ACTA ARITHMETICA
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Another Waring–Goldbach problem

by

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1. We owe to Hooley [4] the elegant theorem that all sufficiently large natural numbers are the sum of two squares of integers and a cube, a biquadrate, a fifth power, a sixth power and a seventh power, all of natural numbers. Thus, for large n, the Diophantine equation

$$x_1^2 + x_2^2 + x_3^3 + x_4^4 + x_5^5 + x_6^6 + x_7^7 = n \tag{1}$$

has solutions in non-negative integers. Note that the sum of the reciprocals of the exponents in (1) exceeds 2, but that this is no longer the case when the seventh power is removed. Hence the representation problem is on the edge of the familiar convexity barrier, a fact that certainly adds to the attraction of Hooley's result.

More recently, Cai and Mu [2] attempted to solve (1) in prime numbers, but with limited success. They showed that for odd n equation (1) has solutions with at most six prime factors in x_1, and all other variables prime. In this note we establish the Waring–Goldbach analogue of Hooley's result.

Theorem. *Let n be odd and sufficiently large. Then equation* (1) *has a solution with all x_j prime.*

Hooley treats (1) as a binary additive problem, with the sum of two squares in the role of one summand. The number of representations of a given number as the sum of two squares is written in terms of the non-trivial Dirichlet character modulo 4, and one then encounters expressions akin to familiar divisor sums. In this way, Hooley obtains an asymptotic formula for the number of solutions of (1) with $x_1, x_2 \in \mathbb{Z}$ and $x_j \in \mathbb{N}$ for $3 \le j \le 7$.

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The argument adapts to the situation where $x_3, x_4, \dots, x_7$ are restricted to be primes, but constraints on x_1, x_2 destroy the multiplicative nature of the problem *ab initio*.

This author [1] gave a new proof of Hooley's result via the circle method. This absorbs constraints on the x_j more easily, and indeed the work of Cai and Mu follows this line of thought. It turns out that one can place a sieving condition on x_1 and force all other variables to be prime. We take a different route and first apply the iterative method of Davenport in an unconventional manner (see Section 3). A cascade of applications of Hölder's inequality produces mean values that are almost optimally understood, or nearly so, with a fraction of the seventh power left for estimation by Vinogradov's method. The latter estimation makes use of the perfect control that we now have on Vinogradov's mean value, and only just compensates losses implied by the imperfect outcome of Davenport's method. Some mild pruning then completes the proof of the theorem.

Notation. Throughout, the letter p denotes a prime, $\phi(q)$ is Euler's totient, and $e(\alpha)$ abbreviates $\exp(2\pi i\alpha)$. Whenever ε occurs in a statement it is asserted that the statement is true for any positive real number ε. Constants implicit in Vinogradov's and Landau's familiar symbols will depend on the value assigned to ε. Note that this convention allows us to deduce from $A \ll n^\varepsilon$, $B \ll n^\varepsilon$ that $AB \ll n^\varepsilon$, for example.

2. Throughout, let n be a large positive integer, and write

$$P_k = n^{1/k}, \quad L = \log n.$$

For $k = 2, 3, 4, 6, 7$ let

$$f_k(\alpha) = \sum_{\frac{1}{2}P_k < p \le P_k} e(\alpha p^k) \log p,$$

but choose $\theta = 230/259$ to define

$$f_5(\alpha) = \sum_{p \le P_5^\theta} e(\alpha p^5) \log p.$$

The reason for the diminished range for p in $f_5(\alpha)$ will be revealed in Section 3. In the interest of brevity, we often write f_k for $f_k(\alpha)$, and put

$$F(\alpha) = f_2(\alpha)^2 f_3(\alpha) f_4(\alpha) f_5(\alpha) f_6(\alpha) f_7(\alpha).$$

By orthogonality, the integral

$$\nu(n) = \int_0^1 F(\alpha) e(-\alpha n)\, d\alpha \tag{2}$$

counts the solutions of (1) in primes constrained to certain intervals, and with weight $\log x_1 \cdots \log x_7$. With

$$\Theta = \frac{1}{3} + \frac{1}{4} + \frac{\theta}{5} + \frac{1}{6} + \frac{1}{7}$$

the goal is to demonstrate that the lower bound

$$\nu(n) \gg n^{\Theta} \tag{3}$$

holds for odd n. This is a quantitative form of the theorem.

Fix a real number $B \geq 1$. Let

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1] : |\alpha - (a/q)| \leq L^B/n\},$$

and let $\mathfrak{M}$ denote the union of the $\mathfrak{M}(q,a)$ with $0 \leq a \leq q$, $(a,q) = 1$ and $1 \leq q \leq L^B$. The contribution from $\mathfrak{M}$ to the integral (3) is readily controlled.

Lemma. *For large odd n one has*

$$\int_{\mathfrak{M}} F(\alpha)e(-\alpha n)\,\mathrm{d}\alpha \gg n^{\Theta}.$$

Experienced workers in this area will have no difficulty finding a proof of this lemma by themselves. For completeness we offer a guided tour along a well trodden path. For $2 \leq k \leq 7$ we write

$$S_k(q,a) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\Big(\frac{ax^k}{q}\Big), \quad v_k(\beta) = \frac{1}{k} \sum_{2^{-k}n < m \leq n} m^{1/k-1} e(\beta m).$$

Now suppose that $\alpha \in \mathfrak{M}(q,a)$ with $q \leq L^B$ and $(a,q) = 1$. Then, for $k \neq 5$, by [5, Lemma 6] and partial summation,

$$f_k(\alpha) = \phi(q)^{-1} S_k(q,a) v_k(\alpha - a/q) + O(P_k L^{-4B}).$$

The treatment of f_5 is rather more direct. For $p \leq P_5^{\theta}$ and $|\beta| \leq L^B/n$, Taylor expansion yields $e(\beta p^5) = 1 + O(L^B n^{-1} P_5^{5\theta})$. We take $\beta = \alpha - a/q$, note that $n^{-1}P_5^{5\theta} = n^{\theta-1}$ and conclude from the definition of f_5 that

$$f_5(\alpha) = f_5(a/q) + O(P_5^{\theta} L^B n^{\theta-1}).$$

Now we apply [5, Lemma 6] to $f_5(a/q)$, again in conjunction with partial summation. This yields

$$f_5(\alpha) = \phi(q)^{-1} S_5(q,a) P_5^{\theta} + O(P_5^{\theta} L^{-4B}).$$

We multiply these approximations together to approximate F. This brings in the products

$$U(q,a) = S_2(q,a)^2 S_3(q,a) S_4(q,a) S_5(q,a) S_6(q,a) S_7(q,a),$$
$$w(\beta) = v_2(\beta)^2 v_3(\beta) v_4(\beta) v_6(\beta) v_7(\beta),$$

and yields

$$F(\alpha) = P_5^\theta \phi(q)^{-7} U(q,a) w(\alpha - a/q) + O(n^{1+\Theta} L^{-4B}).$$

We integrate this over $\mathfrak{M}$, a set of measure $O(L^{3B}/n)$. This shows that

$$(4) \qquad \int_{\mathfrak{M}} F(\alpha) e(-\alpha n)\,\mathrm{d}\alpha = P_5^\theta \sum_{q \le L^B} \frac{A_n(q)}{\phi(q)^7} \int_{-L^B/n}^{L^B/n} w(\beta) e(-\beta n)\,\mathrm{d}\beta + O(n^\Theta L^{-B}),$$

where

$$A_n(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} U(q,a) e(-an/q).$$

Note that the main term on the right hand side of (4) factorises. We handle the sum over q first. Here [5, Lemma 5] yields $U(q,a) \ll q^{7/2+\varepsilon}$ for coprime a,q, and hence $A_n(q) \ll q^{9/2+\varepsilon}$ holds uniformly in n. It follows that the series

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \phi(q)^{-7} A_n(q)$$

converges absolutely, while uniformly in n we also have

$$(5) \qquad \sum_{q \le L^B} \phi(q)^{-7} A_n(q) = \mathfrak{S}(n) + O(L^{-B}).$$

The argument underpinning the proof of [9, Lemma 2.11] reveals that $A_n(q)$ is multiplicative in q, and [5, Lemma 4] implies that $A_n(q) = 0$ unless q is square-free. This yields

$$(6) \qquad \mathfrak{S}(n) = \prod_p \big(1 + (p-1)^{-7} A_n(p)\big).$$

Let $M_n(p)$ denote the number of solutions of (1) in the finite field $\mathbb{F}_p$, with all variables non-zero. By orthogonality modulo p we see that

$$1 + (p-1)^{-7} A_n(p) = p(p-1)^{-7} M_n(p),$$

and a straightforward application of the Cauchy–Davenport theorem (see [9, Lemma 2.14]) yields $M_n(p) \ge 1$ except when $p = 2$ and $2 \mid n$. In particular, the factors in (6) are positive for all odd n, and the uniform convergence relative to n gives us a number p_0 with the property that for odd n one has

$$(7) \qquad \mathfrak{S}(n) \ge \frac{1}{2} \prod_{p \le p_0} \big(1 + (p-1)^{-7} A_n(p)\big) \ge \frac{1}{2} \prod_{p \le p_0} p^{-6}.$$

The integral on the right hand side of (4) is easier to handle. By [9, Lemma 6.2] one has $w(\beta) \ll n^{53/28}(1+n|\beta|)^{-2}$, and then

$$\int_{-L^B/n}^{L^B/n} w(\beta)e(-\beta n)\,\mathrm{d}\beta = \int_{-1/2}^{1/2} w(\beta)e(-\beta n)\,\mathrm{d}\beta + O(n^{25/28}L^{-B}).$$

By orthogonality, the integral on the right hand side is

$$\frac{1}{4^2\cdot 3\cdot 6\cdot 7}\sum (m_1m_2)^{-1/2}m_3^{-2/3}m_4^{-3/4}m_6^{-5/6}m_7^{-6/7} \tag{8}$$

where the sum extends over $m_1, m_2, m_3, m_4, m_6, m_7$ subject to

$$m_1+m_2+m_3+m_4+m_6+m_7 = n, \tag{9}$$

$$2^{-2}n < m_1, m_2 \le n, \quad 2^{-k}n < m_k \le n \quad (3\le k\le 7,\ k\ne 5). \tag{10}$$

If we choose $m_1, m_2 \le \frac{5}{16}n$, $m_3 \le \frac{5}{32}n$, $m_4 \le \frac{5}{64}n$, $m_6 \le \frac{5}{128}n$ in accord with (10) and solve (9) for m_7, then this m_7 satisfies (10). Consequently, the sum in (8) is bounded below by $\gg n^{25/28}$, and we conclude that

$$\int_{-L^B/n}^{L^B/n} w(\beta)e(-\beta n)\,\mathrm{d}\beta \gg n^{25/28}.$$

We now combine this lower bound with (4), (5) and (7). The lemma follows.

3. In view of (2) and the lemma in Section 2, the remaining task is to show that the contribution to (2) that arises from the complement of $\mathfrak{M}$ is negligible. In preparation for this enterprise, we first provide an auxiliary estimate for the number T of solutions of the equation

$$y_1^7 - y_2^7 = z_1^5 + z_2^5 - z_3^5 - z_4^5$$

in integers constrained to the intervals

$$\tfrac12 P_7 < y_i \le P_7, \quad 1 \le z_j \le P_5^{\theta} \quad (1\le i\le 2,\ 1\le j\le 4). \tag{11}$$

We apply Davenport's method [3] in the refined form of Vaughan [8, Lemma 4]. In the latter lemma we take $k=7$, $j=3$, $\lambda=\theta$, $P=\frac12 P_7$ and $R(m)$ as the number of choices for z_1, $z_2 \in [1, P_5^{\theta}]$ with $z_1^5+z_2^5=m$. Then, the quantity T in Vaughan's Lemma 4 coincides with our T. Now note that $z_1^5+z_2^5=m$ implies $z_1+z_2 \mid m$. An elementary divisor counting argument therefore shows that $R(m) \ll m^{\varepsilon}$. Hence we have

$$\sum_m R(m) \le P_5^{2\theta}, \quad \sum_m R(m)^2 \ll P_5^{2\theta+\varepsilon}.$$

With this input, [8, Lemma 4] delivers the estimate

$$T \ll P_7P_5^{2\theta+\varepsilon} + P_7^{\frac12+\frac78(7\theta-6)+\varepsilon}(P_5^{2\theta})^{\frac98} \ll P_7^{1+\varepsilon}P_5^{2\theta}.$$

Note here that θ is chosen to balance the two summands in the initial estimate for T. Our application of Davenport's method is somewhat unorthodox. We "add a seventh power to the sum of two fifth powers" although "adding a fifth power to the sum of a fifth and a seventh power" would be more efficient. However, to put the latter approach into practice, one needs two fifth powers located in different ranges. This is in conflict with (1) where only one fifth power occurs.

Next, we consider the mean value

$$J_1 = \int_0^1 |f_2 f_7|^2 |f_5|^4 \, \mathrm{d}\alpha$$

and show that

$$J_1 \ll P_2^{1+\varepsilon} P_7 P_5^{2\theta}. \tag{12}$$

For a proof of (12), note that by orthogonality one has $J_1 \le L^8 J$ where J is the number of solutions of the equation

$$x_1^2 - x_2^2 = y_1^7 - y_2^7 + z_1^5 + z_2^5 - z_3^5 - z_4^5 \tag{13}$$

in integers satisfying (11) and $\frac{1}{2}P_2 < x_1, x_2 \le P_2$. It is immediate that the number of solutions with $x_1 = x_2$ does not exceed $P_2 T$, which is acceptable. To count the remaining solutions, we choose y_i, z_j such that the right hand side of (13) is non-zero. The number of such choices is $O(P_7^2 P_5^{4\theta})$, and for each such choice, there are $O(P_2^\varepsilon)$ possibilities for x_1, x_2 because $x_1 - x_2$ and $x_1 + x_2$ divide the right hand side of (13). This yields

$$J \ll P_2 P_7^{1+\varepsilon} P_5^{2\theta} + P_2^\varepsilon P_7^2 P_5^{4\theta},$$

and (12) follows.

We also require the means

$$J_2 = \int_0^1 |f_2 f_3 f_6|^2 \, \mathrm{d}\alpha, \quad J_3 = \int_0^1 |f_2|^2 |f_4|^4 \, \mathrm{d}\alpha.$$

Both integrals allow an interpretation as the number of solutions of an underlying Diophantine equation. Similar to the discussion of J_1, this removes the logarithmic weights as well as the restriction to primes. Working in reverse, one can use the counterparts of J_2 and J_3 where the f_j are replaced by ordinary Weyl sums. The latter have been estimated in [1, Lemma 1], and in this way we obtain the bounds

$$J_2 + J_3 \ll n^{1+\varepsilon}. \tag{14}$$

Alternatively, the reader will have no difficulty establishing (14) directly, following the pattern of our discussion of J_1.

We are now well equipped to dispose of the contribution to (2) that arises from the set

$$\mathscr{E} = \{\alpha \in [0,1] : |f_7(\alpha)| \le P_7^{147/148+15\tau}\}; \qquad (15)$$

here and in the remainder of this note, we put $\tau = 2^{-40}$.

Lemma. *One has*

$$\int_{\mathscr{E}} |F(\alpha)| \, \mathrm{d}\alpha \ll n^{\Theta-\tau}.$$

For a proof, apply Hölder's inequality to infer that

$$\int_{\mathscr{E}} |F(\alpha)| \, \mathrm{d}\alpha \le J_1^{1/4} J_2^{1/2} J_3^{1/4} \sup_{\alpha\in\mathscr{E}} |f_7(\alpha)|^{1/2}.$$

By (12), (14), (15) and a numerical calculation, the lemma follows.

In this argument, the integrals J_2 and J_3 lose a marginal factor n^ε against a perfect estimate whilst the exponent $147/148$ in the definition of $\mathscr{E}$ is designed to compensate an intrinsic imperfection in the estimation of J_1.

4. Finally, we explore the consequences for a number $\alpha \in [0,1]$ to be in the complement of $\mathscr{E}$. This is achieved with the aid of Kumchev's estimates for exponential sums over primes ([1]) [6], empowered by recent advances with Vinogradov's mean value theorem. A convenient reference for this is([1]) [7, Lemma 2.2] where we take $k = 7$ and $X = \frac{1}{2}P_7$. With crude estimates, we then see that $[0,1] \setminus \mathscr{E} \subset \mathfrak{K}$ where $\mathfrak{K}$ is the union of the disjoint intervals

$$\mathfrak{K}(q,a) = \{\alpha \in [0,1] : |q\alpha - a| \le n^{-499/500}\}$$

with $0 \le a \le q$, $(q,a) = 1$ and $1 \le q \le n^{1/500}$.

Define a function $\Upsilon : \mathfrak{K} \to [0,1]$ by

$$\Upsilon(\alpha) = (q + |q\alpha - a|)^{-1} \quad (\alpha \in \mathfrak{K}(q,a)).$$

Then, by ([1]) [6, Theorem 2], there is a number $C \ge 1$ with the property that for $\alpha \in \mathfrak{K}$ and $2 \le k \le 7$, $k \ne 5$, one has

$$f_k(\alpha) \ll P_k L^C \Upsilon(\alpha)^{1/2-\varepsilon}.$$

Hence, using a trivial bound for f_5, we infer

$$F(\alpha) \ll n^{1+\Theta} L^{6C} \Upsilon(\alpha)^{5/2}.$$

We integrate this over $\mathfrak{K} \setminus \mathfrak{M}$. Routine calculations show

$$\int_{\mathfrak{K}\setminus\mathfrak{M}} |F(\alpha)| \, \mathrm{d}\alpha \ll n^{1+\Theta} L^{6C} \int_{\mathfrak{K}\setminus\mathfrak{M}} |\Upsilon(\alpha)|^{5/2} \, \mathrm{d}\alpha \ll n^{\Theta} L^{6C-B/2}.$$

([1]) This source considers Weyl sums over primes without the weight $\log p$ that is present in the definitions of f_k. Since the factor $\log p$ is easily removed by partial summation, the estimates in this source apply to our functions f_k as well.

This establishes the following result.

LEMMA. *Let $B \geq 12C + 2$. Then*

$$\int_{\mathfrak{K}\setminus\mathfrak{M}} |F(\alpha)| \, \mathrm{d}\alpha \ll n^{\Theta} L^{-1}.$$

The theorem is now available. We choose B so that the preceding lemma is applicable. The discussion in the opening paragraph of this section shows that the complement of $\mathfrak{M}$ in $[0, 1]$ is contained in the union of $\mathscr{E}$ and $\mathfrak{K} \setminus \mathfrak{M}$. Hence, the desired bound (3) follows from (2), the above lemma and the lemmata in Sections 2 and 3.

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