ACTA ARITHMETICA
Online First version

Transcendence and continued fraction expansion of values of Hecke–Mahler series

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À la mémoire du Professeur Andrzej Schinzel

1. Introduction and main results. Throughout, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are, respectively, the integer part and the upper integer part functions. For a real number θ in $(0,1)$, set

$$h_\theta(z)=\sum_{k\geq 1}\lfloor k\theta\rfloor z^k,$$

where z is a complex number with $|z|<1$, and

$$F_\theta(z_1,z_2)=\sum_{k_1\geq 1}\sum_{k_2=1}^{\lfloor k_1\theta\rfloor} z_1^{k_1}z_2^{k_2},$$

where z_1,z_2 are complex numbers with $|z_1|<1$, $|z_1z_2^\theta|<1$. The series $h_\theta(z)$ were introduced by Hecke [9] in 1922. Böhmer [4] proved in 1927 that if θ has unbounded partial quotients, then $h_\theta(1/b)$ is transcendental for every integer $b\geq 2$. Two years later, in his fundational paper [12], Mahler introduced the two-variable series $F_\theta(z_1,z_2)$ (note that Mahler and most of his followers used ω in place of θ, while we keep the notation from [7]) and, among other results, he established that $h_\theta(\beta)$ is transcendental for every quadratic irrational number θ and every complex nonzero algebraic number β in the open unit disc. This was extended to every irrational number θ in $(0,1)$ by Loxton and van der Poorten [11] (see also [13, Section 2.9]) nearly fifty years later.

We adopt a slightly different point of view to generalize the functions h_θ and F_θ. Let θ and ρ be real numbers with $0\leq\theta,\ \rho<1$ and θ irrational. For

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2020 *Mathematics Subject Classification*: Primary 11J04; Secondary 11J70, 11J81.
Key words and phrases: rational approximation, continued fraction, Mahler's method, transcendence, Sturmian sequence.
Received 23 March 2022; revised 29 November 2022.
Published online 28 February 2023.

DOI: 10.4064/aa220323-18-1 © Instytut Matematyczny PAN, 2023

a complex number z with $|z| < 1$, set

$$h_{\theta,\rho}(z) = \sum_{k\geq 1} \lfloor k\theta + \rho \rfloor z^k.$$

For $n \geq 1$, set

$$s_n := s_n(\theta,\rho) = \lfloor n\theta + \rho \rfloor - \lfloor (n-1)\theta + \rho \rfloor,$$
$$s'_n := s'_n(\theta,\rho) = \lceil n\theta + \rho \rceil - \lceil (n-1)\theta + \rho \rceil.$$

Then the infinite words

$$\mathbf{s}_{\theta,\rho} := s_1 s_2 s_3 \ldots, \qquad \mathbf{s}'_{\theta,\rho} := s'_1 s'_2 s'_3 \ldots$$

are, respectively, the lower and upper Sturmian words of slope θ and intercept ρ, written over the alphabet $\{0,1\}$. Note that $s_n = s'_n$ for all $n \geq 1$ with at most two exceptions. For complex numbers α, β with $|\beta\alpha^\theta| < 1$, write

$$\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha) = \sum_{n\geq 1} s_n \beta^n \alpha^{\sum_{h=1}^n s_h} = \sum_{n\geq 1} s_n \beta^n \alpha^{\lfloor n\theta+\rho \rfloor},$$
$$\xi_{\mathbf{s}'_{\theta,\rho}}(\beta,\alpha) = \sum_{n\geq 1} s'_n \beta^n \alpha^{\sum_{h=1}^n s'_h} = \sum_{n\geq 1} s'_n \beta^n \alpha^{\lfloor n\theta+\rho \rfloor}.$$

If α and β are algebraic, then the transcendence of $\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha)$ is equivalent to the transcendence of $\xi_{\mathbf{s}'_{\theta,\rho}}(\beta,\alpha)$. Observe that setting

$$F_{\theta,\rho}(z_1,z_2) = \sum_{k_1\geq 1} \sum_{k_2=1}^{\lfloor k_1\theta+\rho \rfloor} z_1^{k_1} z_2^{k_2},$$

we have

$$\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha) = (1-\beta) F_{\theta,\rho}(\beta,\alpha)$$

for any β, α satisfying $|\beta|, |\beta\alpha^\theta| < 1$. The notation $\xi_{\mathbf{s}_{\theta,\rho}}$ was introduced in [7] and we keep it in the present work. The transcendence of $F_{\theta,\rho}(\beta,\alpha)$ for nonzero algebraic numbers α, β has been widely studied, after the pioneering works of Mahler [12] and Loxton and van der Poorten [11] for $\rho = 0$. Borwein and Borwein [5, Theorem 0.4] established that if the slope θ has infinitely many partial quotients greater than or equal to 3, then $\xi_{\mathbf{s}_{\theta,\rho}}(1/b, 1/a)$ is transcendental for any positive integers a, b with $b \geq 2$. Nishioka, Shiokawa, and Tamura [14] and Komatsu [10] proved that if the slope θ has unbounded partial quotients, then $\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha)$ is transcendental for any nonzero complex algebraic numbers α, β with $|\beta\alpha^\theta| < 1$, under some technical condition (a first proof was given in [14], taken up again in [10] to correct some inaccuracies). Notice that both papers [14, 10] go further and include also results of algebraic independence of values of inhomogeneous Hecke–Mahler series in the case where the slope θ has unbounded partial quotients. Lastly, Ferenczi and Mauduit [8] used combinatorial properties of Sturmian sequences and Ridout's p-adic extension of Roth's theorem to show that $\xi_{\mathbf{s}_{\theta,\rho}}(1/b,1)$ is transcendental for every integer $b \geq 2$.

Our first main theorem is a considerable extension of all these results.

THEOREM 1.1. *Let θ and ρ be real numbers with $0 \leq \theta,\ \rho < 1$ and θ irrational. Let α, β be nonzero complex algebraic numbers such that $|\beta\alpha^\theta| < 1$ and $\beta \neq 1$. Then the complex numbers $\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha)$ and $\xi_{\mathbf{s}'_{\theta,\rho}}(\beta,\alpha)$ are transcendental. In particular, if $|\beta| < 1$, then the complex numbers*

$$h_{\theta,\rho}(\beta), \qquad F_{\theta,\rho}(\beta,\alpha)$$

are transcendental.

Since, for every α in the open unit disc, we have

$$\xi_{\mathbf{s}_{\theta,\rho}}(1,\alpha) = \frac{\alpha}{1-\alpha},$$

the assumption $\beta \neq 1$ in Theorem 1.1 is necessary.

When the slope θ has unbounded partial quotients in its continued fraction expansion, Theorem 1.1 was proved by Komatsu [10], under some mild additional assumption on α and β. For the sake of completeness, we display a complete proof in Section 5.

Adamczewski and Bugeaud [2, Proposition 11.1] proved that the Diophantine exponent (which measures the repetitions occurring at the beginning or near the beginning of an infinite word, see [2, p. 70]) of a Sturmian sequence is infinite if and only if its slope θ has unbounded partial quotients, independently of the value of its intercept ρ. Under this assumption, the p-adic Schmidt Subspace Theorem applies to show that, for every ρ in $[0,1)$ and every nonzero algebraic number β in the open unit disc, the complex number $\xi_{\mathbf{s}_{\theta,\rho}}(\beta,1)$ is either transcendental, or lies in $\mathbb{Q}(\beta)$; see [1, Theorem 1]. A different, more involved application of the p-adic Schmidt Subspace Theorem allows us to get the same conclusion if θ has bounded partial quotients; the details will be given in a subsequent paper.

The proof of Theorem 1.1 follows Mahler's method and its extension by Loxton and van der Poorten [11]. A key point is the following construction, leading to a chain of functional equations. For an irrational real number θ in $(0,1)$, write

$$\theta = [0; a_1, a_2, \ldots], \qquad \theta_k = [0; a_{k+1}, a_{k+2}, \ldots], \qquad k \geq 0,$$

in such a way that

$$\theta_0 = \theta, \qquad \theta_{k+1} = \frac{1}{\theta_k} - a_{k+1} = \left\{\frac{1}{\theta_k}\right\}, \qquad k \geq 0,$$

where $\{\cdot\}$ denotes the fractional part function. Let $(p_k/q_k)_{k\geq 0}$ denote the sequence of convergents to θ. An elementary calculation yields the equation

$$F_{\theta_k,0}(z_1,z_2) = -F_{\theta_{k+1},0}(z_1^{a_{k+1}} z_2, z_1) + \frac{z_1^{a_{k+1}+1} z_2}{(1-z_1^{a_{k+1}} z_2)(1-z_1)}.$$

When θ is a quadratic irrational, the sequence $(\theta_k)_{k\geq 1}$ is ultimately periodic, and this chain of functional equations yields a single functional equation. Namely, assuming that $\theta_{k+s} = \theta_k$ for $k \geq 0$ and an even positive integer s, we end up with a functional equation of the form

$$F_{\theta,0}(z_1,z_2) = F_{\theta,0}(z_1^{q_s} z_2^{p_s}, z_1^{q_{s-1}} z_2^{p_{s-1}}) + R(z_1,z_2),$$

where $R(z_1,z_2)$ is in $\mathbb{Q}(z_1,z_2)$ and which has been treated by Mahler [12]. In general, we have a system of functional equations

$$F_{\theta,0}(z_1,z_2) = (-1)^k F_{\theta_k,0}(z_1^{q_k} z_2^{p_k}, z_1^{q_{k-1}} z_2^{p_{k-1}}) + R_k(z_1,z_2), \qquad k \geq 1.$$

Loxton and van der Poorten [11] developed a general theory which applies to such chains of equations under some technical constraints. These assumptions may be satisfied (for a suitable subsequence of the indices k) if we assume that the sequence $(a_k)_{k\geq 1}$ is bounded. By means of our new result on the structure of Sturmian sequences [7] (see Proposition 2.2 below), we are able to show that this approach also works for the more general series $F_{\theta,\rho}(z_1,z_2)$. The unbounded case, treated in Section 5, is related to the second part of our paper devoted to continued fraction expansions.

We stress an immediate consequence of Theorem 1.1. For more on β-expansions of real numbers, the reader is directed to [1] and the references given therein.

Corollary 1.2. *Let γ and β be real algebraic numbers with $\beta > 1$. Then the β-expansion of γ is not given by a Sturmian sequence.*

Theorem 1.1 asserts that any power series whose sequence of coefficients is a Sturmian sequence of integers sends nonzero algebraic points in the unit disc to transcendental points. This is not the case for every automatic series, as shown by Adamczewski and Faverjon [3, Section 8.1], who gave the example of an automatic series taking an algebraic value at any point of the form $\phi^{1/3^\ell}$, where $\phi = (1-\sqrt{5})/2$ and $\ell \geq 1$.

Let a and b be positive integers with $b \geq 2$. By Theorem 1.1, the real numbers $\xi_{\mathbf{s}_{\theta,\rho}}(1/b,1/a)$ and $\xi_{\mathbf{s}'_{\theta,\rho}}(1/b,1/a)$ are transcendental. We now deal with the continued fraction expansion of the real numbers ξ of the form

$$(b-1)\xi_{\mathbf{s}_{\theta,\rho}}(1/b,1/a) \quad \text{or} \quad (b-1)\xi_{\mathbf{s}'_{\theta,\rho}}(1/b,1/a).$$

When $a=1$, we will recover the expansion of Sturmian numbers obtained in [7].

We denote by $(b_k)_{k\geq 1}$ the sequence of digits of the number

$$\rho - \theta = \sum_{k\geq 0} b_{k+1}(q_k\theta - p_k), \tag{1.1}$$

written in the Ostrowski numeration system with base θ (normalized as in [7, Theorem 2.1] or in Theorem 4.2 below when ρ is of the form $-m\theta + p$, with m, p nonnegative integers). It satisfies $0 \leq b_1 \leq a_1 - 1$, $0 \leq b_k \leq a_k$ for $k \geq 1$, and $b_{k+1} = a_{k+1}$ implies $b_k = 0$ for every $k \geq 1$. We set (by convention, an empty sum is equal to zero)

$$t_k = \sum_{j=1}^{k} b_j q_{j-1},\ \tilde{t}_k = \sum_{j=1}^{k} b_j p_{j-1}, \quad r_k = q_k - t_k,\ \tilde{r}_k = p_k - \tilde{t}_k, \quad k \geq 0. \tag{1.2}$$

For $k \geq 0$, set

$$c_k = \begin{cases} \frac{b^{a_1-b_1}a-b}{b-1} & \text{when} \quad k = 0, \\ \frac{b^{r_k+q_{k-1}}a^{\tilde{r}_k+p_{k-1}}((b^{q_k}a^{p_k})^{a_{k+1}-b_{k+1}-1}-1)}{b^{q_k}a^{p_k}-1} & \text{when} \quad k \geq 1, \end{cases}$$

$$d_k = b^{t_k}a^{\tilde{t}_k} - 1,$$

$$e_k = b^{r_k}a^{\tilde{r}_k} - 1,$$

$$f_k = b^{t_k}a^{\tilde{t}_k}\frac{(b^{q_k}a^{p_k})^{b_{k+1}} - 1}{b^{q_k}a^{p_k} - 1}.$$

When $a = 1$, the four sequences $(c_k)_{k\geq 0}, (d_k)_{k\geq 0}, (e_k)_{k\geq 0}, (f_k)_{k\geq 0}$ coincide with the corresponding ones introduced in [7]. We point out that some elements of these sequences may be nonpositive, exactly in the same situations as in [7]. For example, f_k is equal to 0 when $b_{k+1} = 0$ and c_{k+1} is equal to 0 when $a_{k+2} = b_{k+2} + 1$. In the case where $a_{k+2} = b_{k+2}$, we have $b_{k+1} = 0$, thus $r_k + q_{k+1} = r_{k+1} + q_k, \tilde{r}_k + p_{k+1} = \tilde{r}_{k+1} + p_k$, so that

$$\begin{aligned} c_{k+1} &= b^{r_{k+1}+q_k}a^{\tilde{r}_{k+1}+p_k}\frac{(b^{q_{k+1}}a^{p_{k+1}})^{-1} - 1}{b^{q_{k+1}}a^{p_{k+1}} - 1} \\ &= \frac{b^{r_k}a^{\tilde{r}_k} - b^{r_k+q_{k+1}}a^{\tilde{r}_k+p_{k+1}}}{b^{q_{k+1}}a^{p_{k+1}} - 1} = -b^{r_k}a^{\tilde{r}_k} = -e_k - 1 \end{aligned}$$

is negative. Notice as well that e_k is always positive, because $b \geq 2$, $a \geq 1$, $r_k \geq 1$, and that d_k is nonnegative and vanishes if and only if $t_k = \tilde{t}_k = 0$, that is, when $b_1 = \cdots = b_k = 0$.

Keeping this in mind, and with some abuse of language, the next theorem asserts that

$$[0; c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots]$$

is an (improper) continued fraction expansion of ξ. In order to rule out nonpositive elements in the sequence

$$c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots$$

we apply to it some contraction rules. The precise statement is as follows.

THEOREM 1.3. *Let a and b be positive integers. Assume that $b \geq 2$ and that a is congruent to 1 modulo $b-1$. Let $A_1, A_2, A_3, \ldots$ be the sequence of positive integers obtained from the sequence $c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots$ after the application of the following rules:*

(i) *For any $k \geq 0$ such that $a_{k+2} = b_{k+2}$, replace the string of the 9 consecutive terms*

$$c_k, d_k, 1, e_k, f_k = 0, c_{k+1} = -e_k - 1, d_{k+1} = d_k, 1, e_{k+1}$$

by the single element $c_k + e_{k+1} + 1$.

(ii) *Replace any three consecutive elements of this new sequence of the form $x, 0, y$ by the integer $x+y$ (x and y may vanish) and continue the reduction until one obtains positive integers.*

Then the continued fraction expansion of ξ is given by

$$\xi = [0; A_1, A_2, A_3, \ldots].$$

Observe that the sequence $(A_j)_{j \geq 1}$ is well-defined. Indeed, c_k and c_{k+1} cannot both be negative, since we cannot have simultaneously $a_{k+1} = b_{k+1}$ and $a_{k+2} = b_{k+2}$ by Ostrowski numeration rules. Procedure (ii) enables us to get rid of the 0 after having ruled out the negative terms using (i). The occurrences of 0, after performing rule (i), are fully described thanks to the six cases displayed in [7, Section 7], which remain unchanged in our setting. For convenience, we reproduce the list below.

$(\mathrm{ii})_1$ $b_k = 0$ and $a_{k+2} = b_{k+2}$ with $k \geq 1$, corresponding to the string

$$1, e_{k-1}, f_{k-1} = 0, c_k + e_{k+1} + 1, f_{k+1},$$

where

$$e_{k-1}, c_k + e_{k+1} + 1, f_{k+1} > 0.$$

$(\mathrm{ii})_2$ $b_{k+1} = 0$, $t_{k+1} \geq 1$ and $a_{k+2} \geq b_{k+2} + 2$ with $k \geq 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1}, d_{k+1}, \quad \text{where} \quad e_k, c_{k+1}, d_{k+1} > 0.$$

$(\mathrm{ii})_3$ $b_{k+1} \geq 1$ and $a_{k+2} = b_{k+2} + 1$ with $k \geq 0$, corresponding to the string

$$e_k, f_k, c_{k+1} = 0, d_{k+1}, 1, \quad \text{where} \quad e_k, f_k, d_{k+1} > 0.$$

$(\mathrm{ii})_4$ $b_{k+1} = 0$, $t_{k+1} \geq 1$ and $a_{k+2} = b_{k+2} + 1$ with $k \geq 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1} = 0, d_{k+1}, 1, \quad \text{where} \quad e_k, d_{k+1} > 0.$$

$(\mathrm{ii})_5$ $t_{k+1} = 0$ and $a_{k+2} \geq b_{k+2} + 2$ with $k \geq 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1}, d_{k+1} = 0, 1, e_{k+1}, \quad \text{where} \quad e_k, c_{k+1}, e_{k+1} > 0.$$

(ii)$_6$ $t_{k+1} = 0$ and $a_{k+2} = b_{k+2} + 1$ with $k \ge 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1} = 0, d_{k+1} = 0, 1, e_{k+1}, \quad \text{where} \quad e_k, e_{k+1} > 0.$$

As a simple example, we obtain

COROLLARY 1.4. *Assume that $a_k - b_k \ge 2$ and $b_k \ge 1$ for every $k \ge 1$. Then the continued fraction expansion of ξ is given by*

$$\xi = [0; c_0 + 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots].$$

Proof. Observe that $d_0 = 0$, while all the other elements of the sequence

$$c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots$$

are positive. ■

When a and b are positive integers with $b \ge 2$ and a not congruent to 1 modulo $b-1$, we get the regular continued fraction expansion of $1/\xi - (c_0+1) = 1/\xi - (b^{a_1-b_1}a - 1)/(b-1)$.

As a consequence of Theorem 1.3, we obtain an expression for the irrationality exponent of any real number ξ as above in terms of its slope and its intercept.

Keeping the notation above, define

$$\nu_k(1) = 2 + \frac{t_k}{r_{k+1}}, \qquad \nu_k(2) = 2 + \frac{r_k}{r_{k+1} + t_k},$$
$$\nu_k(3) = 1 + \frac{q_{k+1}}{r_{k+1} + q_k}, \qquad \nu_k(4) = 1 + \frac{r_{k+2}}{q_{k+1}}.$$

Put

$$\nu(1) = \limsup_{k\to+\infty} \{\nu_k(1) : a_{k+1} - b_{k+1} \ge 1 \text{ and } a_{k+2} - b_{k+2} \ge 1\},$$
$$\nu(2) = \limsup_{k\to+\infty} \{\nu_k(2) : a_{k+2} - b_{k+2} \ge 1\},$$

and, for $j = 3, 4$,

$$\nu(j) = \limsup_{k\to+\infty} \nu_k(j).$$

THEOREM 1.5. *Let $b \ge 2$ and $a \ge 1$ be integers. The irrationality exponent of $\xi_{\mathbf{s}_{\theta,\rho}}(1/b, 1/a)$ and of $\xi_{\mathbf{s}'_{\theta,\rho}}(1/b, 1/a)$ is equal to*

$$\max\{\nu(1), \nu(2), \nu(3), \nu(4)\}.$$

Theorem 1.5 extends [7, Theorem 2.4] which covers the case $a = 1$.

2. Sturmian words. We collect in this section some important properties of the Sturmian words $\mathbf{s}_{\theta,\rho}$ and $\mathbf{s}'_{\theta,\rho}$, obtained in [7]. Recall that $(p_k/q_k)_{k\ge0}$ is the sequence of convergents to $\theta = [0; a_1, a_2, \ldots]$ and that the sequences $(b_k)_{k\ge1}$, $(r_k)_{k\ge0}$, $(t_k)_{k\ge0}$, $(\tilde{r}_k)_{k\ge0}$, $(\tilde{t}_k)_{k\ge0}$ are defined in (1.1) and (1.2).

Lemma 2.1. *We have*

$$r_0 = 1, \quad \tilde r_0 = 0, \quad r_1 = a_1 - b_1, \quad \tilde r_1 = 1,$$

and the following recursion formulae hold for any $k \ge 0$:

$$r_{k+1} = r_k + (a_{k+1} - b_{k+1} - 1)q_k + q_{k-1},$$
$$\tilde r_{k+1} = \tilde r_k + (a_{k+1} - b_{k+1} - 1)p_k + p_{k-1}.$$

It follows that

$$r_{k+1} = 1 - q_k + \sum_{j=0}^{k} (a_{j+1} - b_{j+1})q_j, \quad k \ge 0,$$
$$\tilde r_{k+1} = 1 - p_k + \sum_{j=0}^{k} (a_{j+1} - b_{j+1})p_j, \quad k \ge 0.$$

Moreover, we have $0 \le t_k < q_k$, $0 \le \tilde t_k \le p_k$, $1 \le r_k \le q_k$, and $0 \le \tilde r_k \le p_k$, for every $k \ge 0$.

Proof. The combination of (1.2) with the classical recurrence relations $q_{j+1} = a_{j+1}q_j + q_{j-1}$ and $p_{j+1} = a_{j+1}p_j + p_{j-1}$ for any $j \ge 0$, arising from the theory of continued fractions, gives the formulae of Lemma 2.1. Moreover, the Ostrowski numeration rules ($0 \le b_1 \le a_1 - 1$, $0 \le b_k \le a_k$ for $k \ge 1$, and $b_{k+1} = a_{k+1}$ implies $b_k = 0$ for every $k \ge 1$) yield by induction on k the required inequalities. ∎

Let

$$\mathbf{c}_\theta := \mathbf{s}_{\theta,\theta} = \mathbf{s}'_{\theta,\theta}$$

be the characteristic word of slope θ. For $k \ge 1$, we denote by M_k the prefix of length q_k of $\mathbf{c}_\theta$. Set furthermore $M_0 = 0$.

Proposition 2.2. *Define inductively two sequences $(T_k)_{k\ge0}$ and $(R_k)_{k\ge0}$ of finite words on $\{0,1\}$ by letting T_0 be the empty word, $R_0 = 0$, $R_1 = 0^{a_1-b_1-1}1$, and by the recursion formulae*

$$T_{k+1} = M_k^{b_{k+1}} T_k \tag{2.1}$$

for any $k \ge 0$, and

$$R_{k+1} = \begin{cases} R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} & \text{if } b_{k+1} < a_{k+1}, \\ R_{k-1} & \text{if } b_{k+1} = a_{k+1}, \end{cases} \tag{2.2}$$

for any $k \ge 1$. Then T_k (resp. R_k) has length t_k (resp. r_k) and contains $\tilde t_k$ (resp. $\tilde r_k$) letters 1. Set

$$V_k = R_k T_k, \quad k \ge 0.$$

The word V_k has length q_k, contains p_k letters 1, and its first $q_k - 1$ letters coincide with those of $\mathbf{s}_{\theta,\rho}$ (or $\mathbf{s}'_{\theta,\rho}$). Moreover, $M_k = T_k R_k$ and the sequence

$(V_k)_{k\geq 0}$ *satisfies the recurrence relations*

$$V_0 = 0, \qquad V_1 = 0^{a_1-b_1-1}10^{b_1}, \qquad V_{k+1} = V_k^{a_{k+1}-b_{k+1}}V_{k-1}V_k^{b_{k+1}}, \quad k \geq 1.$$

Proof. Proposition 2.2 is a reformulation of the results of [7, Section 3], with the exception of the assertions concerning the number of letters 1. These follow from Lemma 2.1, combined with the recursion formulae (2.1) and (2.2) established in [7, Lemma 3.3], by observing that the word M_k contains p_k letters 1. Notice that when $a_{k+1} = b_{k+1}$, we have $b_k = 0$, so that

$$\tilde{r}_k = \tilde{r}_{k-1} + (a_k - 1)p_{k-1} + p_{k-2} = \tilde{r}_{k-1} + p_k - p_{k-1},$$
$$\tilde{r}_{k+1} = \tilde{r}_k + (a_{k+1} - b_{k+1} - 1)p_k + p_{k-1} = \tilde{r}_k - p_k + p_{k-1} = \tilde{r}_{k-1}.$$

It follows that $R_{k+1} = R_{k-1}$ contains $\tilde{r}_{k+1} = \tilde{r}_{k-1}$ letters 1, as claimed. ∎

We point out that if the sequence $(b_k)_{k\geq 1}$ is given, then the intercept ρ is uniquely determined.

Definition 2.3. The sequence $(b_k)_{k\geq 1}$ is called the *formal intercept* of the Sturmian word $\mathbf{s}_{\theta,\rho}$ of slope θ and intercept ρ.

The next lemma will be used in Sections 3 and 5.

Lemma 2.4. *As k tends to infinity, we have*

$$r_k\theta - \tilde{r}_k = O(1), \qquad t_k\theta - \tilde{t}_k = O(1).$$

Proof. It follows from (1.2) that

$$\begin{aligned} r_k\theta - \tilde{r}_k &= q_k\theta - p_k - \sum_{j=0}^{k-1} b_{j+1}(q_j\theta - p_j) \\ &= \theta - \rho + (q_k\theta - p_k) + \sum_{j\geq k} b_{j+1}(q_j\theta - p_j), \end{aligned}$$

recalling the Ostrowski expansion

$$\rho - \theta = \sum_{j\geq 0} b_{j+1}(q_j\theta - p_j).$$

This shows that $r_k\theta - \tilde{r}_k = O(1)$. Since $|q_k\theta - p_k| \leq 1$, we get the second estimate. ∎

3. Continued fraction expansion. The main goal of this section is to prove Theorem 1.3, and to give further results on the convergents of ξ.

For a finite word $W = w_1 \dots w_\ell$ over the alphabet $\{0,1\}$ and variables a, b, set $W(b,a) = 0$ if W is the empty word and

$$W(b,a) = \sum_{n=1}^{\ell} w_n b^{\ell-n} a^{\sum_{h=n+1}^{\ell} w_h} = b^\ell a^{\sum_{h=1}^{\ell} w_h} \sum_{n=1}^{\ell} w_n \left(\frac{1}{b}\right)^n \left(\frac{1}{a}\right)^{\sum_{h=1}^{n} w_h},$$

otherwise. Note that the exponent $\sum_{h=1}^n w_h$ counts the number of letters 1 in the prefix of length n of the word W.

Now, if $\mathbf{x} = x_1 x_2 \ldots$ is an infinite word over the alphabet $\{0,1\}$, recall that we have set

$$\xi_{\mathbf{x}}(\beta, \alpha) = \sum_{n\geq 1} x_n \beta^n \alpha^{\sum_{h=1}^n x_h}.$$

When $\mathbf{x}$ is an ultimately periodic word, $\xi_{\mathbf{x}}(\beta,\alpha)$ is a rational function in the two variables β and α. Set $a = 1/\alpha$ and $b = 1/\beta$. More precisely, we have

Lemma 3.1. *Let $Y = y_1 \ldots y_r$ and $Z = z_1 \ldots z_s$ be two finite words over $\{0,1\}$. Put $\tilde{r} = y_1 + \cdots + y_r$ and $\tilde{s} = z_1 + \cdots + z_s$. Then*

$$YZ(b,a) = b^s a^{\tilde{s}} Y(b,a) + Z(b,a),$$

where $YZ = y_1 \ldots y_r z_1 \ldots z_s$ stands for the concatenation of Y and Z. Moreover, if $|b^s a^{\tilde{s}}| > 1$, then

$$\xi_{Z^\infty}(\beta,\alpha) = \frac{Z(b,a)}{b^s a^{\tilde{s}} - 1} \quad \text{and} \quad \xi_{YZ^\infty}(\beta,\alpha) = \frac{YZ(b,a) - Y(b,a)}{b^r a^{\tilde{r}}(b^s a^{\tilde{s}} - 1)},$$

where Z^∞ stands for the concatenation of infinitely many copies of Z.

Proof. The first formula immediately follows from the definition.

By setting $\mathbf{x} = YZ^\infty$ and writing $n = r + js + m$ for $n \geq r+1$, we obtain by periodicity

$$\begin{aligned}
\xi_{\mathbf{x}}(\beta,\alpha) &= \sum_{n\geq 1} x_n \beta^n \alpha^{\sum_{h=1}^n x_h} \\
&= \sum_{n=1}^{r} y_n \beta^n \alpha^{\sum_{h=1}^n y_h} + \sum_{j\geq 0} \sum_{m=1}^{s} z_m \beta^{r+js+m} \alpha^{\tilde{r}+j\tilde{s}+\sum_{h=1}^m z_h} \\
&= \sum_{n=1}^{r} y_n \beta^n \alpha^{\sum_{h=1}^n y_h} + \frac{\beta^r \alpha^{\tilde{r}} \sum_{m=1}^s z_m \beta^m \alpha^{\sum_{h=1}^m z_h}}{1 - \beta^s \alpha^{\tilde{s}}} \\
&= \frac{Y(b,a)}{b^r a^{\tilde{r}}} + \frac{Z(b,a)}{b^r a^{\tilde{r}}(b^s a^{\tilde{s}} - 1)} \\
&= \frac{Y(b,a)(b^s a^{\tilde{s}} - 1) + Z(b,a)}{b^r a^{\tilde{r}}(b^s a^{\tilde{s}} - 1)} = \frac{YZ(b,a) - Y(b,a)}{b^r a^{\tilde{r}}(b^s a^{\tilde{s}} - 1)}.
\end{aligned}$$

When Y is the empty word, we obtain the formula $\xi_{Z^\infty}(\beta,\alpha) = \frac{Z(b,a)}{b^s a^{\tilde{s}} - 1}$. ∎

We use Lemma 3.1 in order to construct rational fractions in a and b associated to four sequences of periodic words which approach the Sturmian word $\mathbf{s} = \mathbf{s}_{\theta,\rho}$. For any $k \geq 0$, define

$$(1)_k = \frac{(b-1)(R_{k+1}(b,a) - R_k(b,a))}{b^{r_k} a^{\tilde{r}_k}(b^{r_{k+1}-r_k} a^{\tilde{r}_{k+1}-\tilde{r}_k} - 1)},$$

which is associated to the word $R_k(M_k^{a_{k+1}-b_{k+1}-1}M_{k-1})^\infty$ whenever $k \geq 1$ and $a_{k+1}-b_{k+1} \geq 1$. Next, set

$$(2)_k = \frac{(b-1)(R_{k+1}T_k)(b,a)}{b^{r_{k+1}+t_k}a^{\tilde{r}_{k+1}+\tilde{t}_k}-1},$$

associated to the purely periodic word $(R_{k+1}T_k)^\infty$. The third approximation is

$$(3)_k = \frac{(b-1)\big((R_{k+1}M_k)(b,a)-R_{k+1}(b,a)\big)}{b^{r_{k+1}}a^{\tilde{r}_{k+1}}(b^{q_k}a^{p_k}-1)},$$

associated to the word $R_{k+1}M_k^\infty$. Finally, put

$$(4)_k = \frac{(b-1)V_{k+1}(b,a)}{b^{q_{k+1}}a^{p_{k+1}}-1},$$

associated to the purely periodic word $V_{k+1}^\infty = (R_{k+1}T_{k+1})^\infty$.

We now give an analogue of [7, Lemma 7.1] in our framework. We use the notation $\frac{P}{Q} = c \cdot \frac{P'}{Q'} \dotplus \frac{P''}{Q''}$ between fractions to mean that both relations $P = cP' + P''$ and $Q = cQ' + Q''$ hold true. Similarly, $(2)_k \dot{-} (1)_k$ stands below for the fraction whose numerator (resp. denominator) is the difference between the numerators (resp. denominators) of $(2)_k$ and $(1)_k$. It is convenient to define formally $(3)_{-1} = \frac{b-1}{0}$ and $(4)_{-1} = \frac{0}{b-1}$.

LEMMA 3.2. *For any $k \geq 0$, we have the following relations:*

$$\begin{aligned}(1)_k &= c_k \cdot (4)_{k-1} \dotplus (3)_{k-1},\\ (2)_k \dot{-} (1)_k &= d_k \cdot (1)_k \dotplus (4)_{k-1},\\ (2)_k &= 1 \cdot \big((2)_k \dot{-} (1)_k\big) \dotplus (1)_k,\\ (3)_k &= e_k \cdot (2)_k \dotplus \big((2)_k \dot{-} (1)_k\big),\\ (4)_k &= f_k \cdot (3)_k \dotplus (2)_k.\end{aligned}$$

Proof. We compute

$$(1)_0 = \frac{b-1}{b^{a_1-b_1}a-b}, \qquad (2)_0 \dot{-} (1)_0 = \frac{0}{b-1}, \qquad (2)_0 = \frac{b-1}{b^{a_1-b_1}a-1},$$
$$(3)_0 = \frac{(b-1)^2}{b^{a_1-b_1}a(b-1)}, \qquad (4)_0 = \frac{b^{b_1}(b-1)}{b^{a_1}a-1}.$$

We have

$$c_0 = \frac{b^{a_1-b_1}a-b}{b-1}, \qquad d_0 = 0, \qquad e_0 = b-1, \qquad f_0 = \frac{b^{b_1}-1}{b-1},$$

so that the above five relations are verified for $k = 0$.

Assume now that $k \geq 1$. The third relation is obvious. Let us check the remaining four relations. The denominators of $(1)_k, (2)_k \dot{-} (1)_k, (2)_k, (3)_k, (4)_k$ are respectively

$$\begin{aligned}
Q_{(1)_k} &= b^{r_{k+1}} a^{\tilde r_{k+1}} - b^{r_k} a^{\tilde r_k},\\
Q_{(2)_k \dot- (1)_k} &= b^{r_{k+1}+t_k} a^{\tilde r_{k+1}+\tilde t_k} - 1 - (b^{r_{k+1}} a^{\tilde r_{k+1}} - b^{r_k} a^{\tilde r_k}),\\
Q_{(2)_k} &= b^{r_{k+1}+t_k} a^{\tilde r_{k+1}+\tilde t_k} - 1,\\
Q_{(3)_k} &= b^{r_{k+1}} a^{\tilde r_{k+1}} (b^{q_k} a^{p_k} - 1),\\
Q_{(4)_k} &= b^{q_{k+1}} a^{p_{k+1}} - 1.
\end{aligned}$$

Using Lemma 2.1, we check that

$$\begin{aligned}
\frac{Q_{(1)_k} - Q_{(3)_{k-1}}}{Q_{(4)_{k-1}}} &= \frac{b^{r_{k+1}} a^{\tilde r_{k+1}} - b^{r_k} a^{\tilde r_k} - b^{r_k+q_{k-1}} a^{\tilde r_k+p_{k-1}} + b^{r_k} a^{\tilde r_k}}{b^{q_k} a^{p_k} - 1}\\
&= \frac{b^{r_k+q_{k-1}} a^{\tilde r_k+p_{k-1}} \big((b^{q_k} a^{p_k})^{a_{k+1}-b_{k+1}-1} - 1\big)}{b^{q_k} a^{p_k} - 1} = c_k,
\end{aligned}$$

as required. Similarly, we have to check that

$$d_k = \frac{Q_{(2)_k} - Q_{(1)_k} - Q_{(4)_{k-1}}}{Q_{(1)_k}} \quad \text{or equivalently} \quad d_k + 1 = \frac{Q_{(2)_k} - Q_{(4)_{k-1}}}{Q_{(1)_k}}.$$

Now,

$$\frac{Q_{(2)_k} - Q_{(4)_{k-1}}}{Q_{(1)_k}} = \frac{b^{r_{k+1}+t_k} a^{\tilde r_{k+1}+\tilde t_k} - 1 - (b^{q_k} a^{p_k} - 1)}{b^{r_{k+1}} a^{\tilde r_{k+1}} - b^{r_k} a^{\tilde r_k}} = b^{t_k} a^{\tilde t_k} = d_k + 1,$$

since $q_k = r_k + t_k$ and $p_k = \tilde r_k + \tilde t_k$.

For the fourth relation, we have to show that

$$e_k = \frac{Q_{(3)_k} - Q_{(2)_k} + Q_{(1)_k}}{Q_{(2)_k}} \quad \text{or equivalently} \quad e_k + 1 = \frac{Q_{(3)_k} + Q_{(1)_k}}{Q_{(2)_k}}.$$

But

$$\frac{Q_{(3)_k} + Q_{(1)_k}}{Q_{(2)_k}} = \frac{b^{r_{k+1}+q_k} a^{\tilde r_{k+1}+p_k} - b^{r_k} a^{\tilde r_k}}{b^{r_{k+1}+t_k} a^{\tilde r_{k+1}+\tilde t_k} - 1} = b^{r_k} a^{\tilde r_k} = e_k + 1,$$

writing again $q_k = r_k + t_k$ and $p_k = \tilde r_k + \tilde t_k$.

For the fifth relation, we compute

$$\begin{aligned}
\frac{Q_{(4)_k} - Q_{(2)_k}}{Q_{(3)_k}} &= \frac{b^{q_{k+1}} a^{p_{k+1}} - b^{r_{k+1}+t_k} a^{\tilde r_{k+1}+\tilde t_k}}{b^{r_{k+1}} a^{\tilde r_{k+1}} (b^{q_k} a^{p_k} - 1)}\\
&= \frac{b^{t_k+b_{k+1}q_k} a^{\tilde t_k+b_{k+1}p_k} - b^{t_k} a^{\tilde t_k}}{b^{q_k} a^{p_k} - 1} = f_k,
\end{aligned}$$

since

$$\begin{aligned}
q_{k+1} &= r_{k+1} + t_{k+1} = r_{k+1} + t_k + b_{k+1} q_k,\\
p_{k+1} &= \tilde r_{k+1} + \tilde t_{k+1} = \tilde r_{k+1} + \tilde t_k + b_{k+1} p_k.
\end{aligned}$$

It remains to deal with the numerators. For the first relation, we have to show that

$$c_k V_k(b,a) + R_k M_{k-1}(b,a) - R_k(b,a) = R_{k+1}(b,a) - R_k(b,a).$$

Assume first that $a_{k+1} - b_{k+1} \geq 1$. Using Lemma 3.1 and Proposition 2.2, and noting that $(R_k T_k)^{a_{k+1}-b_{k+1}-1} R_k M_{k-1} = R_{k+1}$ by (2.2), we compute

$$\begin{aligned} c_k V_k(b,a) &= V_k(b,a) \times b^{r_k+q_{k-1}} a^{\tilde{r}_k+p_{k-1}} \frac{(b^{q_k}a^{p_k})^{a_{k+1}-b_{k+1}-1} - 1}{b^{q_k}a^{p_k} - 1} \\ &= R_k T_k(b,a) \times b^{r_k+q_{k-1}} a^{\tilde{r}_k+p_{k-1}} (1 + b^{q_k}a^{p_k} + \cdots + (b^{q_k}a^{p_k})^{a_{k+1}-b_{k+1}-2}) \\ &= b^{r_k+q_{k-1}} a^{\tilde{r}_k+p_{k-1}} (R_k T_k)^{a_{k+1}-b_{k+1}-1}(b,a) \\ &= (R_k T_k)^{a_{k+1}-b_{k+1}-1} R_k M_{k-1}(b,a) - R_k M_{k-1}(b,a) \\ &= R_{k+1}(b,a) - R_k M_{k-1}(b,a), \end{aligned}$$

as required. Assume finally that $a_{k+1} = b_{k+1}$. Then $c_k = -b^{r_{k-1}} a^{\tilde{r}_{k-1}}$ and we have

$$\begin{aligned} V_k(b,a) \times (-b^{r_{k-1}} a^{\tilde{r}_{k-1}}) &= -V_k R_{k-1}(b,a) + R_{k-1}(b,a) \\ &= -R_k T_k R_{k-1}(b,a) + R_{k-1}(b,a) \\ &= -R_k M_{k-1}(b,a) + R_{k+1}(b,a), \end{aligned}$$

since $T_k = T_{k-1}$ in that case, thanks to (2.1).

The second relation for the numerators reads

$$(d_k + 1)(R_{k+1}(b,a) - R_k(b,a)) = R_{k+1} T_k(b,a) - V_k(b,a).$$

To see this, write

$$\begin{aligned} &b^{t_k} a^{\tilde{t}_k} (R_{k+1}(b,a) - R_k(b,a)) \\ &\qquad = b^{t_k} a^{\tilde{t}_k} R_{k+1}(b,a) + T_k(b,a) - \big(b^{t_k} a^{\tilde{t}_k} R_k(b,a) + T_k(b,a)\big) \\ &\qquad = R_{k+1} T_k(b,a) - R_k T_k(b,a) = R_{k+1} T_k(b,a) - V_k(b,a). \end{aligned}$$

The third relation for the numerators is obvious, while the fourth reads

$$\begin{aligned} (e_k + 1) R_{k+1} T_k(b,a) &= \big(R_{k+1} M_k(b,a) - R_{k+1}(b,a)\big) \\ &\quad + \big(R_{k+1}(b,a) - R_k(b,a)\big) \\ &= R_{k+1} M_k(b,a) - R_k(b,a), \end{aligned}$$

which follows from the equalities

$$b^{r_k} a^{\tilde{r}_k} R_{k+1} T_k(b,a) + R_k(b,a) = R_{k+1} T_k R_k(b,a) = R_{k+1} M_k(b,a),$$

by Lemma 3.1 (with $Z = R_k$ and $Y = R_{k+1} T_k$) and Proposition 2.2. The fifth relation reads

$$f_k\big((R_{k+1} M_k)(b,a) - R_{k+1}(b,a)\big) = V_{k+1}(b,a) - (R_{k+1} T_k)(b,a).$$

Notice that

$$V_{k+1} = R_{k+1} T_{k+1} = R_{k+1} M_k^{b_{k+1}} T_k,$$

so that

$$\begin{aligned} V_{k+1}(b,a)-&(R_{k+1}T_k)(b,a) \\ &= \big(V_{k+1}(b,a)-T_k(b,a)\big)-\big((R_{k+1}T_k)(b,a)-T_k(b,a)\big) \\ &= b^{t_k}a^{\tilde{t}_k}\big((R_{k+1}M_k^{b_{k+1}})(b,a)-R_{k+1}(b,a)\big), \end{aligned}$$

thanks to Lemma 3.1. Now, we can write

$$\begin{aligned} (R_{k+1}M_k^{b_{k+1}})(b,a) &- R_{k+1}(b,a) \\ &= \sum_{j=0}^{b_{k+1}-1}\big((R_{k+1}M_k^{j+1})(b,a)-(R_{k+1}M_k^{j})(b,a)\big) \\ &= \Big(\sum_{j=0}^{b_{k+1}-1} b^{jq_k}a^{jp_k}\Big)\big((R_{k+1}M_k)(b,a)-R_{k+1}(b,a)\big), \end{aligned}$$

by factoring M_k^j on the right and applying again Lemma 3.1. The fifth relation immediately follows and Lemma 3.2 has been fully checked. ∎

We now have all the tools to prove Theorems 1.3 and 1.5.

Proof of Theorem 1.3. At this stage, the proof of Theorem 1.3 follows the argument of [7, Section 7]. We briefly repeat it.

Let $\alpha_1, \alpha_2, \dots$ be the cyclic sequence

$$c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, d_2, 1, e_2, f_2, \dots$$

and define two sequences $(P_j)_{j\ge -1}$ and $(Q_j)_{j\ge -1}$ by the recurrence relations

$$\begin{aligned} P_{-1} &= b-1, & P_0 &= 0, & P_j &= \alpha_j P_{j-1} + P_{j-2}, & j &\ge 1, \\ Q_{-1} &= 0, & Q_0 &= b-1, & Q_j &= \alpha_j Q_{j-1} + Q_{j-2}, & j &\ge 1. \end{aligned}$$

Equivalently, we have the matrix equalities

$$\begin{pmatrix} Q_j & Q_{j-1} \\ P_j & P_{j-1} \end{pmatrix} = \begin{pmatrix} b-1 & 0 \\ 0 & b-1 \end{pmatrix} \begin{pmatrix} \alpha_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} \alpha_j & 1 \\ 1 & 0 \end{pmatrix}, \qquad j \ge 1.$$

Lemma 3.2 tells us that the sequence $(P_j)_{j\ge 1}$ (resp. $(Q_j)_{j\ge 1}$) coincides with the sequence of numerators (resp. denominators) of

$$(1)_0, (2)_0 \dot{-} (1)_0, (2)_0, (3)_0, (4)_0, (1)_1, (2)_1 \dot{-} (1)_1, (2)_1, (3)_1, (4)_1, \dots.$$

Now, we reduce by associativity the (formal) infinite product

$$\begin{pmatrix} \alpha_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_3 & 1 \\ 1 & 0 \end{pmatrix} \dots,$$

thanks to procedures (i) and (ii) of Theorem 1.3. For brevity, write

$$E(\alpha) = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) If $a_{k+2} = b_{k+2}$, we reduce the product of the nine consecutive factors:

$$\begin{aligned} E(c_k)E(d_k)E(1)E(e_k)E(f_k)E(c_{k+1})E(d_{k+1})E(1)E(e_{k+1}) \\ = E(c_k)E(d_k)E(1)E(e_k)E(0)E(-e_k-1)E(d_k)E(1)E(e_{k+1}) \\ = E(c_k+e_{k+1}+1). \end{aligned}$$

If $c_k = \alpha_l$, we thus have

$$\begin{pmatrix} Q_{l+8} & Q_{l+7} \\ P_{l+8} & P_{l+7} \end{pmatrix} = \begin{pmatrix} Q_{l-1} & Q_{l-2} \\ P_{l-1} & P_{l-2} \end{pmatrix} \begin{pmatrix} c_k+e_{k+1}+1 & 1 \\ 1 & 0 \end{pmatrix},$$

and we jump from $\frac{P_{l-1}}{Q_{l-1}} = (4)_{k-1}$ to $\frac{P_{l+8}}{Q_{l+8}} = (3)_{k+1}$ thanks to the elementary matrix $E(c_k+e_{k+1}+1)$.

(ii) Reduction (i) allows transforming the infinite product $E(\alpha_1)E(\alpha_2)\cdots$ into a product

$$E(\alpha_1)E(\alpha_2)\cdots = E(\alpha'_1)E(\alpha'_2)\cdots$$

where $\alpha'_1, \alpha'_2, \dots$ are nonnegative. We may encounter some zeroes (these occur precisely in the six cases displayed after Theorem 1.3). If say $\alpha'_l = 0$, we replace $\alpha'_{l-1}, 0, \alpha'_{l+1}$ by $\alpha'_{l-1} + \alpha'_{l+1}$, thanks to the matrix equality

$$E(\alpha'_{l-1})E(0)E(\alpha'_{l+1}) = E(\alpha'_{l-1} + \alpha'_{l+1}).$$

We end up with a product

$$E(\alpha_1)E(\alpha_2)\cdots = E(A_1)E(A_2)\cdots$$

where $A_1, A_2, \dots$ are positive. Moreover, every convergent $[0; A_1, \dots, A_n]$ equals one of the five fractions $(1)_k$, $(2)_k \dotdiv (1)_k$, $(2)_k$, $(3)_k$, $(4)_k$, and infinitely many of these convergents are of the form $(j)_k$ for some $1 \le j \le 4$. Now, by Lemma 3.1, $(j)_k = (b-1)\xi_{R_{k+1}\dots}(1/b, 1/a)$ for some ultimately periodic word $R_{k+1}\dots$ sharing a large prefix with $\mathbf{s}_{\theta,\rho}$ (or $\mathbf{s}'_{\theta,\rho}$). It follows that

$$\xi = [0, A_1, A_2, \dots].$$

Since a and b are integers, observe that the numbers

$$c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_2, f_2, \dots$$

are integers, except possibly $c_0 = \frac{b^{a_1-b_1}a-b}{b-1}$. But c_0 is a nonnegative integer if we assume that a is congruent to 1 modulo $b-1$. Then the A_j are positive integers and $A_1, A_2, \dots$ is the sequence of partial quotients of ξ. Theorem 1.3 is proved. ∎

Proof of Theorem 1.5. Assume first that a is congruent to 1 modulo $b-1$. Let ξ denote one of the numbers $(b-1)\xi_{\mathbf{s}_{\theta,\rho}}(1/b, 1/a)$ or $(b-1)\xi_{\mathbf{s}'_{\theta,\rho}}(1/b, 1/a)$ and let ξ' be the corresponding number with $a = 1$. We denote by $(P_j/Q_j)_{j\ge 1}$

(resp. $(P'_j/Q'_j)_{j\ge 1}$) the sequence of convergents to ξ (resp. ξ'). We claim that

$$Q'_j \gg\ll Q_j^{\varphi}, \quad \text{where} \quad \varphi = \frac{\log b}{\log ba^{\theta}}, \tag{3.1}$$

as j tends to infinity. Indeed, it follows from the proof of Theorem 1.3 that each convergent P_j/Q_j coincides with one of the fractions

$$(1)_k, (2)_k \dotplus (1)_k, (2)_k, (3)_k, (4)_k,$$

for some k. Moreover, if say $P_j/Q_j = (3)_k$, then $P'_j/Q'_j = (3)'_k$, where $(3)'_k$ stands for the corresponding fraction with $a = 1$, observing that reductions (i) and (ii) occurring in Theorem 1.3 are independent of a and b (they depend only on the two sequences $(a_k)_{k\ge 1}$ and $(b_k)_{k\ge 1}$). It follows that

$$(ba^{\theta})^{r_{k+1}+q_k} \ll Q_j = \frac{b^{r_{k+1}} a^{\tilde{r}_{k+1}} (b^{q_k} a^{p_k} - 1)}{b-1} \ll (ba^{\theta})^{r_{k+1}+q_k},$$

by using Lemma 2.4, while

$$b^{r_{k+1}+q_k} \ll Q'_j = \frac{b^{r_{k+1}} (b^{q_k} - 1)}{b-1} \le b^{r_{k+1}+q_k}.$$

Thus, (3.1) holds true in this case. The other cases are similar.

Now, the theory of continued fractions and (3.1) imply that the irrationality exponents $\mu(\xi)$ and $\mu(\xi')$ of ξ and ξ' are equal, since they are given by the formulae

$$\mu(\xi) = 1 + \limsup_{j\to+\infty} \frac{\log Q_{j+1}}{\log Q_j} = 1 + \limsup_{j\to+\infty} \frac{\log Q'_{j+1}}{\log Q'_j} = \mu(\xi').$$

As already mentioned, Theorem 1.5 holds true for ξ', and thus for ξ, by [7, Theorem 2.4].

If a is not assumed to be congruent to 1 modulo $b-1$, then A_1 is a positive rational number whose denominator divides $b-1$, while $A_2, A_3, \ldots$ are positive integers. Define

$$\frac{P_j}{Q_j} = [0; A_1, \ldots, A_j], \quad j \ge 1,$$

or equivalently

$$\begin{pmatrix} Q_j & Q_{j-1} \\ P_j & P_{j-1} \end{pmatrix} = \begin{pmatrix} A_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} A_j & 1 \\ 1 & 0 \end{pmatrix}, \quad j \ge 1.$$

Then the P_j are integers and the Q_j are rational numbers with denominators dividing $b-1$. The sequence $(P_j/Q_j)_{j\ge 1}$ does not necessarily coincide with the sequence of convergents to ξ. However, the inequalities

$$\left| \xi - \frac{P_j}{Q_j} \right| \le \frac{1}{Q_j Q_{j+1}}, \quad j \ge 1$$

remain true, and the above argument remains valid. ∎

4. Functional equations and expansions of Hecke–Mahler series. In this Section we give analytical formulae involving Hecke–Mahler series which will reveal to be useful for the proof of Theorem 1.1.

Let us begin with a relation between the fractions $(3)_k$ and $(4)_k$. Here, unless otherwise stated, we consider α and β as variables and work in the ring of power series $\mathbb{Q}[[\alpha,\beta]]$. We have

$$(3)_k = (4)_{k-1} + (-1)^k\left(\frac{1}{\beta}-1\right)^2\frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}}{1-\beta^{q_k}\alpha^{p_k}}. \tag{4.1}$$

We stress that here and in this section the + sign between fractions denotes the usual addition and not the Farey addition, denoted by $\dot{+}$ in the previous section.

The equality (4.1) is proved in [6, Lemma 2.2] for $\alpha = 1$. The general case is similar.

For $k \geq 0$, set $\mathbf{v}_k = V_k^\infty$ and

$$\gamma_k = \beta^{q_k}\alpha^{p_k}, \qquad \sigma_k = \sum_{n=1}^{q_k} v_n^{(k)}\beta^n\alpha^{\sum_{h=1}^n v_h^{(k)}},$$

where $V_k := v_1^{(k)}v_2^{(k)}\dots v_{q_k}^{(k)}$ (while the first q_k-1 letters of V_k coincide with those of $\mathbf{s}_{\theta,\rho}$, the last letter of V_k may differ from s_{q_k}), so that, by Lemma 3.1,

$$\xi_{\mathbf{v}_k}(\beta,\alpha) = \frac{\sigma_k}{1-\gamma_k} = \frac{(4)_{k-1}}{\frac{1}{\beta}-1}$$

for $k \geq 1$. We have

$$\sigma_0 = 0, \qquad \sigma_1 = \beta^{a_1-b_1}\alpha, \quad \gamma_0 = \beta, \qquad \gamma_1 = \beta^{a_1}\alpha.$$

The recursion relations between the words V_k yield

Lemma 4.1. *For any $k \geq 1$, the numerators σ_k satisfy the linear recurrence relation*

$$\sigma_{k+1} = \frac{1-\gamma_{k+1}-\gamma_k^{a_{k+1}-b_{k+1}}(1-\gamma_{k-1})}{1-\gamma_k}\sigma_k + \gamma_k^{a_{k+1}-b_{k+1}}\sigma_{k-1}.$$

It follows that

$$\begin{aligned}(4)_k - (4)_{k-1} &= (-1)^k\frac{\alpha\beta\left(\frac{1}{\beta}-1\right)^2\prod_{h=0}^k\gamma_h^{a_{h+1}-b_{h+1}}}{(1-\gamma_{k+1})(1-\gamma_k)}\\ &= (-1)^k\frac{\alpha\beta\left(\frac{1}{\beta}-1\right)^2\beta^{\sum_{h=0}^k(a_{h+1}-b_{h+1})q_h}\alpha^{\sum_{h=0}^k(a_{h+1}-b_{h+1})p_h}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})}\\ &= (-1)^k\left(\frac{1}{\beta}-1\right)^2\frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})}\end{aligned}$$

and

$$(3)_{k+1}-(3)_k=(-1)^k\left(\frac{1}{\beta}-1\right)^2 \times \frac{\beta^{r_{k+1}+q_{k+1}+q_k}\alpha^{\tilde r_{k+1}+p_{k+1}+p_k}-\beta^{r_{k+2}+q_{k+1}}\alpha^{\tilde r_{k+2}+p_{k+1}}(1-\beta^{q_k}\alpha^{p_k})}{(1-\beta^{q_k}\alpha^{p_k})(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})}.$$

Proof. Note that V_k has length q_k and contains p_k letters 1. Then, we deduce from the word equation

$$V_{k+1}=V_k^{a_{k+1}-b_{k+1}}V_{k-1}V_k^{b_{k+1}}$$

the equality

$$\begin{aligned}\sigma_{k+1}&=(1+\gamma_k+\cdots+\gamma_k^{a_{k+1}-b_{k+1}-1})\sigma_k+\gamma_k^{a_{k+1}-b_{k+1}}\sigma_{k-1}\\&\quad+\gamma_k^{a_{k+1}-b_{k+1}}\gamma_{k-1}(1+\cdots+\gamma_k^{b_{k+1}-1})\sigma_k\\&=\frac{1-\gamma_k^{a_{k+1}-b_{k+1}}+\gamma_k^{a_{k+1}-b_{k+1}}\gamma_{k-1}(1-\gamma_k^{b_{k+1}})}{1-\gamma_k}\sigma_k+\gamma_k^{a_{k+1}-b_{k+1}}\sigma_{k-1}\\&=\frac{1-\gamma_{k+1}-\gamma_k^{a_{k+1}-b_{k+1}}(1-\gamma_{k-1})}{1-\gamma_k}\sigma_k+\gamma_k^{a_{k+1}-b_{k+1}}\sigma_{k-1},\end{aligned}$$

since $\gamma_k^{a_{k+1}}\gamma_{k-1}=\gamma_{k+1}$. Observe now that the denominators $1-\gamma_k$ satisfy obviously the same linear relation

$$\begin{aligned}1-\gamma_{k+1}=&\frac{1-\gamma_{k+1}-\gamma_k^{a_{k+1}-b_{k+1}}(1-\gamma_{k-1})}{1-\gamma_k}(1-\gamma_k)\\&+\gamma_k^{a_{k+1}-b_{k+1}}(1-\gamma_{k-1}).\end{aligned}$$

It follows that

$$(1-\gamma_k)\sigma_{k+1}-(1-\gamma_{k+1})\sigma_k=-\gamma_k^{a_{k+1}-b_{k+1}}\big((1-\gamma_{k-1})\sigma_k-(1-\gamma_k)\sigma_{k-1}\big).$$

Going down inductively to $k=1$, we obtain

$$\begin{aligned}(1-\gamma_k)\sigma_{k+1}-(1-\gamma_{k+1})\sigma_k&=(-1)^k\big((1-\gamma_0)\sigma_1-(1-\gamma_1)\sigma_0\big)\prod_{h=1}^{k}\gamma_h^{a_{h+1}-b_{h+1}}\\&=(-1)^k(1-\beta)\alpha\prod_{h=0}^{k}\gamma_h^{a_{h+1}-b_{h+1}}\\&=(-1)^k\left(\frac{1}{\beta}-1\right)\beta^{r_{k+1}+q_k}\alpha^{\tilde r_{k+1}+p_k},\end{aligned}$$

by using Lemma 2.1 and noting that $\sigma_0=0$ and $\sigma_1=\beta^{a_1-b_1}\alpha=\gamma_0^{a_1-b_1}\alpha$. The formulae for $(4)_k-(4)_{k-1}$ immediately follow. For the difference $(3)_{k+1}-(3)_k$, we use moreover equality (4.1) to obtain

$$\begin{aligned}
(3)_{k+1} - (3)_k &= (4)_k - (4)_{k-1} \\
&\quad + (-1)^k \left(\frac{1}{\beta} - 1\right)^2 \left(-\frac{\beta^{r_{k+2}+q_{k+1}} \alpha^{\tilde{r}_{k+2}+p_{k+1}}}{1-\beta^{q_{k+1}}\alpha^{p_{k+1}}} - \frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}}{1-\beta^{q_k}\alpha^{p_k}}\right) \\
&= (-1)^k \left(\frac{1}{\beta} - 1\right)^2 \left(\frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}}{(1-\beta^{q_k}\alpha^{p_k})(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})}\right. \\
&\quad \left. - \frac{\beta^{r_{k+2}+q_{k+1}}\alpha^{\tilde{r}_{k+2}+p_{k+1}}(1-\beta^{q_k}\alpha^{p_k}) + \beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})}{(1-\beta^{q_k}\alpha^{p_k})(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})}\right) \\
&= (-1)^k \left(\frac{1}{\beta} - 1\right)^2 \\
&\quad \times \frac{\beta^{r_{k+1}+q_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_{k+1}+p_k} - \beta^{r_{k+2}+q_{k+1}}\alpha^{\tilde{r}_{k+2}+p_{k+1}}(1-\beta^{q_k}\alpha^{p_k})}{(1-\beta^{q_k}\alpha^{p_k})(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})}.
\end{aligned}$$

The proof is complete. ∎

COROLLARY 4.2. *We have the following formulae for the Hecke–Mahler series:*

$$\begin{aligned}
\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha) &= (1-\beta)\alpha \sum_{k\geq 0} (-1)^k \frac{\prod_{h=0}^k \gamma_h^{a_{h+1}-b_{h+1}}}{(1-\gamma_{k+1})(1-\gamma_k)} \\
&= (1-\beta)\alpha \sum_{k\geq 0} (-1)^k \frac{\beta^{\sum_{h=0}^k (a_{h+1}-b_{h+1})q_h}\alpha^{\sum_{h=0}^k (a_{h+1}-b_{h+1})p_h}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})} \\
&= \frac{1-\beta}{\beta} \sum_{k\geq 0} (-1)^k \frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})}
\end{aligned}$$

and

$$\begin{aligned}
\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha) &= \frac{1-\beta}{\beta}\left(\frac{\beta^{r_1+1}\alpha - \beta^{r_2+q_1}\alpha^{\tilde{r}_2+p_1}(1-\beta^{q_0}\alpha^{p_0})}{(1-\beta^{q_1}\alpha^{p_1})(1-\beta^{q_0}\alpha^{p_0})}\right. \\
&\left. + \sum_{k\geq 1}(-1)^k \frac{\beta^{r_{k+1}+q_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_{k+1}+p_k} - \beta^{r_{k+2}+q_{k+1}}\alpha^{\tilde{r}_{k+2}+p_{k+1}}(1-\beta^{q_k}\alpha^{p_k})}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})}\right).
\end{aligned}$$

Proof. We use the telescopic sums

$$\begin{aligned}
\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha) &= \frac{(4)_0}{\frac{1}{\beta}-1} + \sum_{k\geq 1} \frac{(4)_k - (4)_{k-1}}{\frac{1}{\beta}-1} \\
&= \frac{(3)_1}{\frac{1}{\beta}-1} + \sum_{k\geq 1} \frac{(3)_{k+1} - (3)_k}{\frac{1}{\beta}-1},
\end{aligned}$$

which, combined with Lemma 4.1, give rise to the terms in the sums with index $k \geq 1$. It remains to compute $(4)_0/(1/\beta - 1)$ and $(3)_1/(1/\beta - 1)$. We have

$$\frac{(4)_0}{\frac{1}{\beta}-1} = \frac{\sigma_1}{1-\gamma_1} = \frac{\alpha\beta^{a_1-b_1}}{1-\alpha\beta^{a_1}} = \alpha(1-\beta)\frac{\gamma_0^{a_1-b_1}}{(1-\gamma_0)(1-\gamma_1)}$$
$$= \frac{1-\beta}{\beta}\,\frac{\beta^{r_1+q_0}\alpha^{\tilde{r}_1+p_0}}{(1-\beta^{q_1}\alpha^{p_1})(1-\beta^{q_0}\alpha^{p_0})},$$

which establishes the first three expressions for $\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha)$.

We now deal with

$$\frac{(3)_1}{\frac{1}{\beta}-1} = \xi_{R_2 M_1^\infty}(\beta,\alpha).$$

Assume first that $a_2-b_2\geq 1$. Then

$$R_2M_1^\infty = R_1M_1^{a_2-b_2-1}M_0M_1^\infty = 0^{a_1-b_1-1}1(0^{a_1-1}1)^{a_2-b_2-1}0(0^{a_1-1}1)^\infty.$$

It follows that

$$\xi_{R_2M_1^\infty}(\beta,\alpha) = \beta^{a_1-b_1}\alpha\big(1+\beta^{a_1}\alpha+\cdots+(\beta^{a_1}\alpha)^{a_2-b_2-1}\big)$$
$$+\beta^{a_1-b_1+a_1(a_2-b_2-1)+1}\alpha^{a_2-b_2}\frac{\beta^{a_1}\alpha}{1-\beta^{a_1}\alpha}$$
$$= \frac{\beta^{u_1}\alpha^{v_1}-\beta^{u_2}\alpha^{v_2}+\beta^{u_3}\alpha^{v_2}}{1-\beta^{a_1}\alpha}$$

with

$$u_1 = a_1-b_1 = r_1, \qquad v_1 = 1,$$
$$u_2 = a_1-b_1+a_1(a_2-b_2) = r_2+q_1-1, \qquad v_2 = a_2-b_2+1 = \tilde{r}_2+p_1,$$
$$u_3 = a_1a_2+1+a_1-b_1-b_2a_1 = r_2+q_1.$$

Thus,

$$\frac{(3)_1}{\frac{1}{\beta}-1} = \frac{1-\beta}{\beta}\,\frac{\beta^{r_1+1}\alpha-\beta^{r_2+q_1}\alpha^{\tilde{r}_2+p_1}(1-\beta)}{(1-\beta^{a_1}\alpha)(1-\beta)},$$

as asserted. The fourth expression for $\xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha)$ is established when $a_2-b_2\geq 1$. For $a_2=b_2$, we have $R_2=0$, $r_1=a_1$. The computations are similar and simpler. ∎

As an example, for the characteristic Sturmian word $\mathbf{c}_\theta$ we have $b_k=0$ for every $k\geq 1$. Then, it follows from

$$\sum_{j=0}^{k} a_{j+1}q_j = a_1+\sum_{j=1}^{k}(q_{j+1}-q_{j-1}) = q_{k+1}+q_k-1, \qquad k\geq 0, \tag{4.2}$$

$$\sum_{j=0}^{k} a_{j+1}p_j = \sum_{j=1}^{k}(p_{j+1}-p_{j-1}) = p_{k+1}+p_k-1, \qquad k\geq 0 \tag{4.3}$$

that

$$\sum_{h=0}^{k}(a_{h+1}-b_{h+1})q_h = q_{k+1}+q_k-1, \qquad \sum_{h=0}^{k}(a_{h+1}-b_{h+1})p_h = p_{k+1}+p_k-1.$$

Thus, we recover the known formula

$$\xi_{\mathbf{c}_\theta}(\beta,\alpha)=\left(\frac{1}{\beta}-1\right)\sum_{k\geq 0}(-1)^k\frac{\beta^{q_{k+1}+q_k}\alpha^{p_{k+1}+p_k}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})},$$

which is usually obtained as a consequence of the functional equation for the Hecke–Mahler series.

Conversely, a functional chain of equations of Mahler's type can be deduced from our formula for an arbitrary Sturmian word $\mathbf{s}$. For $m\geq 0$, put $\theta_m=[0,a_{m+1},a_{m+2},\dots]$ and denote by $\mathbf{s}_m$ the Sturmian word with slope θ_m and formal intercept $b_{m+1},b_{m+2},\dots$ (see Definition 2.3). Observe that $\mathbf{s}=\mathbf{s}_0$. With our notation, we have $\xi_{\mathbf{s}}(\beta,\alpha)=\xi_{\mathbf{s}_0}(\gamma_0,\gamma_{-1})$, where $\gamma_{-1}=\beta^{q_{-1}}\alpha^{p_{-1}}=\alpha$.

Proposition 4.3. *For any $m\geq 1$, we have the relation of Mahler's type*

$$\begin{aligned}\xi_{\mathbf{s}}(\beta,\alpha)&=(1-\beta)\alpha\sum_{k=0}^{m-1}(-1)^k\frac{\beta^{\sum_{h=0}^{k}(a_{h+1}-b_{h+1})q_h}\alpha^{\sum_{h=0}^{k}(a_{h+1}-b_{h+1})p_h}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})}\\&\quad+(-1)^m\frac{(1-\beta)\alpha}{(1-\beta^{q_m}\alpha^{p_m})\beta^{q_{m-1}}\alpha^{p_{m-1}}}\beta^{\sum_{h=0}^{m-1}(a_{h+1}-b_{h+1})q_h}\alpha^{\sum_{h=0}^{m-1}(a_{h+1}-b_{h+1})p_h}\\&\qquad\times\xi_{\mathbf{s}_m}(\beta^{q_m}\alpha^{p_m},\beta^{q_{m-1}}\alpha^{p_{m-1}})\\&=(1-\beta)\alpha\\&\quad\times\left(\sum_{k=0}^{m-1}(-1)^k\frac{\prod_{h=0}^{k}\gamma_h^{a_{h+1}-b_{h+1}}}{(1-\gamma_{k+1})(1-\gamma_k)}+(-1)^m\frac{\prod_{h=0}^{m-1}\gamma_h^{a_{h+1}-b_{h+1}}}{(1-\gamma_m)\gamma_{m-1}}\xi_{\mathbf{s}_m}(\gamma_m,\gamma_{m-1})\right)\\&=(1-\beta)\alpha\left(\frac{\sigma_m}{1-\gamma_m}+(-1)^m\frac{\prod_{h=0}^{m-1}\gamma_h^{a_{h+1}-b_{h+1}}}{(1-\gamma_m)\gamma_{m-1}}\xi_{\mathbf{s}_m}(\gamma_m,\gamma_{m-1})\right).\end{aligned}$$

Proof. We truncate the sum giving $\xi_{\mathbf{s}}(\beta,\alpha)$ at the order m and consider the remaining terms

$$\begin{aligned}(1-\beta)\alpha&\sum_{k\geq m}(-1)^k\frac{\beta^{\sum_{h=0}^{k}(a_{h+1}-b_{h+1})q_h}\alpha^{\sum_{h=0}^{k}(a_{h+1}-b_{h+1})p_h}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})}\\&=(-1)^m(1-\beta)\alpha\beta^{\sum_{h=0}^{m-1}(a_{h+1}-b_{h+1})q_h}\alpha^{\sum_{h=0}^{m-1}(a_{h+1}-b_{h+1})p_h}\\&\quad\times\sum_{k\geq 0}(-1)^k\frac{\beta^{\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})q_{m+h}}\alpha^{\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})p_{m+h}}}{(1-\beta^{q_{m+k+1}}\alpha^{p_{m+k+1}})(1-\beta^{q_{m+k}}\alpha^{p_{m+k}})}.\end{aligned}$$

Now, we claim that the last sum over $k \geq 0$ is equal to

$$(1-\beta^{q_m}\alpha^{p_m})^{-1}(\beta^{q_{m-1}}\alpha^{p_{m-1}})^{-1}\xi_{\mathbf{s}_m}(\beta^{q_m}\alpha^{p_m}, \beta^{q_{m-1}}\alpha^{p_{m-1}}).$$

Indeed, let $(u_n^{(m)}/v_n^{(m)})_{n\geq 0}$ be the convergents of θ_m. We have

$$\frac{u_0^{(m)}}{v_0^{(m)}} = \frac{0}{1}, \quad \frac{u_1^{(m)}}{v_1^{(m)}} = \frac{1}{a_{m+1}}, \quad \frac{u_2^{(m)}}{v_2^{(m)}} = \frac{a_{m+2}}{a_{m+1}a_{m+2}+1}, \ \ldots,$$

and we easily check that, for any $h \geq 0$,

$$q_{m+h} = v_h^{(m)} q_m + u_h^{(m)} q_{m-1} \quad \text{and} \quad p_{m+h} = v_h^{(m)} p_m + u_h^{(m)} p_{m-1}.$$

It follows that we can write the exponents in a form involving the convergents of θ_m:

$$\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})q_{m+h}$$
$$= \Big(\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})v_h^{(m)}\Big)q_m + \Big(\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})u_h^{(m)}\Big)q_{m-1}$$

and

$$\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})p_{m+h}$$
$$= \Big(\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})v_h^{(m)}\Big)p_m + \Big(\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})u_h^{(m)}\Big)p_{m-1}.$$

Thus

$$\beta^{\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})q_{m+h}}\alpha^{\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})p_{m+h}}$$
$$= (\beta^{q_m}\alpha^{p_m})^{\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})v_h^{(m)}} \times (\beta^{q_{m-1}}\alpha^{p_{m-1}})^{\sum_{h=0}^{k}(a_{m+h+1}-b_{m+h+1})u_h^{(m)}},$$

and

$$1-\beta^{q_{m+k+1}}\alpha^{p_{m+k+1}} = 1-(\beta^{q_m}\alpha^{p_m})^{v_{k+1}^{(m)}}(\beta^{q_{m-1}}\alpha^{p_{m-1}})^{u_{k+1}^{(m)}},$$
$$1-\beta^{q_{m+k}}\alpha^{p_{m+k}} = 1-(\beta^{q_m}\alpha^{p_m})^{v_k^{(m)}}(\beta^{q_{m-1}}\alpha^{p_{m-1}})^{u_k^{(m)}}.$$

The last claim follows from the equalities

$$(1-\beta)\alpha\Big(\sum_{k=0}^{m-1}(-1)^k\frac{\prod_{h=0}^{k}\gamma_h^{a_{h+1}-b_{h+1}}}{(1-\gamma_{k+1})(1-\gamma_k)}\Big) = \frac{\sigma_m}{1-\gamma_m} - \frac{\sigma_0}{1-\gamma_0} = \frac{\sigma_m}{1-\gamma_m}.$$

The proof is complete. ∎

5. When the slope has unbounded partial quotients. The purpose of this section is to establish Theorem 1.1 when the slope θ has unbounded partial quotients. In this case, an application of Liouville's inequality is sufficient to conclude. We use the logarithmic Weil height h and Liouville's inequality in the form

$$\log|\zeta| \geq -[\mathbb{Q}(\zeta):\mathbb{Q}]h(\zeta) \tag{5.1}$$

for any nonzero algebraic number ζ. There is some similarity with the proof of [10, Theorem 6].

We make use of the approximations $\frac{\beta}{1-\beta}(4)_{k-1}$ and $\frac{\beta}{1-\beta}(3)_k$ to $\xi = \xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha)$, considered in Section 4. If U and V are positive quantities depending upon k, let us write $U \asymp V$ to indicate that there exist positive constants c, c' such that $cU \leq V \leq c'U$ for large k.

LEMMA 5.1. *We have the estimates*

$$\left|\xi - \frac{\beta}{1-\beta}(4)_{k-1}\right| \asymp |\beta\alpha^\theta|^{u_k+q_k}$$

with

$$u_k = \begin{cases} r_{k+1} & \text{if } a_{k+2} - b_{k+2} \geq 1, \\ r_k + q_{k+1} & \text{if } a_{k+2} = b_{k+2}, \end{cases}$$

and

$$\left|\xi - \frac{\beta}{1-\beta}(3)_k\right| \asymp |\beta\alpha^\theta|^{v_k+q_{k+1}}$$

with

$$v_k = \begin{cases} r_{k+1} + q_k & \text{if } a_{k+2} - b_{k+2} \geq 2, \\ r_{k+1} + 2q_k & \text{if } a_{k+2} - b_{k+2} = 1, a_{k+3} - b_{k+3} \geq 1, \\ r_{k+1} + q_k & \text{if } a_{k+2} = 1, b_{k+2} = 0, a_{k+3} = b_{k+3}, \\ r_k & \text{if } a_{k+2} = b_{k+2}. \end{cases}$$

Proof. Let us set, for $k \geq 1$,

$$\Gamma_k = \frac{1-\beta}{\beta}(-1)^k \frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})},$$

and

$$\Delta_k = \frac{1-\beta}{\beta}(-1)^k \times \frac{\beta^{r_{k+1}+q_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_{k+1}+p_k} - \beta^{r_{k+2}+q_{k+1}}\alpha^{\tilde{r}_{k+2}+p_{k+1}}(1-\beta^{q_k}\alpha^{p_k})}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})}.$$

Recalling Lemma 4.1 and the representations of $\xi = \xi_{\mathbf{s}_{\theta,\rho}}(\beta,\alpha)$ given in Corollary 4.2, we have

$$\xi - \frac{\beta}{1-\beta}(4)_{k-1} = \sum_{h\geq k}\Gamma_h \quad \text{and} \quad \xi - \frac{\beta}{1-\beta}(3)_k = \sum_{h\geq k}\Delta_h.$$

We now estimate the above two sums. For $\sum_{h\ge k}\varGamma_h$, observe that the two sequences of exponents

$$r_{h+1}+q_h = 1+\sum_{j=0}^{h}(a_{j+1}-b_{j+1})q_j, \qquad h=k,k+1,\dots,$$

and

$$\tilde{r}_{h+1}+p_h = 1+\sum_{j=0}^{h}(a_{j+1}-b_{j+1})p_j, \qquad h=k,k+1,\dots,$$

occurring in the quantities $\varGamma_h$, are nondecreasing. Moreover, $r_{h+1}+q_h = r_{h+2}+q_{h+1}$ if and only if $a_{h+2}=b_{h+2}$, and $r_{h+2}+q_{h+1}\ge r_{h+1}+q_h+q_{h+1}$ if $a_{h+2}>b_{h+2}$. Notice also that we cannot have $r_{h+1}+q_h = r_{h+2}+q_{h+1} = r_{h+3}+q_{h+2}$, since the simultaneous equalities $a_{h+2}=b_{h+2}$ and $a_{h+3}=b_{h+3}$ are forbidden by Ostrowski's rules.

In view of Lemma 2.4, we have

$$|\varGamma_h| \asymp |\beta\alpha^\theta|^{r_{h+1}+q_h}, \qquad h\ge k.$$

In order to estimate $\sum_{h\ge k}\varGamma_h$, we distinguish two cases. Assume first that $a_{k+2}-b_{k+2}\ge 1$. Then

$$|\varGamma_k| \asymp |\beta\alpha^\theta|^{r_{k+1}+q_k} \quad \text{and} \quad \Big|\sum_{h\ge k+1}\varGamma_h\Big| \ll |\beta\alpha^\theta|^{r_{k+1}+q_k+q_{k+1}}.$$

Taking into account the preceding observations, it follows that

$$\Big|\sum_{h\ge k}\varGamma_h\Big| \asymp |\beta\alpha^\theta|^{r_{k+1}+q_k}.$$

Assume secondly that $a_{k+2}=b_{k+2}$. Then

$$\begin{aligned}|\varGamma_k+\varGamma_{k+1}| &= \left|\frac{1-\beta}{\beta}\right|\left|\frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}}{1-\beta^{q_{k+1}}\alpha^{p_{k+1}}}\left(\frac{1}{1-\beta^{q_k}\alpha^{p_k}}-\frac{1}{1-\beta^{q_{k+2}}\alpha^{p_{k+2}}}\right)\right| \\ &= \left|\frac{1-\beta}{\beta}\cdot\frac{\beta^{r_{k+1}+q_k}\alpha^{\tilde{r}_{k+1}+p_k}(\beta^{q_{k+2}}\alpha^{p_{k+2}}-\beta^{q_k}\alpha^{p_k})}{(1-\beta^{q_{k+1}}\alpha^{p_{k+1}})(1-\beta^{q_k}\alpha^{p_k})(1-\beta^{q_{k+2}}\alpha^{p_{k+2}})}\right|,\end{aligned}$$

so that

$$|\varGamma_k+\varGamma_{k+1}| \asymp |\beta\alpha^\theta|^{r_{k+1}+2q_k} = |\beta\alpha^\theta|^{r_k+q_k+q_{k+1}}.$$

Now, since $a_{k+3}>b_{k+3}$, we get

$$r_{k+3}+q_{k+2}-(r_k+q_{k+1}) = r_{k+3}+q_{k+2}-(r_{k+2}+q_{k+1}) \ge q_{k+2},$$

so that

$$\Big|\sum_{h\ge k+2}\varGamma_h\Big| \ll |\beta\alpha^\theta|^{r_k+q_{k+1}+q_{k+2}}.$$

It follows that

$$\Big|\sum_{h\ge k}\varGamma_h\Big| \asymp |\beta\alpha^\theta|^{r_k+q_k+q_{k+1}}.$$

We now briefly deal with the sum $\sum_{h\geq k} \Delta_h$. Observe that

$$|\Delta_h| \asymp \begin{cases} |\beta\alpha^\theta|^{r_{h+1}+q_{h+1}+q_h} & \text{if } a_{h+2}-b_{h+2}\geq 2,\\ |\beta\alpha^\theta|^{r_{h+1}+q_{h+1}+2q_h} & \text{if } a_{h+2}-b_{h+2}=1,\\ |\beta\alpha^\theta|^{r_h+q_{h+1}} & \text{if } a_{h+2}=b_{h+2}. \end{cases}$$

Looking at the absolute value of Δ_k and Δ_{k+1} according to the above cases, we check that

$$\Big|\sum_{h\geq k}\Delta_h\Big| \asymp |\Delta_k|$$

unless $a_{k+2}=1, b_{k+2}=0$ and $a_{k+3}=b_{k+3}$, in which case

$$\Big|\sum_{h\geq k}\Delta_h\Big| \asymp |\Delta_{k+1}|.$$

It follows that

$$\Big|\sum_{h\geq k}\Delta_h\Big| \asymp |\beta\alpha^\theta|^{v_k+q_{k+1}},$$

as asserted. Lemma 5.1 is proved. ∎

We are now able to prove Theorem 1.1 when θ has unbounded partial quotients. Assume on the contrary that ξ is algebraic. We distinguish two cases.

Assume first that r_{k+1}/q_k takes arbitrarily large values and set

$$\zeta = \xi - \frac{\beta}{1-\beta}(4)_{k-1}.$$

Lemma 5.1 shows, for large k, that ζ is nonzero and that

$$\log|\zeta| \ll -(u_k+q_k) \ll -r_{k+1},$$

since we always have $u_k \geq r_{k+1}$. But the algebraic number ζ has height $h(\zeta) \ll q_k$. This contradicts Liouville's inequality (5.1) provided that we have chosen k such that r_{k+1}/q_k is large enough.

Assume now that the sequence $(r_{k+1}/q_k)_{k\geq 1}$ is bounded. Set

$$\zeta = \xi - \frac{\beta}{1-\beta}(3)_k.$$

Again Lemma 5.1 implies that ζ is nonzero and

$$\log|\zeta| \ll -(v_k+q_{k+1}) \ll -q_{k+1}$$

when k is large enough. But the algebraic number ζ has now height

$$h(\zeta) \ll r_{k+1}+q_k \ll q_k,$$

by assumption. Since θ has unbounded partial quotients, the quotient q_{k+1}/q_k takes arbitrarily large values. This yields a contradiction with Liouville's inequality (5.1).

6. Functional transcendence. A general idea underlying Mahler's method is that the transcendence over $\mathbb{Q}(z)$ of a function $f(z)$ in $\mathbb{Z}[[z]]$ is transferred to the transcendence of the value of f at every nonzero algebraic point in the open unit disc. Therefore, we need a functional transcendence statement.

PROPOSITION 6.1. *Let θ, ρ be real numbers such that $0 \le \theta, \rho < 1$ and θ is irrational. Then the function $z \mapsto \xi_{\mathbf{s}_{\theta,\rho}}(z, 1)$ is transcendental over $\mathbb{C}(z)$. Consequently, the function $(z_1, z_2) \mapsto \xi_{\mathbf{s}_{\theta,\rho}}(z_1, z_2)$ is transcendental over $\mathbb{C}(z_1, z_2)$.*

Proof. Observe that an algebraic function, say $f(z)$, holomorphic in the open unit disc, can be analytically extended over a neighborhood of a point z_0 on the unit circle if we assume that z_0 is neither a root of the discriminant nor of the highest coefficient of the minimal polynomial of $f(z)$ over $\mathbb{C}[z]$.

Therefore, it is sufficient to show that $z \mapsto \xi_{\mathbf{s}_{\theta,\rho}}(z, 1)$ cannot be extended beyond the unit circle. The case of $\rho = 0$ has been treated by Hecke [9]. His argument extends easily to an arbitrary value of ρ. For the sake of completeness, we give the details below. Set

$$F(z) = \sum_{n\ge 1} \{n\theta + \rho\} z^n.$$

Recall that if, for a power series $\sum_{n\ge 1} c_n z^n$, we have

$$\lim_{t\to+\infty} \frac{1}{t} \sum_{n=1}^{t} c_n = c,$$

then

$$\lim_{r\to 1_-} (1 - r) \sum_{n=1}^{+\infty} c_n r^n = c,$$

where $r \to 1_-$ means that the real number r tends to 1 and is less than 1. Let t be a positive integer. Write

$$S(t) = \sum_{n=1}^{t} \{n\theta + \rho\} \mathrm{e}^{2\mathrm{i}\pi n\alpha}$$

and take $\alpha = q\theta$ for a nonzero integer q. We have

$$S(t) = \sum_{n=1}^{t} \{n\theta + \rho\} \mathrm{e}^{2\mathrm{i}\pi nq\theta} = \mathrm{e}^{-2\mathrm{i}\pi q\rho} \sum_{n=1}^{t} \{n\theta + \rho\} \mathrm{e}^{2\mathrm{i}\pi q\{n\theta+\rho\}}.$$

As θ is irrational, the sequence $(\{n\theta + \rho\})_{n\ge 1}$ is equidistributed in $[0, 1]$, thus

$$\lim_{t\to+\infty} \frac{1}{t} \sum_{n=1}^{t} f(\{n\theta + \rho\}) = \int_0^1 f(x)\,\mathrm{d}x$$

for every continuous function $f(x)$. Consequently,

$$\lim_{t\to+\infty}\frac{1}{t}S(t)=\frac{\mathrm{e}^{-2\mathrm{i}\pi q\rho}}{2\mathrm{i}\pi q}.$$

It then follows that

$$\lim_{r\to1_-}(1-r)\sum_{n=1}^{+\infty}\{n\theta+\rho\}(r\mathrm{e}^{2\mathrm{i}\pi q\theta})^n=\lim_{r\to1_-}(1-r)F(r\mathrm{e}^{2\mathrm{i}\pi q\theta})=\frac{\mathrm{e}^{-2\mathrm{i}\pi q\rho}}{2\mathrm{i}\pi q}.$$

Since the set of points of the form $\{q\theta\}$ is dense in $[0,1]$, the function F cannot be extended beyond the unit circle. The same conclusion holds for the function $z\mapsto\sum_{n\ge1}\lfloor n\theta+\rho\rfloor z^n$. ∎

7. Transcendence of Hecke–Mahler series at algebraic points. Loxton and van der Poorten [11] (see also [13, Section 2.9]) obtained a general transcendence theorem for chains of functional equations of Mahler's type, from which they deduced [11, Theorem 8] the transcendence of $F_{\theta,0}(\beta,\alpha)$, for every irrational number θ in $(0,1)$ and every nonzero complex algebraic numbers α,β with $|\beta\alpha^\theta|<1$ and $\beta^{q_k}\alpha^{p_k}\ne1$ for $k\ge1$, where p_k/q_k is the kth convergent to θ.

We follow the presentation of Nishioka [13], with some simplification. In her book, the size $\|\alpha\|$ of an algebraic number α is the maximum of the absolute values of the conjugates of α and of its denominator. The function $\log\|\cdot\|$ is thus comparable to the logarithmic Weil height h, which we are using.

For a 2×2 matrix $\Omega=(\omega_{i,j})$ with nonnegative integer coefficients and a point (z_1,z_2) in $\mathbb{C}^2$, we define a map $\Omega:\mathbb{C}^2\to\mathbb{C}^2$ by

$$\Omega(z_1,z_2)=(z_1^{\omega_{1,1}}z_2^{\omega_{1,2}},z_1^{\omega_{2,1}}z_2^{\omega_{2,2}}).$$

Let $(\Omega_k)_{k\ge1}$ be a sequence of matrices with nonnegative integer coefficients. Let K be a number field and α_1,α_2 be nonzero elements in K. Write

$$(\alpha_1^{(k)},\alpha_2^{(k)})=\Omega_k(\alpha_1,\alpha_2),\qquad k\ge1.$$

Let

$$f_k(z_1,z_2)=\sum_{\lambda_1,\lambda_2\ge0}\sigma^{(k)}_{\lambda_1,\lambda_2}z_1^{\lambda_1}z_2^{\lambda_2},\qquad k\ge0,$$

be power series. We assume that $\underline{\sigma}^{(k)}:=(\sigma^{(k)}_{\lambda_1,\lambda_2})_{\lambda_1,\lambda_2\ge0}$ are bounded sequences of integers. For a collection $\underline{s}=(s_{\lambda_1,\lambda_2})_{\lambda_1,\lambda_2\ge0}$ of variables, set

$$F(z_1,z_2;\underline{s})=\sum_{\lambda_1,\lambda_2\ge0}s_{\lambda_1,\lambda_2}z_1^{\lambda_1}z_2^{\lambda_2}.$$

Then we have

$$F(z_1,z_2;\underline{\sigma}^{(k)})=f_k(z_1,z_2),\qquad k\ge0.$$

Assume that there exist positive real numbers $r_1, r_2, \dots$ such that $(r_k)_{k\geq 1}$ tends to infinity and the following conditions are satisfied:

(i) Every coefficient of Ω_k is $\ll r_k$ for $k \geq 1$.

(ii) There exist positive real numbers η_1, η_2 which are linearly independent over the rationals and such that

$$\log|\alpha_i^{(k)}| \sim -\eta_i r_k, \quad i = 1, 2, \text{ as } k \to +\infty.$$

(iii) For $k \geq 1$, there exist u_k, v_k in K such that

$$f_k(\Omega_k(\alpha_1, \alpha_2)) = u_k f_0(\alpha_1, \alpha_2) + v_k, \quad h(u_k), h(v_k) \ll r_k.$$

(iv) If p is a positive integer, $P_0(z_1, z_2; \underline{s}), \dots, P_p(z_1, z_2; \underline{s})$ are polynomials in z_1, z_2 and in the variables s_{λ_1,λ_2}, with coefficients in K, and

$$E(z_1, z_2; \underline{s}) = \sum_{j=0}^{p} P_j(z_1, z_2; \underline{s}) F(z_1, z_2; \underline{s})^j = \sum_{\lambda_1,\lambda_2 \geq 0} P_{\lambda_1,\lambda_2}(\underline{s}) z_1^{\lambda_1} z_2^{\lambda_2},$$

then there exist nonnegative λ_1, λ_2 with the following property: If k is sufficiently large and $P_0(z_1, z_2; \underline{\sigma}^{(k)}), \dots, P_p(z_1, z_2; \underline{\sigma}^{(k)})$ are not all zero, then $P_{\lambda_1,\lambda_2}(\underline{\sigma}^{(k)})$ is nonzero.

Assumptions (i), (ii), and (iii) correspond exactly to Assumptions (I), (II), and (III) in [13]. Assumption (IV) in [13] is clearly satisfied since the coefficients of the series f_k are integers and are bounded. We can simplify Assumption (V) in [13] to (iv) since $\underline{\sigma}^{(k)}, k \geq 0$, take only finitely many values in our case.

Theorem 7.1 (Loxton–van der Poorten). *Under assumptions* (i)–(iv) *above, the complex number $f_0(\alpha_1, \alpha_2)$ is transcendental.*

Our presentation slightly differs from that of [11], where the authors have to cope with admissibility conditions on α_1 and α_2. Here, we have formulated assumption (iii) with u_k, v_k in K, and not with functions $u_k(\alpha_1, \alpha_2), v_k(\alpha_1, \alpha_2)$ in $K(\alpha_1, \alpha_2)$, in which case we should have excluded the pairs (α_1, α_2) at which these functions are not defined.

We show how Theorem 7.1 applies to establish Theorem 1.1 when the slope θ has bounded partial quotients.

For $m \geq 0$, recall that $\theta_m = [0, a_{m+1}, a_{m+2}, \dots]$ and that $\mathbf{s}_m$ denotes the Sturmian word with slope θ_m and formal intercept $b_{m+1}, b_{m+2}, \dots$ (see Definition 2.3), as in Section 4. Let us start with the following estimates.

Proposition 7.2. *Let α and β be complex algebraic numbers such that $0 < |\beta\alpha^\theta| < 1$ and $\beta \neq 1$. If there exists ℓ such that $\beta^{q_\ell}\alpha^{p_\ell} = 1$, then put $m_0 = \ell + 1$, otherwise put $m_0 = 1$. For any $m \geq m_0$, there exist A_m and B_m such that*

$$\xi_{\mathbf{s}_m}(\beta^{q_m}\alpha^{p_m}, \beta^{q_{m-1}}\alpha^{p_{m-1}}) = A_m + B_m \xi_{\mathbf{s}}(\beta, \alpha)$$

and

$$h(A_m), h(B_m) \leq c q_m$$

for some positive constant c depending only upon α and β.

Proof. Here, α and β denote complex algebraic numbers satisfying $0 < |\beta\alpha^\theta| < 1$ and $\beta \neq 1$. Recall that $\gamma_k = \beta^{q_k}\alpha^{p_k}$ for $k \geq 0$. Let $m \geq m_0$ be an integer. Then $\gamma_m \neq 1$ and

$$A'_m = \frac{\sigma_m}{1-\gamma_m}, \qquad B'_m = (-1)^m \frac{\prod_{j=0}^{m-1} \gamma_j^{a_{j+1}-b_{j+1}}}{(1-\gamma_m)\gamma_{m-1}}$$

are well defined. Proposition 4.3 asserts that

$$\xi_{\mathbf{s}}(\beta,\alpha) = (1-\beta)\alpha\big(A'_m + B'_m \xi_{\mathbf{s}_m}(\gamma_m,\gamma_{m-1})\big).$$

It is sufficient to prove that

$$h(A'_m), h(B'_m) \ll_{\alpha,\beta} q_m$$

to establish the proposition, where $\ll_{\alpha,\beta}$ means that the implied constant depends only on α and on β. We have

$$\begin{aligned} h\Big(\prod_{j=0}^{m-1} \gamma_j^{a_{j+1}-b_{j+1}}\Big) &\leq \sum_{j=0}^{m-1} a_{j+1} h(\gamma_j) \leq \sum_{j=0}^{m-1} a_{j+1}(q_j h(\beta) + p_j h(\alpha)) \\ &\leq (q_m + q_{m-1})h(\beta) + (p_m + p_{m-1})h(\alpha), \end{aligned}$$

by (4.2) and (4.3). This implies that $h(B'_m) \ll_{\alpha,\beta} q_m$.

To estimate the height of

$$\sigma_m = \sum_{n=1}^{q_m} v_n^{(m)} \beta^n \alpha^{\sum_{h=1}^n v_h^{(m)}},$$

first note that its denominator is bounded from above by the q_mth power of the product of the denominators of α and β. Let $M \geq 2$ be an upper bound for the moduli of the conjugates of α and β. Then the modulus of any conjugate of σ_m is at most $M^2 + \cdots + M^{2q_m}$, thus less than M^{2q_m+1}. We conclude that $h(A'_m) \ll_{\alpha,\beta} q_m$, as asserted.

Hence, the heights of the coefficients

$$A_m = -\frac{A'_m}{B'_m} \quad \text{and} \quad B_m = \frac{1}{(1-\beta)\alpha B'_m}$$

satisfy the required estimate. ∎

We are now equipped to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 when θ has bounded partial quotients. Let θ and ρ be as in the statement of the theorem. Assume that θ has bounded partial quotients. We apply Theorem 7.1 with $\alpha_1 = \beta$ and $\alpha_2 = \alpha$, which yields Theorem 1.1, once we have checked that assumptions (i)–(iv) are satisfied.

Let M be a positive integer such that $a_k \le M$ for $k \ge 1$. By Ostrowski's numeration rules, the bound $b_k \le M$ holds for any $k \ge 1$ as well. Recall that p_k/q_k denotes the kth convergent to θ. Let m_0 be as in Proposition 7.2. By compactness, there exist an increasing sequence $(\nu_k)_{k\ge1}$ of positive integers, with $\nu_1 \ge m_0$, integers $g_1, g_2, \dots, a'_1, a'_2, \dots$ in $\{1, \dots, M\}$, and integers $b'_1, b'_2, \dots$ in $\{0, \dots, M\}$ such that

$$\begin{aligned}
(a_{\nu_k}, a_{\nu_k-1}, a_{\nu_k-2}, \dots) &\to (g_1, g_2, g_3, \dots), \qquad k \to \infty,\\
(a_{\nu_k+1}, a_{\nu_k+2}, a_{\nu_k+3}, \dots) &\to (a'_1, a'_2, a'_3, \dots), \qquad k \to \infty,\\
(b_{\nu_k+1}, b_{\nu_k+2}, b_{\nu_k+3}, \dots) &\to (b'_1, b'_2, b'_3, \dots), \qquad k \to \infty.
\end{aligned}$$

Note that $0 \le b'_1 \le a'_1 - 1$, $0 \le b'_k \le a'_k$ for $k \ge 1$, and $b'_{k+1} = a'_{k+1}$ implies $b'_k = 0$ for every $k \ge 1$, so that Ostrowski numeration rules hold as well for the limit sequences. As k tends to infinity, the Sturmian word $\mathbf{s}_{\nu_k}$ with slope θ_{ν_k} and formal intercept $b_{\nu_k+1}, b_{\nu_k+2}, \dots$ tends to the Sturmian word $\mathbf{t}$ with slope $[0; a'_1, a'_2, \dots]$ and formal intercept $b'_1, b'_2, \dots$.

Set

$$\phi := [0; g_1, g_2, \dots].$$

Observe that ϕ is irrational and, by the theory of continued fractions,

$$\lim_{k\to+\infty} \frac{p_{\nu_k-1}}{p_{\nu_k}} = \lim_{k\to+\infty} \frac{q_{\nu_k-1}}{q_{\nu_k}} = \phi.$$

Set

$$\Omega_k = \begin{pmatrix} q_{\nu_k} & p_{\nu_k} \\ q_{\nu_k-1} & p_{\nu_k-1} \end{pmatrix} \quad \text{and} \quad r_k = q_{\nu_k},$$

for any $k \ge 1$. Thus, assumption (i) is satisfied.

Since $0 < |\alpha_1 \alpha_2^{\theta}| < 1$,

$$\begin{aligned}
\lim_{k\to+\infty} \frac{\log|\alpha_1^{(k)}|}{r_k} &= \lim_{k\to+\infty} \frac{q_{\nu_k}\log|\alpha_1| + p_{\nu_k}\log|\alpha_2|}{r_k} = \log|\alpha_1| + \theta\log|\alpha_2|,\\
\lim_{k\to+\infty} \frac{\log|\alpha_2^{(k)}|}{r_k} &= \lim_{k\to+\infty} \frac{q_{\nu_k-1}\log|\alpha_1| + p_{\nu_k-1}\log|\alpha_2|}{r_k}\\
&= \phi(\log|\alpha_1| + \theta\log|\alpha_2|),
\end{aligned}$$

and ϕ is irrational, assumption (ii) is satisfied.

Put

$$f_0(z_1, z_2) = \xi_{\mathbf{s}}(z_1, z_2), \qquad f_k(z_1, z_2) = \xi_{\mathbf{s}_{\nu_k}}(z_1, z_2), \qquad k \ge 1,$$

and write

$$\begin{aligned}
f_k(z_1, z_2) &= \sum_{\lambda_1, \lambda_2 \ge 0} \sigma^{(k)}_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2}, \qquad k \ge 1,\\
\xi_{\mathbf{t}}(z_1, z_2) &= \sum_{\lambda_1, \lambda_2 \ge 0} \sigma_{\lambda_1,\lambda_2} z_1^{\lambda_1} z_2^{\lambda_2},
\end{aligned}$$

where all the coefficients $\sigma^{(k)}_{\lambda_1,\lambda_2}$ and $\sigma_{\lambda_1,\lambda_2}$ belong to $\{0,1\}$. Set

$$\underline{\sigma}^{(k)} = (\sigma^{(k)}_{\lambda_1,\lambda_2})_{\lambda_1,\lambda_2\geq 0} \quad \text{and} \quad \underline{\sigma} = (\sigma_{\lambda_1,\lambda_2})_{\lambda_1,\lambda_2\geq 0}.$$

By construction, the sequences $\underline{\sigma}^{(k)}$ converge to $\underline{\sigma}$ for the discrete topology on $\{0,1\}^{\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}}$ as k tends to infinity.

Moreover, it follows from Proposition 7.2 that assumption (iii) is satisfied with $u_k = B_{q_{\nu_k}}$ and $v_k = A_{q_{\nu_k}}$.

As noted above, we have

$$\lim_{k\to+\infty} f_k(z_1,z_2) = \xi_{\mathbf{t}}(z_1,z_2).$$

Furthermore, it follows from Proposition 6.1 that $\xi_{\mathbf{t}}(z_1,z_2)$ is a transcendental function over $\mathbb{C}(z_1,z_2)$, since $\theta' = [0; a'_1, a'_2, \ldots]$ is irrational.

Let p be a positive integer and $P_0(z_1,z_2;\underline{s}),\ldots,P_p(z_1,z_2;\underline{s})$ be polynomials as in (iv). Let $E(z_1,z_2;\underline{s})$ be as above. We have

$$\lim_{k\to+\infty} P_j(z_1,z_2;\underline{\sigma}^{(k)}) = P_j(z_1,z_2;\underline{\sigma}), \qquad \lim_{k\to+\infty} P_{\lambda_1,\lambda_2}(\underline{\sigma}^{(k)}) = P_{\lambda_1,\lambda_2}(\underline{\sigma}),$$

since the sequences $\underline{\sigma}^{(k)}$ converge to the sequence $\underline{\sigma}$.

If $P_j(z_1,z_2;\underline{\sigma})$, $0\leq j\leq p$, are all zero, then the polynomials $P_j(z_1,z_2;\underline{\sigma}^{(k)})$ vanish identically for k suficiently large, thus assumption (iv) is clearly satisfied. Otherwise, $E(z_1,z_2;\underline{\sigma})$ is not zero, since $\xi_{\mathbf{t}}$ is a transcendental function. Consequently, there exist nonnegative λ_1,λ_2 such that $P_{\lambda_1,\lambda_2}(\underline{\sigma})$ is nonzero. Hence, for all k sufficiently large, $P_{\lambda_1,\lambda_2}(\underline{\sigma}^{(k)})$ is not zero, and assumption (iv) is satisfied.

All this shows that Theorem 7.1 applies and yields Theorem 1.1 when the slope θ has bounded partial quotients. ∎

Acknowledgements. We are grateful to the referee for a very careful reading.

References

[1] B. Adamczewski and Y. Bugeaud, *Dynamics for β-shifts and Diophantine approximation*, Ergodic Theory Dynam. Systems 27 (2007), 1695–1711.

[2] B. Adamczewski et Y. Bugeaud, *Nombres réels de complexité sous-linéaire: mesures d'irrationalité et de transcendance*, J. Reine Angew. Math. 658 (2011), 65–98.

[3] B. Adamczewski et C. Faverjon, *Méthode de Mahler: relations linéaires, transcendance et applications aux nombres automatiques*, Proc. London Math. Soc. 115 (2017), 55–90.

[4] P. E. Böhmer, *Über die Transzendenz gewisser dyadischer Brüche*, Math. Ann. 96 (1927), 367–377.

[5] J. M. Borwein and P. B. Borwein, *On the generating function of the integer part: $[n\alpha+\gamma]$*, J. Number Theory 43 (1993), 293–318.

[6] Y. Bugeaud, D. H. Kim, M. Laurent, and A. Nogueira, *On the Diophantine nature of the elements of Cantor sets arising in the dynamics of contracted rotations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) 22 (2021), 1691–1704.

[7] Y. Bugeaud and M. Laurent, *Combinatorial structure of Sturmian words and continued fraction expansions of Sturmian numbers*, Ann. Inst. Fourier (Grenoble), to appear.

[8] S. Ferenczi and Ch. Mauduit, *Transcendence of numbers with a low complexity expansion*, J. Number Theory 67 (1997), 146–161.

[9] E. Hecke, *Über analytische Funktionen und die Verteilung von Zahlen mod. eins*, Abh. Math. Sem. Univ. Hamburg 1 (1922), 54–76.

[10] T. Komatsu, *A certain power series and the inhomogeneous continued fraction expansions*, J. Number Theory 59 (1996), 291–312.

[11] J. H. Loxton and A. J. van der Poorten, *Arithmetic properties of certain functions in several variables III*, Bull. Austral. Math. Soc. 16 (1977), 15–47.

[12] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. 101 (1929), 342–366.

[13] Ku. Nishioka, *Mahler Functions and Transcendence*, Lecture Notes in Math. 1631, Springer, 1996.

[14] Ku. Nishioka, I. Shiokawa and J. Tamura, *Arithmetical properties of a certain power series*, J. Number Theory 42 (1992), 61–87.

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