\nu^{1-p'}\chi_Q)(x)}{(\nu^{1-p'})_Q}\right).$$

It is well known that $g\in\mathrm{BMO}(\mathbb{R}^n)$. Moreover, the Kolmogorov inequality yields

$$\int_Q(M(f\chi_Q))^\delta\lesssim\left(\frac{1}{|Q|}\int_Q|f|\right)^\delta|Q|,\qquad 0<\delta<1.$$

We then have $g_Q\lesssim 1$. According to Lemma 3.2, there is a function φ satisfying $\varphi=g-g_Q$ on Q, $\varphi=0$ outside $2Q$ and $\|\varphi\|_{\mathrm{BMO}(\mathbb{R}^n)}\lesssim 1$. Choose a ball B such that the center of B is the same as that of the cube Q and $r=\operatorname{diam}Q$. Then by Lemma 3.1, there is a ball $\tilde{B}$ such that the center of $\tilde{B}$ is the same as that of the cube Q and $\operatorname{dist}(B,\tilde{B})\sim Ar$, where $A\geq 4$ will be determined later.

Now, we return to the proof of our theorem. By duality, we find that the condition

$$\|I_\alpha^{b,m}f\|_{L^{q,\infty}(\mu)}\lesssim\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^m\|f\|_{L^p(\nu)}$$

is equivalent to

$$\|(I_\alpha^{b,m})^*f\|_{L^{p'}(\nu^{1-p'})}\lesssim\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^m\|f/\mu\|_{L^{q',1}(\mu)}.\tag{3.1}$$

One can check that $(I_\alpha^{b,m})^*=(-1)^mI_\alpha^{b,m}$, hence, we can still deal with (3.1) by considering $I_\alpha^{b,m}$. Let $b=\varphi$. Then for $x\in B$ and a nonnegative function f,

$$I_\alpha^{b,m}(f\chi_{\tilde B})(x)=\int_{\tilde B}(b(x)-b(y))^m\frac{f(y)}{|x-y|^{n-\alpha}}\,dy=\varphi(x)^m\int_{\tilde B}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

By (3.1), we immediately get

$$\Big(\int_B\Big(\int_{\tilde B}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy\Big)^{p'}|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\lesssim\|f\chi_{\tilde B}/\mu\|_{L^{q',1}(\mu)}.$$

This, combined with Lemma 3.1, yields

$$\begin{aligned}
\frac{1}{A^{n-\alpha}}&\Big(\int_B|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}f_{\tilde B}\\
&=\frac{r^{n-\alpha}}{(Ar)^{n-\alpha}}\Big(\int_B|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}f_{\tilde B}\\
&=r^{-\alpha}\Big(\int_B\Big(\int_{\tilde B}\frac{f(y)}{|x_0-y_0|^{n-\alpha}}\,dy\Big)^{p'}|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\\
&\le r^{-\alpha}\Big(\int_B\Big(\int_{\tilde B}\Big|\frac{1}{|x_0-y_0|^{n-\alpha}}-\frac{1}{|x-y|^{n-\alpha}}\Big|f(y)\,dy\Big)^{p'}\\
&\qquad\qquad\times|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\\
&\quad+r^{-\alpha}\Big(\int_B\Big(\int_{\tilde B}\frac{1}{|x-y|^{n-\alpha}}f(y)\,dy\Big)^{p'}|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\\
&\lesssim\frac{\epsilon_A}{A^{n-\alpha}}\Big(\int_B|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}f_{\tilde B}+r^{-\alpha}\|f\chi_{\tilde B}/\mu\|_{L^{q',1}(\mu)}.
\end{aligned}$$

Choosing A large enough, we have

$$\Big(\int_B|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}f_{\tilde B}\lesssim r^{-\alpha}\|f\chi_{\tilde B}/\mu\|_{L^{q',1}(\mu)}.$$

Taking $f=\mu$ and using the fact that $\|\chi_{\tilde B}\|_{L^{q',1}(\mu)}\sim(\int_{\tilde B}\mu)^{1/q'}$, we obtain

$$r^\alpha|\tilde B|^{-1}\Big(\int_B|\varphi(x)|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\Big(\int_{\tilde B}\mu(x)\,dx\Big)^{1/q}\lesssim 1. \tag{3.2}$$

Similarly, let $b=\chi_B$. Following the arguments for (3.2), we get

$$r^\alpha|\tilde B|^{-1}\Big(\int_B\nu(x)^{1-p'}\,dx\Big)^{1/p'}\Big(\int_{\tilde B}\mu(x)\,dx\Big)^{1/q}\lesssim 1. \tag{3.3}$$

Observe that $|\tilde{B}| \sim |Q|$ and $Q \subset \theta\tilde{B}$, where θ depends only on A and n. Combining these facts and the doubling property of μ, we can replace (3.2) and (3.3) by

$$r^\alpha |Q|^{-1}\Big(\int_Q |g(x)-g_Q|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\Big(\int_Q \mu(x)\,dx\Big)^{1/q} \lesssim 1$$

and

$$r^\alpha |Q|^{-1}\Big(\int_Q \nu(x)^{1-p'}\,dx\Big)^{1/p'}\Big(\int_Q \mu(x)\,dx\Big)^{1/q} \lesssim 1,$$

respectively. Keeping in mind that $g_Q \lesssim 1$, we finally have

$$\begin{aligned} r^\alpha |Q|^{-1}&\Big(\int_Q g(x)^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\Big(\int_Q \mu(x)\,dx\Big)^{1/q} \\ &\lesssim r^\alpha |Q|^{-1}\Big(\int_Q |g(x)-g_Q|^{mp'}\nu(x)^{1-p'}\,dx\Big)^{1/p'}\Big(\int_Q \mu(x)\,dx\Big)^{1/q} \\ &\quad + r^\alpha |Q|^{-1}\Big(\int_Q \nu(x)^{1-p'}\,dx\Big)^{1/p'}\Big(\int_Q \mu(x)\,dx\Big)^{1/q} \lesssim 1, \end{aligned}$$

which implies that

$$\begin{aligned} \sup_Q |Q|^{\alpha/n+1/q-1/p}&\Big(\frac{1}{|Q|}\int_Q \mu(x)\,dx\Big)^{1/q} \\ &\times \Big(\frac{1}{|Q|}\int_Q \nu(x)^{1-p'}\Big[\log\Big(\frac{v(x)^{1-p'}}{(v(x)^{1-p'})_Q}+e\Big)\Big]^{mp'}dx\Big)^{1/p'} < \infty. \end{aligned}$$

Therefore, using the following fact proved in [30]:

$$\|f\|_{L(\log L)^\alpha,Q} \sim \frac{1}{|Q|}\int_Q |f(x)|[\log(|f(x)|/|f|_Q+e)]^\alpha\,dx,$$

we get the desired result. ∎

4. Converse to Bloom type estimate for $I_\alpha^{b,m}$. This section is concerned with the proof of Theorem 1.8. First, we recall or establish some lemmas, which are key tools for our arguments.

LEMMA 4.1 ([19]). *Let η_1, η_2 be weights such that $\eta_1/\eta_2 \notin L^\infty$. Then there exists $b \in \mathrm{BMO}_{\eta_1}\setminus\mathrm{BMO}_{\eta_2}$.*

LEMMA 4.2. *Let λ, μ be arbitrary weights satisfying* (1.6) *and p, q, m, α be given in Theorem* 1.8. *Then for each ball $B := B(y_0, r)$, there exists a disjoint ball $\tilde{B} := B(x_0, r)$ with $\mathrm{dist}(B, \tilde{B}) \sim Ar$ such that for any nonnegative* *measurable function f,*

$$\Big(\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\,dx\Big)^{1/q} f_B \lesssim r^{-\alpha}\Big(\int_B f(x)^p\mu(x)^p\,dx\Big)^{1/p}.$$

Proof. Let $\tilde B$ be given in Lemma 3.1 and $b=\eta\chi_{\tilde B}$. Then for $x\in\tilde B$,

$$I_\alpha^{b,m}(f\chi_B)(x)=\int_B (b(x)-b(y))^m\frac{f(y)}{|x-y|^{n-\alpha}}\,dy=\eta(x)^m\int_B\frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

By (1.6), we have

$$\bigg(\int_{\tilde B}\Big(\int_B\frac{f(y)}{|x-y|^{n-\alpha}}\,dy\Big)^q\eta(x)^{mq}\lambda(x)^q\,dx\bigg)^{1/q}\lesssim\Big(\int_B f(x)^p\mu(x)^p\,dx\Big)^{1/p}.$$

From this and Lemma 3.1, we deduce that

$$\begin{aligned}
&\frac{1}{A^{n-\alpha}}\Big(\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\,dx\Big)^{1/q} f_B\\
&=\frac{r^{n-\alpha}}{(Ar)^{n-\alpha}}\bigg[\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\Big(\frac{1}{|B|}\int_B f(y)\,dy\Big)^q dx\bigg]^{1/q}\\
&=r^{-\alpha}\bigg[\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\Big(\int_B\frac{f(y)}{|x_0-y_0|^{n-\alpha}}\,dy\Big)^q dx\bigg]^{1/q}\\
&\le r^{-\alpha}\bigg[\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\Big(\int_B\Big|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y_0|^{n-\alpha}}\Big|f(y)\,dy\Big)^q dx\bigg]^{1/q}\\
&\quad+r^{-\alpha}\bigg[\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\Big(\int_B\frac{1}{|x-y|^{n-\alpha}}f(y)\,dy\Big)^q dx\bigg]^{1/q}\\
&\lesssim\frac{\epsilon_A}{A^{n-\alpha}}\Big(\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\,dx\Big)^{1/q} f_B+r^{-\alpha}\Big(\int_B f(x)^p\mu(x)^p\,dx\Big)^{1/p}.
\end{aligned}$$

Then the desired result directly follows by letting $A\to\infty$. ∎

LEMMA 4.3. *Let λ,μ be arbitrary weights satisfying* (1.6) *and p,q,m,α be given in Theorem* 1.8. *Then*

$$\lambda(x)\eta(x)^m\lesssim\mu(x).$$

Proof. Let $f=1$ in Lemma 4.2. Keep in mind that $1/p-1/q=\alpha/n$; then

$$\bigg(\frac{1}{|\tilde B|}\int_{\tilde B}\eta(x)^{mq}\lambda(x)^q\,dx\bigg)^{1/q}\lesssim\bigg(\frac{1}{|B|}\int_B\mu(x)^p\,dx\bigg)^{1/p}.$$

By the Lebesgue differentiation theorem, we get the desired result. ∎

Proof of Theorem 1.8. By Lemma 4.3, it suffices to prove that

$$\mu \lesssim \lambda\eta^m \tag{4.1}$$

almost everywhere. Suppose that (4.1) is not true. Denote $\tilde{\eta} = (\mu/\lambda)^{1/m}$. Then $\tilde{\eta}/\eta \notin L^\infty$. Note that when λ, $\mu \in A_{p,q}$, Accomazzo et al. [1] proved that for $b \in \mathrm{BMO}_{\tilde{\eta}}$,

$$\|I_\alpha^{b,m} f\|_{L^q(\lambda^q)} \lesssim \|f\|_{L^p(\mu^p)}.$$

This, together with Lemma 4.1, implies that $b \notin \mathrm{BMO}_\eta$, which contradicts (1.7) and completes the proof of Theorem 1.8. ∎

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Yongming Wen (corresponding author)
School of Mathematics and Statistics
Minnan Normal University
363000 Zhangzhou, China
E-mail: wenyongmingxmu@163.com

Huoxiong Wu
School of Mathematical Sciences
Xiamen University
361005 Xiamen, China
E-mail: huoxwu@xmu.edu.cn