

The Thue–Morse continued fractions in characteristic 2 are algebraic

by

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1. Introduction. An infinite sequence $(a_n)_{n \geq 0}$ over a finite alphabet \mathcal{A} is *k-automatic* for an integer $k \geq 2$ if it can be generated by a finite automaton which reads the representation in base k of a non-negative integer n from right to left and outputs an element a_n in \mathcal{A} . It is called *automatic* if it is *k-automatic* for some integer $k \geq 2$. In 1968, Cobham [14] conjectured that, for any integer $b \geq 2$, the base- b expansion of an irrational real number never forms an automatic sequence. This was confirmed in 2007 by Adamczewski and Bugeaud [1] (see [30, 3] for alternative proofs), who also established that if the Hensel expansion of an irrational p -adic number ξ is an automatic sequence, then ξ is transcendental. In a similar spirit, Bugeaud [10] proved that the sequence of partial quotients of an algebraic real number of degree at least 3 never forms an automatic sequence.

Analogous questions can be asked for power series over a finite field, but the answers are different. Throughout the paper, we let q denote a power of a prime number p , and \mathbb{F}_q denote the field with q elements. In 1979–1980 Christol et al. [12, 13] established that a power series in $\mathbb{F}_q((z^{-1}))$ is algebraic over $\mathbb{F}_q(z)$ if and only if the sequence of its coefficients is q -automatic (or equivalently p -automatic).

In analogy with the continued fraction algorithm for real numbers, there is a well-studied continued fraction algorithm for power series in $\mathbb{F}_q((z^{-1}))$, the partial quotients being non-constant polynomials in $\mathbb{F}_q[z]$. In both settings, eventually periodic expansions correspond to quadratic elements, but much more is known on the continued fraction expansion of algebraic power

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series than on that of algebraic real numbers. In 1949 Mahler [26] established that, for $p \geq 3$, the root

$$z^{-1} + z^{-p} + z^{-p^2} + \dots$$

in $\mathbb{F}_q((z^{-1}))$ of the polynomial $zX^p - zX + 1$ has unbounded partial quotients, in the sense that their degrees are unbounded. In the opposite direction, there exist power series which are algebraic of degree at least 3 and have bounded partial quotients (recall that analogous statements have not yet been proved, nor disproved, for real numbers): in 1976, Baum and Sweet [9] proved that the continued fraction expansion of the unique solution ξ_{BS} in $\mathbb{F}_2((z^{-1}))$ of the equation

$$zX^3 + X + z = 0$$

has all its partial quotients of degree at most 2.

In the late 80s, Mendès France asked whether the sequence of partial quotients of an algebraic power series in $\mathbb{F}_q((z^{-1}))$ is q -automatic, as soon as it takes only finitely many different values. A positive answer has been given by Allouche [5] and Allouche, Bétréma, and Shallit [6] for the algebraic power series of degree at least 3 in $\mathbb{F}_p((z^{-1}))$ (here, $p \geq 3$) which have been constructed by Mills and Robbins [27] and whose partial quotients are polynomials of degree 1; see also Lasjaunias and Yao [23, 24, 25]. However, in 1995 Mkaouar [28] (see also Yao [33] for an alternative proof) gave a negative answer to the question of Mendès France by establishing that the sequence of partial quotients of the Baum–Sweet power series ξ_{BS} is morphic, but not automatic. Recall that a sequence is called *morphic* if it is the image under a coding of a fixed point of a substitution. If the substitution can be chosen of constant length k , then the sequence is k -automatic. The equivalence between the two definitions of automatic sequences given here was established by Cobham [15]; see [8].

All this shows that the sequence of partial quotients of an algebraic power series may or may not be automatic. Conversely, very little is known about the Diophantine nature of a power series whose sequence of partial quotients is automatic, but not ultimately periodic. The only contribution to this question is a recent work of Hu and Han [19]. They proved that, for any distinct non-constant polynomials a and b in $\mathbb{F}_2[z]$ whose sum of degrees is at most 7, the power series whose sequence of partial quotients is the Thue–Morse sequence written over $\{a, b\}$ is algebraic of degree 4. Recall that the Thue–Morse word

$$\mathbf{t} = t_0 t_1 t_2 \dots = \text{abbabaabbaababbabaababbaabbabaab} \dots$$

over $\{a, b\}$ is defined by $t_0 = a$, $t_{2k} = t_k$ and $t_{2k+1} = a$ (resp., b) if $t_k = b$ (resp., a), for $k \geq 0$. The Thue–Morse sequence is the most famous automatic sequence and is not ultimately periodic.

To establish their result, Hu and Han made use of a Guess 'n' Prove method and implemented a program which takes a pair (a, b) of distinct non-constant polynomials as input and outputs its minimal defining polynomial and a complete proof. The time needed grows with the degrees of a and b . Hu and Han conjectured that their result holds more generally for every pair (a, b) of distinct non-constant polynomials.

In the present paper, built on [19], we confirm this conjecture. Our main results are stated in Section 2 and proved in Sections 4 and 5. We consider higher degree exponents of approximation in Section 3. Additional remarks are gathered in Section 6.

2. Results. Let a and b be distinct non-constant polynomials in $\mathbb{F}_q[z]$ and let $\xi_{q,a,b}$ denote the power series in $\mathbb{F}_q((z^{-1}))$ whose sequence of partial quotients is the Thue–Morse sequence \mathbf{t} over the alphabet $\{a, b\}$. Since \mathbf{t} is not ultimately periodic, $\xi_{q,a,b}$ is transcendental or algebraic of degree at least 3 over $\mathbb{F}_q(z)$. Furthermore, as \mathbf{t} begins with infinitely many palindromes, an argument given in the proof of Theorem 3.2 shows that $\xi_{q,a,b}$ and its square are very well simultaneously approximable by rational fractions with the same denominator, and consequently $\xi_{q,a,b}$ cannot be algebraic of degree 3. This proves that $\xi_{q,a,b}$ is transcendental or algebraic of degree at least 4 over $\mathbb{F}_q(z)$.

Our main result asserts that, in the case $q = 2$, any Thue–Morse continued fraction $\xi_{2,a,b} = \xi_{a,b}$ is algebraic of degree 4 over $\mathbb{F}_2(z)$.

THEOREM 2.1. *Let a, b be non-constant, distinct polynomials in $\mathbb{F}_2[z]$ and let*

$$\xi_{a,b} = [0; a, b, b, a, b, a, a, b, b, a, a, b, a, b, b, a, \dots]$$

be the power series in $\mathbb{F}_2((z^{-1}))$ whose sequence of partial quotients is the Thue–Morse sequence \mathbf{t} over the alphabet $\{a, b\}$. Then $\xi_{a,b}$ is algebraic of degree 4 over $\mathbb{F}_2(z)$. More precisely, setting

$$\begin{aligned} A_0 &= b(a+b)(a^2b^2 + a^2 + b^2) + a^2b^4, \\ A_1 &= ab(a+b)(a^2b^2 + a^2 + b^2), \\ A_2 &= a^2b^2(a^2b^2 + a^2 + b^2), \\ A_3 &= A_1, \\ A_4 &= a(a+b)(a^2b^2 + a^2 + b^2) + a^2b^4, \end{aligned}$$

we have

$$A_4\xi_{a,b}^4 + A_3\xi_{a,b}^3 + A_2\xi_{a,b}^2 + A_1\xi_{a,b} + A_0 = 0.$$

REMARK. In Theorem 2.1 the ground field \mathbb{F}_2 can be replaced by any field K of characteristic 2, and a and b by any non-constant, distinct polynomials in $K[z]$.

We define an absolute value $|\cdot|$ on the field $\mathbb{F}_q((z^{-1}))$ by setting $|0| = 0$ and, for any non-zero power series $\xi = \xi(z) = \sum_{h=-m}^{+\infty} a_h z^{-h}$ with $a_{-m} \neq 0$, by setting $|\xi| = q^m$. We write $\|\xi\|$ for the norm of the fractional part of ξ , that is, of the part of the series which comprises only the negative powers of z .

For a power series ξ in $\mathbb{F}_q((z^{-1}))$, let

$$C(\xi) = \liminf_{Q \in \mathbb{F}_q[z] \setminus \{0\}} |Q| \cdot \|Q\xi\|$$

denote its Lagrange constant. Clearly, $C(\xi)$ is an element of

$$\mathcal{L}_q = \{q^{-k} : k \geq 1\} \cup \{0\},$$

and it is positive if and only if the degrees of the partial quotients of ξ are bounded.

If $[a_0; a_1, a_2, \dots]$ denotes the continued fraction expansion of ξ and $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$ for $n \geq 1$, then

$$|q_n \xi - p_n| = |q_{n+1}|^{-1} = |a_{n+1} q_n|^{-1}.$$

Consequently, we have

$$C(\xi) = 2^{-\limsup_{n \rightarrow +\infty} \deg a_n}.$$

Exactly as in [26], we can check that the root $z^{-1} + z^{-4} + z^{-4^2} + \dots$ in $\mathbb{F}_2((z^{-1}))$ of $zX^4 + zX + 1$ is algebraic of degree 4 and has unbounded partial quotients, thus its Lagrange constant is 0. Since the degrees of a and b can be arbitrarily chosen in Theorem 2.1, we derive at once the following corollary.

COROLLARY 2.2. *There is algebraic power series of degree 4 in $\mathbb{F}_2((z^{-1}))$ with an arbitrarily prescribed Lagrange constant in \mathcal{L}_2 .*

A power series ξ in $\mathbb{F}_q((z^{-1}))$ is called *badly approximable* when the degrees of its partial quotients are uniformly bounded. We define the *spectrum* of a badly approximable power series ξ in $\mathbb{F}_q((z^{-1}))$ as the set of positive integers k such that ξ has infinitely many partial quotients of degree k .

PROBLEM 2.3. *Let \mathcal{N} be a finite set of positive integers. Does there exist a badly approximable power series ξ in $\mathbb{F}_q((z^{-1}))$ which is algebraic of degree at least 3 and whose spectrum is equal to \mathcal{N} ?*

Theorem 2.1 gives an affirmative answer to Problem 2.3 for $q = 2$ and any set \mathcal{N} of cardinality 2.

For a power q of a prime number p the class $H(q)$ of *hyperquadratic power series* in $\mathbb{F}_q((z^{-1}))$ has attracted special attention; see e.g. [21, 31]. It is composed of the irrational power series ξ in $\mathbb{F}_q((z^{-1}))$ for which there exist polynomials A, B, C, D in $\mathbb{F}_q[z]$ such that $AD - BC \neq 0$ and an integer s for which

$$\xi = \frac{A\xi^{p^s} + B}{C\xi^{p^s} + D}.$$

Once a power series is known to be algebraic, a natural question is to determine whether it belongs to the restricted class of hyperquadratic power series.

In another direction, since the pioneering works of Kolchin [20] and Os-good [29], formal derivation has been used to study rational approximation to power series. By differentiating with respect to z the minimal defining polynomial satisfied by an algebraic power series ξ in $\mathbb{F}_q((z^{-1}))$ of degree d , we see that ξ' , where the $'$ indicates the derivation on $\mathbb{F}_q((z^{-1}))$ with respect to z extending the derivation on $\mathbb{F}_q[z]$, can be expressed as a polynomial in ξ of degree at most $d - 1$.

We say that a power series ξ in $\mathbb{F}_q((z^{-1}))$ satisfies a Riccati differential equation if there are A, B, C in $\mathbb{F}_q(z)$ such that

$$\xi' = A\xi^2 + B\xi + C.$$

Clearly, any cubic power series satisfies a Riccati equation. This is also the case for any hyperquadratic power series, but the converse does not hold; see e.g. [22, Section 4].

PROPOSITION 2.4. *Let a, b be non-constant, distinct polynomials in $\mathbb{F}_2[z]$. Then the Thue–Morse continued fraction $\xi_{a,b}$ in $\mathbb{F}_2((z^{-1}))$ is differential-quadratic: it satisfies the Riccati equation*

$$[(ab(a + b))x]' = (ab)'(1 + x^2).$$

Furthermore, $\xi_{a,b}$ is not hyperquadratic.

REMARK. The Riccati equation in Proposition 2.4 is equivalent to

$$[(ab(a + b))x]' = [ab(1 + x^2)]'.$$

Its solutions are the power series ξ such that $ab(a + b)\xi + ab(1 + \xi^2)$ is a square, that is, such that $1 + (a + b)\xi + \xi^2$ is a linear combination of terms $a^i b^j$, with i and j odd.

The strategy of the proof of Theorem 2.1 is the following. After a careful study of the minimal defining polynomials $P_{a,b}(X)$ of $\xi_{a,b}$ found by Hu and Han [19] in the case where the sum of the degrees of a and b is at most 7, we have guessed the coefficients, expressed in terms of a and b , of the quartic polynomial $P_{a,b}(X)$ which vanishes at $\xi_{a,b}$, for *any* distinct, non-constant polynomials a and b . It then only remained for us to check that $P_{a,b}(\xi_{a,b}) = 0$. This step is, however, much more difficult than it may seem to be. Denoting by p_ℓ/q_ℓ (we drop the letters a, b) the ℓ th convergent of $\xi_{a,b}$ for $\ell \geq 1$, it is sufficient to check that $P_{a,b}(p_\ell/q_\ell)$ tends to 0 as ℓ tends to infinity along a subsequence of the integers. We focus on the indices ℓ which are powers of 4. We heavily use the properties of symmetry of the Thue–Morse sequence and we proceed by induction to show that $|P_{a,b}(p_{4^k}/q_{4^k})|$ is, for $k \geq 1$, less than $|q_{4^k}|^{-2}$ times some constant independent of k .

3. Higher degree exponents of approximation. Besides the rational approximation to a power series ξ in $\mathbb{F}_q((z^{-1}))$, we often consider the simultaneous rational approximation of ξ, ξ^2, \dots, ξ^n by rational fractions with the same denominator, as well as small values of the linear form $b_0 + b_1\xi + \dots + b_n\xi^n$ with coefficients in $\mathbb{F}_q[z]$. This leads us to introduce the exponents of approximation w_n and λ_n , defined below. For a survey of recent results on these exponents evaluated at real numbers, the reader is directed to [11].

The *height* $H(P)$ of a polynomial $P(X) = b_n(z)X^n + \dots + b_1(z)X + b_0(z)$ over $\mathbb{F}_q[z]$ is the maximum of the absolute values of its coefficients, that is, of $|b_0|, |b_1|, \dots, |b_n|$. Recall that the ‘fractional part’ $\|\cdot\|$ is defined by

$$\left\| \sum_{h=-m}^{+\infty} a_h z^{-h} \right\| = \left| \sum_{h=1}^{+\infty} a_h z^{-h} \right|,$$

for every power series $\xi = \sum_{h=-m}^{+\infty} a_h z^{-h}$ in $\mathbb{F}_q((z^{-1}))$.

DEFINITION 3.1. Let ξ be in $\mathbb{F}_q((z^{-1}))$. Let $n \geq 1$ be an integer. We let $w_n(\xi)$ denote the supremum of the real numbers w for which

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in polynomials $P(X)$ over $\mathbb{F}_q[z]$ of degree at most n . We let $\lambda_n(\xi)$ denote the supremum of the real numbers λ for which

$$0 < \max \{ \|Q(z)\xi\|, \dots, \|Q(z)\xi^n\| \} < q^{-\lambda \deg Q}$$

has infinitely many solutions in polynomials $Q(z)$ in $\mathbb{F}_q[z]$. For positive real numbers w, λ , set

$$B_n(\xi, w) = \liminf_{H(P) \rightarrow +\infty} H(P)^w \cdot |P(\xi)|,$$

$$B'_n(\xi, \lambda) = \liminf_{|Q| \rightarrow +\infty} |Q|^\lambda \cdot \max \{ \|Q(z)\xi\|, \dots, \|Q(z)\xi^n\| \}.$$

Let ξ be an algebraic power series in $\mathbb{F}_q((z^{-1}))$ of degree $d \geq 2$. Let $n \geq 1$ be an integer. We briefly show how a Liouville-type argument allows us to bound $w_n(\xi)$ from above. Denote by $\xi_1 = \xi, \xi_2, \dots, \xi_d$ the Galois conjugates of ξ . Let $P(X)$ be a non-zero polynomial in $\mathbb{F}_q[z](X)$. Then the product $P(\xi_1) \dots P(\xi_d)$ is a non-zero element of $\mathbb{F}_q(z)$, whose absolute value is bounded from below by $c_1(\xi)$ and which (as $c_2(\xi)$ and $c_3(\xi)$ below) is positive and depends only on ξ . Since $|P(\xi_j)|$, $j = 2, \dots, d$, are bounded from above by $c_2(\xi)$ times $H(P)$, we get

$$|P(\xi)| > c_3(\xi) H(P)^{-d+1}$$

and we derive

$$(3.1) \quad w_n(\xi) \leq d - 1, \quad n \geq 1.$$

In the particular case of $\xi_{a,b}$, this gives $w_3(\xi_{a,b}) \leq 3$, by Theorem 2.1. Actually, this inequality is an equality.

THEOREM 3.2. *The Thue–Morse power series $\xi_{a,b}$ in $\mathbb{F}_2((z^{-1}))$ satisfies*

$$\lambda_2(\xi_{a,b}) = 1, \quad w_2(\xi_{a,b}) = 3, \quad w_n(\xi_{a,b}) = 3, \quad \lambda_n(\xi_{a,b}) = 1/3, \quad n \geq 3.$$

Moreover, there are positive constants c_4, c_5 , depending only on a and b , such that

$|P(\xi_{a,b})| > c_4 H(P)^{-3}$ for every non-zero $P(X)$ in $\mathbb{F}_2[z](X)$ of degree ≤ 3 , and there are polynomials $P(X)$ in $\mathbb{F}_2[z](X)$ of degree 2 of arbitrarily large height such that

$$|P(\xi_{a,b})| < c_5 H(P)^{-3}.$$

Proof. Since the continued fraction expansion of $\xi_{a,b}$ begins with arbitrarily large palindromes, $\xi_{a,b}$ and its square are simultaneously well approximable by rational fractions with the same denominator. This argument appeared in [2] and we recall it below for the sake of completeness. For $k \geq 0$, let p_k/q_k denote the k th convergent to $\xi_{a,b}$. Recall that the Thue–Morse word $\mathbf{t} = t_0 t_1 t_2 \dots$ over $\{a, b\}$ is the fixed point starting with a of the uniform morphism τ defined by $\tau(a) = ab$ and $\tau(b) = ba$. Since the words $\tau^2(a) = abba$ and $\tau^2(b) = baab$, and the prefix of length 4 of $\mathbf{t} = abba \dots$, are palindromes, every prefix of \mathbf{t} of length a power of 4 is a palindrome. Put

$$\mathcal{M}_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{M}_b = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, for $k \geq 1$, we have

$$\mathcal{M}_{t_0} \mathcal{M}_{t_1} \dots \mathcal{M}_{t_{k-1}} = \begin{pmatrix} q_k & q_{k-1} \\ p_k & p_{k-1} \end{pmatrix}.$$

Take $k = 4^\ell$ for some positive integer ℓ . The matrix $\mathcal{M}_{t_0} \mathcal{M}_{t_1} \dots \mathcal{M}_{t_{k-1}}$ is symmetric since \mathcal{M}_a and \mathcal{M}_b are symmetric and $t_0 t_1 \dots t_{k-1}$ is a palindrome. Consequently, $p_k = q_{k-1}$. It then follows from $t_{k-1} = a$, $t_k = b$ and the theory of continued fractions in $\mathbb{F}_2((z^{-1}))$ that

$$\left| \xi_{a,b} - \frac{p_k}{q_k} \right| = \frac{1}{|b| |q_k|^2}, \quad \left| \xi_{a,b} - \frac{p_{k-1}}{q_{k-1}} \right| = \frac{1}{|a| |q_{k-1}|^2} = \frac{1}{|q_{k-1}| \cdot |q_k|},$$

thus

$$\left| \xi_{a,b}^2 - \frac{p_{k-1}}{q_k} \right| = \left| \left(\xi_{a,b} - \frac{p_k}{q_k} \right) \left(\xi_{a,b} + \frac{p_{k-1}}{q_{k-1}} \right) + \frac{\xi_{a,b}}{q_{k-1} q_k} \right| = \frac{1}{|q_k|^2},$$

since $|\xi_{a,b}| = 1/|a|$. We conclude that

$$\max \{ \|q_k \xi_{a,b}\|, \|q_k \xi_{a,b}^2\| \} = \frac{1}{|q_k|}.$$

This gives $\lambda_2(\xi_{a,b}) \geq 1$ and there is equality since $\lambda_1(\xi_{a,b}) = 1$. Furthermore, the quantity $B_2'(\xi_{a,b}, 1)$ is finite, since $\xi_{a,b}$ has bounded partial quotients.

By the power series field analogue of a classical transference inequality established by Aggarwal [4], we immediately deduce that $B_2(\xi_{a,b}, 3)$ is finite, thus $w_2(\xi_{a,b}) \geq 3$.

This lower bound holds for $w_2(\xi_{q,a,b})$ as well. Combined with (3.1), this shows that $\xi_{q,a,b}$ is transcendental or algebraic of degree at least 4, thereby establishing the assertion stated at the end of the first paragraph of Section 2.

Since $\xi_{a,b}$ is algebraic of degree 4, by Theorem 2.1, the Liouville-type result obtained below Definition 3.1 combined with the lower bound $w_2(\xi_{a,b}) \geq 3$ implies that $w_2(\xi_{a,b}) = w_3(\xi_{a,b}) = 3$ and

$$w_n(\xi_{a,b}) = 3, \quad \lambda_n(\xi_{a,b}) = 1/3, \quad n \geq 3,$$

the value of $\lambda_3(\xi_{a,b})$ being a consequence of a transference inequality established in [4]. ■

4. Proof of Theorem 2.1. We keep the notation of Theorem 2.1 and check that $\xi_{a,b}$ is a root of the equation

$$A_4 X^4 + \cdots + A_1 X + A_0 = 0.$$

In view of the discussion at the beginning of Section 2, this implies that $\xi_{a,b}$ is algebraic of degree 4.

The computation is easier if we replace a and b by their inverses, that is, if we consider the continued fraction

$$(4.1) \quad \zeta = [0; a^{-1}, b^{-1}, b^{-1}, a^{-1}, b^{-1}, a^{-1}, a^{-1}, b^{-1}, \dots].$$

DEFINITION 4.1. Set

$$M_0(a, b) = a \begin{pmatrix} a^{-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a & 0 \end{pmatrix}$$

and, for $k \geq 1$,

$$\begin{aligned} M_k(a, b) &= M_{k-1}(a, b) \cdot M_{k-1}(b, a)^2 \cdot M_{k-1}(a, b), \\ N_k(a, b) &= M_{k-1}(a, b) \cdot M_{k-1}(b, a). \end{aligned}$$

Observe that

$$M_1(a, b) = \begin{pmatrix} a^2 b^2 + b^2 + 1 & a^2 b + ab^2 + a \\ a^2 b + ab^2 + a & a^2 b^2 + a^2 \end{pmatrix}$$

and that, for $k \geq 1$, the matrix $M_k(a, b)$ is symmetric and of the form

$$\begin{pmatrix} 1 + \cdots + (ab)^{2^{2k-1}} & a + \cdots + (a+b)^{(2^{2k-1}+1)/3} (ab)^{2^{2k-2}} \\ a + \cdots + (a+b)^{(2^{2k-1}+1)/3} (ab)^{2^{2k-2}} & a^2 + \cdots + (ab)^{2^{2k-1}} \end{pmatrix}.$$

An immediate induction shows that its entries are polynomials in a and b , whose degrees in a (resp., in b) are at most equal to 2^{2k-1} .

For a polynomial $M(a, b)$ (or a matrix $M(a, b)$ whose coefficients are polynomials) in the variables a and b , write $\widetilde{M}(a, b)$ for the image of $M(a, b)$ under the involution which exchanges a and b , that is, $\widetilde{M}(a, b) = M(b, a)$.

It follows from the definition of $N_k(a, b)$ that its upper left and lower right entries are symmetrical in a and b , while its upper right entry is obtained from its lower left one by exchanging a and b . We introduce some further notation.

DEFINITION 4.2. Write

$$M_k(a, b) = \begin{pmatrix} Q_k(a, b) & P_k(a, b) \\ P_k(a, b) & R_k(a, b) \end{pmatrix}, \quad k \geq 0,$$

$$N_k(a, b) = \begin{pmatrix} U_k(a, b) & V_k(a, b) \\ V_k(b, a) & W_k(a, b) \end{pmatrix}, \quad k \geq 1.$$

To shorten the notation, we simply write

$$M_k = \begin{pmatrix} Q_k & P_k \\ P_k & R_k \end{pmatrix}, \quad N_k = \begin{pmatrix} U_k & V_k \\ \widetilde{V}_k & W_k \end{pmatrix}, \quad k \geq 1,$$

and observe that

$$U_k = \widetilde{U}_k, \quad W_k = \widetilde{W}_k, \quad k \geq 1.$$

Set

$$\tau = 1 + a + b, \quad n_k = 2^{2k-1}, \quad k \geq 1.$$

First, we establish several relations between the entries of M_k and \widetilde{M}_k .

PROPOSITION 4.3. For $k \geq 1$, we have

$$Q_k + R_k = \tau^{(2n_k+2)/3},$$

$$P_k + \widetilde{P}_k = (1 + \tau)\tau^{(2n_k-4)/3},$$

$$Q_k + \widetilde{R}_k = \tau^{(2n_k-4)/3}.$$

Proof. We proceed by induction. Check that

$$Q_1 + R_1 = a^2 + b^2 + 1 = \tau^2,$$

$$P_1 + \widetilde{P}_1 = a + b = 1 + \tau = (1 + \tau)\tau^0,$$

$$Q_1 + \widetilde{R}_1 = 2a^2b^2 + 2b^2 + 1 = 1 = \tau^0.$$

Set

$$m_k = (2n_k + 2)/3 = (2^{2k} + 2)/3, \quad k \geq 0,$$

and observe that

$$m_k = 4m_{k-1} - 2, \quad k \geq 1.$$

Since $M_k = N_k \tilde{N}_k$, we have

$$Q_k = U_k^2 + V_k^2, \quad R_k = \tilde{V}_k^2 + W_k^2, \quad P_k = U_k \tilde{V}_k + V_k W_k, \quad k \geq 1.$$

Let $k \geq 2$ be an integer and assume that the proposition holds for the index $k-1$. Since τ is invariant by exchanging a and b , it follows from the induction hypothesis that

$$R_{k-1} + Q_{k-1} = \tilde{R}_{k-1} + \tilde{Q}_{k-1} = \tau^{m_{k-1}}.$$

Consequently, we get

$$\begin{aligned} V_k + \tilde{V}_k &= Q_{k-1} \tilde{P}_{k-1} + P_{k-1} \tilde{R}_{k-1} + \tilde{Q}_{k-1} P_{k-1} + \tilde{P}_{k-1} R_{k-1} \\ &= \tilde{P}_{k-1} (Q_{k-1} + R_{k-1}) + P_{k-1} (\tilde{Q}_{k-1} + \tilde{R}_{k-1}) \\ &= (P_{k-1} + \tilde{P}_{k-1}) (Q_{k-1} + R_{k-1}) \\ &= (1 + \tau) \tau^{m_{k-1}-2} \tau^{m_{k-1}}. \end{aligned}$$

Likewise, we have

$$\begin{aligned} U_k + W_k &= Q_{k-1} \tilde{Q}_{k-1} + R_{k-1} \tilde{R}_{k-1} \\ &= Q_{k-1} (\tau^{m_{k-1}} + \tilde{R}_{k-1}) + \tilde{R}_{k-1} (Q_{k-1} + \tau^{m_{k-1}}) \\ &= \tau^{m_{k-1}} (Q_{k-1} + \tilde{R}_{k-1}) = \tau^{2m_{k-1}-2}. \end{aligned}$$

This gives

$$P_k + \tilde{P}_k = (V_k + \tilde{V}_k)(U_k + W_k) = (1 + \tau) \tau^{4m_{k-1}-4} = (1 + \tau) \tau^{m_k-2},$$

as expected.

We get

$$\begin{aligned} Q_k + R_k &= U_k^2 + V_k^2 + \tilde{V}_k^2 + W_k^2 = (U_k + W_k)^2 + (V_k + \tilde{V}_k)^2 \\ &= \tau^{4m_{k-1}-4} + (1 + \tau^2) \tau^{4m_{k-1}-4} = \tau^{4m_{k-1}-2} = \tau^{m_k}. \end{aligned}$$

Finally, we check that

$$Q_k + \tilde{R}_k = U_k^2 + W_k^2 = (U_k + W_k)^2 = \tau^{4m_{k-1}-4} = \tau^{m_k-2}. \quad \blacksquare$$

In the next lemma, we express P_k, Q_k, R_k in terms of a new auxiliary quantity denoted by Z_k .

LEMMA 4.4. *Let k be a positive integer and define*

$$Z_k = U_k + V_k.$$

Then

$$Q_k = Z_k^2, \quad R_k = Z_k^2 + \tau^{(2n_k+2)/3}, \quad P_k = Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k},$$

and

$$\tilde{Z}_k = Z_k + (1 + \tau) \tau^{(n_k-2)/3}.$$

Proof. Since $Q_k = U_k^2 + V_k^2$, we immediately get $Q_k = Z_k^2$. The expression of R_k in terms of Z_k then follows from Proposition 4.3.

The matrix M_k is the product, in a suitable order, of n_k copies of $M_0(a, b)$ and n_k copies of $M_0(b, a)$, whose determinants are respectively equal to a^2 and b^2 . The computation of the determinant of M_k then gives the relation

$$Q_k R_k + P_k^2 = (ab)^{2n_k}.$$

Consequently,

$$P_k^2 = Q_k(Q_k + \tau^{(2n_k+2)/3}) + (ab)^{2n_k},$$

thus

$$P_k = Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k}.$$

It follows from Proposition 4.3 that

$$\tilde{Q}_k + Q_k = \tau^{(2n_k+2)/3} + \tau^{(2n_k-4)/3} = (1 + \tau^2)\tau^{(2n_k-4)/3},$$

which implies the last equality of the lemma. ■

We are now in a position to express Z_{k+1} in terms of n_k and Z_k .

PROPOSITION 4.5. *For $k \geq 1$, we have*

$$Z_{k+1} = \tau^{n_k} Z_k + \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}.$$

Proof. We proceed by induction. Observe that

$$\begin{aligned} Z_1 &= U_1 + V_1 = ab + b + 1, \\ Z_2 &= a^4 b^4 + a^3 b + a^2 b^2 + a^2 + ab + ab^3 + b^2 + 1 \\ &\quad + a^3 b^2 + a^2 b + a^2 b^3 + b + b^3. \end{aligned}$$

Since

$$\tau^2 Z_1 + \tau(ab)^2 + (ab)^4 = (1 + a^2 + b^2)(ab + b + 1) + (1 + a + b)a^2 b^2 + a^4 b^4,$$

we see that the proposition holds for $k = 1$. Assuming that it holds for a positive integer k , we get

$$\begin{aligned} Z_{k+1} &= P_k \tilde{P}_k + Q_k \tilde{Q}_k + Q_k \tilde{P}_k + P_k \tilde{R}_k \\ &= P_k(\tilde{Q}_k + \tilde{R}_k) + (P_k + Q_k)(\tilde{Q}_k + \tilde{P}_k) \\ &= (Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k})\tau^{(2n_k+2)/3} \\ &\quad + (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})(\tau^{(n_k+1)/3} \tilde{Z}_k + (ab)^{n_k}) \\ &= \tau^{(2n_k+2)/3} Z_k^2 + \tau^{n_k+1} Z_k + (ab)^{n_k} \tau^{(2n_k+2)/3} \\ &\quad + \tau^{(n_k+1)/3} Z_k (\tau^{(n_k+1)/3} Z_k + (1 + \tau)\tau^{(2n_k-1)/3} + (ab)^{n_k}) \\ &\quad + (ab)^{n_k} (\tau^{(n_k+1)/3} Z_k + (1 + \tau)\tau^{(2n_k-1)/3} + (ab)^{n_k}) \\ &= \tau^{n_k+1} Z_k + (ab)^{n_k} \tau^{(2n_k+2)/3} + \tau^{n_k} Z_k (1 + \tau) \\ &\quad + (ab)^{n_k} ((1 + \tau)\tau^{(2n_k-1)/3} + (ab)^{n_k}) \\ &= \tau^{n_k} Z_k + \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}. \quad \blacksquare \end{aligned}$$

We deduce from Proposition 4.5 an equation of degree 4 satisfied by Z_k .

PROPOSITION 4.6. *For $k \geq 1$, we have*

$$Z_k^4 + \tau^{(2n_k-1)/3} Z_k^2 + (a+b)\tau^{n_k-1} Z_k + (b(a+b)\tau^2 + a^2)\tau^{(4n_k-8)/3} \\ + (a+b)(ab)^{n_k} \tau^{(2n_k-4)/3} + (ab)^{2n_k} = 0.$$

Proof. For $k \geq 1$, set

$$\Delta_k = Z_k^4 + \tau^{(2n_k-1)/3} Z_k^2 + (a+b)\tau^{n_k-1} Z_k + (b(a+b)\tau^2 + a^2)\tau^{(4n_k-8)/3}.$$

Note that

$$\Delta_1 = (1 + b^4 + a^4 b^4) + (1 + a + b)(1 + b^2 + a^2 b^2 + (a+b)(1 + b + ab)) \\ + b(a+b)(1 + a^2 + b^2) + a^2 \\ = a^4 b^4 + a^2 b^3 + a^3 b^2 = (a+b)(ab)^{n_1} \tau^{(2n_1-4)/3} + (ab)^{2n_1}.$$

Thus, the conclusion holds for $k = 1$. Let k be a positive integer. Since $n_{k+1} = 4n_k$, it follows from Proposition 4.5 that

$$\Delta_{k+1} = Z_{k+1}^4 + \tau^{(8n_k-1)/3} Z_{k+1}^2 + (a+b)\tau^{4n_k-1} Z_{k+1} \\ + (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} \\ = (\tau^{n_k} Z_k + \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k})^4 \\ + \tau^{(8n_k-1)/3} (\tau^{n_k} Z_k + \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k})^2 \\ + (a+b)\tau^{4n_k-1} (\tau^{n_k} Z_k + \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}) \\ + (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} \\ = \tau^{4n_k} Z_k^4 + \tau^{(8n_k-4)/3} (ab)^{4n_k} + (ab)^{8n_k} \\ + \tau^{(8n_k-1)/3} (\tau^{2n_k} Z_k^2 + \tau^{(4n_k-2)/3} (ab)^{2n_k} + (ab)^{4n_k}) \\ + (a+b)\tau^{4n_k-1} (\tau^{n_k} Z_k + \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}) \\ + (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} \\ = \tau^{4n_k} Z_k^4 + \tau^{(8n_k-1)/3} (\tau^{2n_k} Z_k^2) + (a+b)\tau^{4n_k-1} (\tau^{n_k} Z_k) \\ + (b(a+b)\tau^2 + a^2)\tau^{(16n_k-8)/3} + \tau^{(8n_k-4)/3} (ab)^{4n_k} \\ + \tau^{(8n_k-1)/3} (\tau^{(4n_k-2)/3} (ab)^{2n_k} + (ab)^{4n_k}) \\ + (a+b)\tau^{4n_k-1} (\tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}) + (ab)^{8n_k} \\ = \tau^{4n_k} (Z_k^4 + \tau^{(2n_k-1)/3} Z_k^2 + (a+b)\tau^{n_k-1} Z_k \\ + (b(a+b)\tau^2 + a^2)\tau^{(4n_k-8)/3}) + \tau^{(8n_k-4)/3} (1 + \tau)(ab)^{4n_k} \\ + \tau^{(8n_k-1)/3} \tau^{(4n_k-2)/3} (ab)^{2n_k} \\ + (a+b)\tau^{4n_k-1} (\tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}) + (ab)^{8n_k}$$

$$\begin{aligned}
&= \tau^{4n_k} \Delta_k + \tau^{(8n_k-4)/3} (a+b)(ab)^{4n_k} + \tau^{4n_k} (ab)^{2n_k} \\
&\quad + (a+b)\tau^{4n_k-1} (\tau^{(2n_k-1)/3} (ab)^{n_k}) + (ab)^{8n_k}.
\end{aligned}$$

Assuming

$$\Delta_k = (a+b)(ab)^{n_k} \tau^{(2n_k-4)/3} + (ab)^{2n_k},$$

we deduce that

$$\begin{aligned}
\Delta_{k+1} &= \tau^{4n_k} (a+b)(ab)^{n_k} \tau^{(2n_k-4)/3} + \tau^{4n_k} (ab)^{2n_k} \\
&\quad + \tau^{(8n_k-4)/3} (a+b)(ab)^{4n_k} + \tau^{4n_k} (ab)^{2n_k} \\
&\quad + (a+b)\tau^{4n_k-1} (\tau^{(2n_k-1)/3} (ab)^{n_k}) + (ab)^{8n_k} \\
&= \tau^{(8n_k-4)/3} (a+b)(ab)^{4n_k} + (ab)^{8n_k}.
\end{aligned}$$

Since $n_{k+1} = 4n_k$, this shows that the conclusion holds for every positive integer k . ■

Set

$$\begin{aligned}
a_2 &= (1+a+b)^2, & a_1 &= a_3 = (a+b)a_2, \\
a_0 &= a(a+b)a_2 + a^2, & a_4 &= b(a+b)a_2 + a^2.
\end{aligned}$$

To prove that

$$a_4 \zeta^4 + a_3 \zeta^3 + a_2 \zeta^2 + a_1 \zeta + a_0 = 0,$$

with ζ as in (4.1), it is sufficient to show that the absolute value of

$$\delta_k := a_4 P_k^4 + a_3 P_k^3 Q_k + a_2 P_k^2 Q_k^2 + a_1 P_k Q_k^3 + a_0 Q_k^4$$

tends to 0 as k tends to infinity. The next proposition is more precise.

PROPOSITION 4.7. *For $k \geq 1$, there exists a polynomial $T_k(X, Y)$ with coefficients in \mathbb{F}_2 such that $\delta_k = (ab)^{2n_k} T_k(a, b)$.*

Proof. We express δ_k in terms of Z_k . A lengthy computation based on Lemma 4.4 gives

$$\begin{aligned}
\delta_k &= a_4 P_k^4 + a_3 P_k^3 Q_k + a_2 P_k^2 Q_k^2 + a_1 P_k Q_k^3 + a_0 Q_k^4 \\
&= a_4 (P_k^4 + Q_k^4) + (a+b)^2 \tau^2 Q_k^4 + (a+b) \tau^2 P_k Q_k (P_k^2 + Q_k^2) + \tau^2 P_k^2 Q_k^2 \\
&= a_4 (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^4 + (a+b)^2 \tau^2 Q_k^4 \\
&\quad + (a+b) \tau^2 P_k Q_k (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 + \tau^2 P_k^2 Q_k^2 \\
&= a_4 (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^4 + (a+b)^2 \tau^2 Z_k^8 \\
&\quad + (a+b) \tau^2 P_k Z_k^2 (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 + \tau^2 P_k^2 Z_k^4 \\
&= a_4 (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^4 + (a+b)^2 \tau^2 Z_k^8 \\
&\quad + (a+b) \tau^2 (Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k}) Z_k^2 (\tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 \\
&\quad + \tau^2 (Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k})^2 Z_k^4
\end{aligned}$$

$$\begin{aligned}
&= a_4 \tau^{(4n_k+4)/3} Z_k^4 + a_4 (ab)^{4n_k} + (a+b)^2 \tau^2 Z_k^8 \\
&\quad + (a+b) \tau^2 Z_k^2 (Z_k^2 + \tau^{(n_k+1)/3} Z_k + (ab)^{n_k}) (\tau^{(2n_k+2)/3} Z_k^2 + (ab)^{2n_k}) \\
&\quad + \tau^2 (Z_k^4 + \tau^{(2n_k+2)/3} Z_k^2 + (ab)^{2n_k}) Z_k^4 \\
&= a_4 \tau^{(4n_k+4)/3} Z_k^4 + a_4 (ab)^{4n_k} + (a+b)^2 \tau^2 Z_k^8 \\
&\quad + (a+b) \tau^2 Z_k^2 (\tau^{(2n_k+2)/3} Z_k^4 + \tau^{3n_k+1} Z_k^3 + \tau^{(2n_k+2)/3} Z_k^2 (ab)^{n_k}) \\
&\quad + (a+b) \tau^2 Z_k^2 ((ab)^{2n_k} Z_k^2 + (ab)^{2n_k} \tau^{(n_k+1)/3} Z_k + (ab)^{3n_k}) \\
&\quad + \tau^2 (Z_k^8 + \tau^{(2n_k+2)/3} Z_k^6 + (ab)^{2n_k} Z_k^4) \\
&= a_4 \tau^{(4n_k+4)/3} Z_k^4 + a_4 (ab)^{4n_k} + (a+b)^2 \tau^2 Z_k^8 + (a+b) \tau^2 \tau^{(2n_k+2)/3} Z_k^6 \\
&\quad + (a+b) \tau^2 \tau^{3n_k+1} Z_k^5 + (a+b) \tau^2 \tau^{(2n_k+2)/3} Z_k^4 (ab)^{n_k} \\
&\quad + (a+b) \tau^2 (ab)^{2n_k} Z_k^4 + (a+b) \tau^2 (ab)^{2n_k} \tau^{(n_k+1)/3} Z_k^3 \\
&\quad + (a+b) \tau^2 Z_k^2 (ab)^{3n_k} + \tau^2 Z_k^8 + \tau^2 \tau^{(2n_k+2)/3} Z_k^6 + \tau^2 (ab)^{2n_k} Z_k^4 \\
&= (a_4 \tau^{(4n_k+4)/3} + \tau^2 (ab)^{2n_k} \\
&\quad + (a+b) \tau^2 \tau^{(2n_k+2)/3} (ab)^{n_k} + (a+b) \tau^2 (ab)^{2n_k}) Z_k^4 \\
&\quad + a_4 (ab)^{4n_k} + [(a+b)^2 \tau^2 + \tau^2] Z_k^8 \\
&\quad + [(a+b) \tau^2 \tau^{(2n_k+2)/3} + \tau^2 \tau^{(2n_k+2)/3}] Z_k^6 + (a+b) \tau^2 \tau^{3n_k+1} Z_k^5 \\
&\quad + (a+b) \tau^2 (ab)^{2n_k} \tau^{(n_k+1)/3} Z_k^3 + (a+b) \tau^2 Z_k^2 (ab)^{3n_k} \\
&= ((b(a+b) \tau^2 + a^2) \tau^{(4n_k+4)/3} + (a+b) \tau^{(2n_k+8)/3} (ab)^{n_k} + \tau^3 (ab)^{2n_k}) Z_k^4 \\
&\quad + (b(a+b) \tau^2 + a^2) (ab)^{4n_k} + \tau^4 Z_k^8 + \tau^{(2n_k+8)/3} Z_k^6 + (a+b) \tau^{3n_k+3} Z_k^5 \\
&\quad + (a+b) (ab)^{2n_k} \tau^{(n_k+7)/3} Z_k^3 + (a+b) \tau^2 Z_k^2 (ab)^{3n_k} \\
&= \tau^4 Z_k^8 + \tau^{(2n_k+11)/3} Z_k^6 + (a+b) \tau^{n_k+3} Z_k^5 \\
&\quad + [(b(a+b) \tau^2 + a^2) \tau^{(4n_k-5)/3} + (a+b) \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}] \tau^3 Z_k^4 \\
&\quad + (a+b) (ab)^{2n_k} \tau^{(n_k+7)/3} Z_k^3 + (a+b) \tau^2 (ab)^{3n_k} Z_k^2 \\
&\quad + (b(a+b) \tau^2 + a^2) (ab)^{4n_k}.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
\delta_k &= \tau^4 \Delta_k Z_k^4 + [(a+b) \tau^{(2n_k-1)/3} (ab)^{n_k} + (ab)^{2n_k}] \tau^3 Z_k^4 \\
&\quad + (a+b) (ab)^{2n_k} \tau^{(n_k+7)/3} Z_k^3 + (a+b) \tau^2 (ab)^{3n_k} Z_k^2 \\
&\quad + (b(a+b) \tau^2 + a^2) (ab)^{4n_k},
\end{aligned}$$

where

$$\Delta_k = Z_k^4 + \tau^{(2n_k-1)/3} Z_k^2 + (a+b) \tau^{n_k-1} Z_k + (b(a+b) \tau^2 + a^2) \tau^{(4n_k-8)/3}$$

has already been introduced in the proof of Proposition 4.6.

It follows from Proposition 4.6 that

$$\tau^4 \Delta_k + (a+b)\tau^{(2n_k-1)/3}(ab)^{n_k}\tau^3 = \tau^4(ab)^{2n_k}.$$

Consequently, we get

$$\begin{aligned} \delta_k &= \tau^4(ab)^{2n_k}Z_k^4 + (ab)^{2n_k}\tau^3Z_k^4 + (a+b)(ab)^{2n_k}\tau^{(n_k+1)/3+2}Z_k^3 \\ &\quad + (a+b)\tau^2(ab)^{3n_k}Z_k^2 + (b(a+b)\tau^2 + a^2)(ab)^{4n_k}. \end{aligned}$$

This shows that δ_k can be written as $(ab)^{2n_k}$ times a polynomial in a and b , which depends on k . ■

To conclude the proof of Theorem 2.1, we explain the relationship between P_k, Q_k and the convergents $p_\ell/q_\ell = p_\ell(a, b)/q_\ell(a, b)$ of the Thue–Morse continued fraction

$$\xi_{a,b} = [0; a, b, b, a, b, a, a, b, \dots].$$

We have

$$q_{4k}(a, b) = (ab)^{n_k}Q_k(a^{-1}, b^{-1}), \quad p_{4k}(a, b) = (ab)^{n_k}P_k(a^{-1}, b^{-1}), \quad k \geq 1,$$

and we check that

$$A_j = (ab)^4 a_j(a^{-1}, b^{-1}), \quad 0 \leq j \leq 4.$$

For $k \geq 1$, put

$$\begin{aligned} \varepsilon_{4k} &= A_4 \left(\frac{p_{4k}}{q_{4k}} \right)^4 + A_3 \left(\frac{p_{4k}}{q_{4k}} \right)^3 + A_2 \left(\frac{p_{4k}}{q_{4k}} \right)^2 + A_1 \frac{p_{4k}}{q_{4k}} + A_0 \\ &= \frac{(ab)^{-2n_k+4} T_k(a^{-1}, b^{-1})}{(ab)^{-4n_k} q_{4k}^4} = \frac{(ab)^{2n_k+4} T_k(a^{-1}, b^{-1})}{q_{4k}^4}. \end{aligned}$$

Set $d = \deg a + \deg b$. Since

$$|q_{4k}| = 2^{dn_k}, \quad |T_k(a^{-1}, b^{-1})| \leq 1, \quad |ab|^{2n_k} = 2^{2dn_k},$$

we get

$$|\varepsilon_{4k}| \leq 2^{-2dn_k} = |q_{4k}|^{-2}.$$

Recalling $|\xi_{a,b} - p_{4k}/q_{4k}| < |q_{4k}|^{-2}$, we derive that

$$|A_4 \xi_{a,b}^4 + A_3 \xi_{a,b}^3 + A_2 \xi_{a,b}^2 + A_1 \xi_{a,b} + A_0| \leq \max\{|A_1|, \dots, |A_4|\} \cdot 2^{-2dn_k}.$$

Since n_k is arbitrarily large, this gives

$$A_4 \xi_{a,b}^4 + A_3 \xi_{a,b}^3 + A_2 \xi_{a,b}^2 + A_1 \xi_{a,b} + A_0 = 0,$$

and concludes the proof of the theorem.

5. Proof of Proposition 2.4. Since

$$A_0 + A_4 = (a+b)^2(ab+a+b)^2$$

is a square, its derivative is 0 and we get $(A_0)' = (A_4)'$. By differentiating the minimal defining polynomial $A_4X^4 + \cdots + A_1X + A_0$ of $\xi_{a,b}$, we obtain

$$\sum_{j=0}^4 (A_j)'X^j + A_1X'(1 + X^2) = 0,$$

hence

$$(A_0)'(1 + X^4) + (A_1)'(X + X^3) = A_1X'(1 + X^2)$$

and

$$(A_0)'(1 + X^2) + (A_1)'X = A_1X',$$

that is,

$$(A_0)'(1 + X^2) = (A_1X)'$$

Since

$$(A_0)' = (ab)'(a + b + ab)^2, \quad A_1 = ab(a + b)(a + b + ab)^2,$$

the last equation becomes

$$[(ab(a + b))X]' = (ab)'(1 + X^2).$$

This establishes that $\xi_{a,b}$ is differential-quadratic and satisfies a simple Riccati equation.

Assume now that there are an integer $s \geq 2$ and polynomials A, B, C, D in $\mathbb{F}_2[z]$ such that $\xi_{a,b}$ satisfies

$$A\xi_{a,b}^{2^s+1} + B\xi_{a,b}^{2^s} + C\xi_{a,b} + D = 0.$$

Then the polynomial $AX^{2^s+1} + BX^{2^s} + CX + D$ must be a multiple of the minimal defining polynomial $A_4X^4 + \cdots + A_1X + A_0$ of $\xi_{a,b}$ and there exist $c_0, c_1, \dots, c_{2^s-3}$ in $\mathbb{F}_2[z]$ such that

$$\begin{aligned} (A_4X^4 + \cdots + A_1X + A_0)(c_{2^s-3}X^{2^s-3} + \cdots + c_1X + c_0) \\ = AX^{2^s+1} + BX^{2^s} + CX + D. \end{aligned}$$

We thus get $2^s - 2$ linear forms in c_0, \dots, c_{2^s-3} which vanish. The associated matrix is a pentadiagonal (if $s \geq 3$, the case $s = 2$ being immediate) Toeplitz matrix of the form

$$M_s = \begin{pmatrix} A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_3 & A_2 & A_1 & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_4 & A_3 & A_2 & A_1 & A_0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & A_4 & A_3 & A_2 \end{pmatrix}.$$

To compute its determinant, we simply expand it and observe that A_2 is the only coefficient of the minimal defining polynomial of $\xi_{a,b}$ which involves the term a^4b^4 . All the other terms involved are of the form $a^i b^j$ with $0 \leq i, j \leq 4$ and $i + j \leq 7$. Consequently, the determinant of M_s is equal to $(a^4b^4)^{2^s-2}$ plus a linear combination of terms $a^i b^j$ with $0 \leq i, j \leq 4(2^s - 2)$ and $(i, j) \neq (4(2^s - 2), 4(2^s - 2))$. In particular, it does not vanish and the system of equations has only the trivial solution $c_0 = \dots = c_{2^s-3} = 0$. This shows that $\xi_{a,b}$ cannot be hyperquadratic.

6. Concluding remarks. In this concluding section, we gather several additional results, without providing their proofs.

6.1. Explicit expressions. By Proposition 4.5, we can derive an explicit formula for Z_k :

$$Z_k = \tau^{(2^{2k-1}-2)/3} \left(1 + b + \sum_{j=0}^{2k-2} \tau^{(2-2^{j+1}-\chi(j))/3} (ab)^{2^j} \right), \quad k \geq 1,$$

where $\chi(j) := j \pmod{2}$. Then, by Lemma 4.4, we obtain the following explicit formula for P_k/Q_k .

PROPOSITION 6.1. *For $k \geq 2$, we have*

$$\frac{P_k}{Q_k} = \frac{a + a^2b + ab^2 + (a + b)\alpha_k}{1 + b^2 + a^2b^2 + \alpha_k + \alpha_k^2},$$

where

$$\alpha_k = \sum_{j=1}^{k-1} \tau^{(2-2^{2j+1})/3} (ab)^{2^{2j}}.$$

Notice that α_k satisfies the relation

$$\alpha_k^4 + \tau^2 \alpha_k + \tau^{(8-2^{2k+1})/3} (ab)^{2^{2k}} + a^4b^4 = 0.$$

Proposition 6.1 implies the following theorem by taking the limit as k goes to infinity.

THEOREM 6.2. *Set*

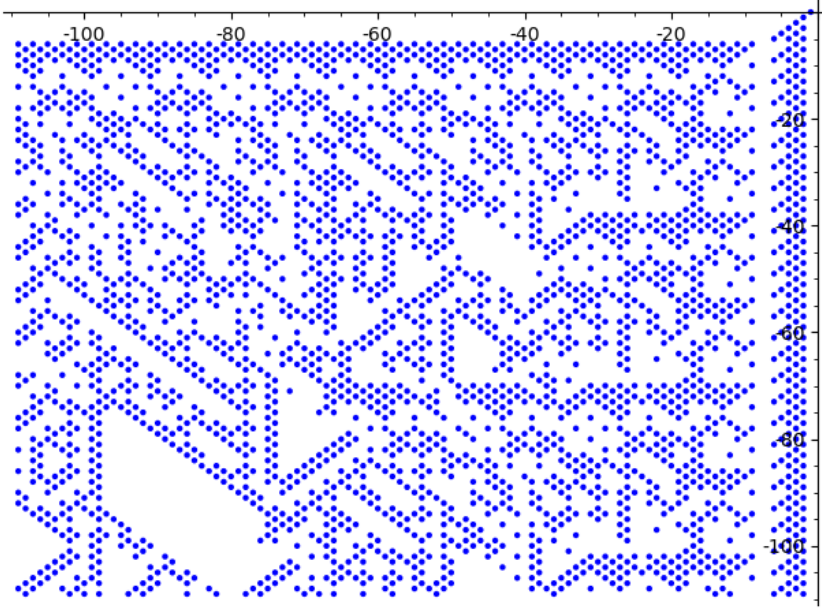
$$\alpha = \sum_{j=1}^{\infty} \tau^{(2-2^{2j+1})/3} (ab)^{2^{2j}}.$$

Then α is algebraic and satisfies

$$\alpha^4 + \tau^2 \alpha + a^4b^4 = 0.$$

Furthermore, the Thue–Morse continued fraction $\xi_{a,b}$ can be expressed as

$$\xi_{a,b} = \frac{a + a^2b + ab^2 + (a + b)\alpha}{1 + b^2 + a^2b^2 + \alpha + \alpha^2}.$$

Fig. 1. Coefficients of $\xi_{a,b}$

We stress that Theorem 6.2 implies that $\xi_{a,b}$ is algebraic.

6.2. Even and odd sections. To understand the structure of $\xi_{a,b}$, we let

$$\xi_{a,b} = \xi^{ee} + \xi^{eo} + \xi^{oe} + \xi^{oo},$$

where

$$\begin{aligned} \xi^{ee} &= \sum a^{2i}b^{2j}, \\ \xi^{eo} &= \sum a^{2i}b^{2j+1}, & \xi^{oe} &= \sum a^{2i+1}b^{2j}, \\ \xi^{oo} &= \sum a^{2i+1}b^{2j+1}. \end{aligned}$$

We can deduce, by Theorem 2.1, that

$$\xi^{ee} = \xi^{oo} = 0,$$

and ξ^{eo}, ξ^{oe} are algebraic. The coefficients of $\xi_{a,b}$ are reproduced in Figure 1 in the following manner. Let $S = \{(-1, 0), (-2, -1), (-2, -3), \dots\}$ be the set of the blue dots in Figure 1. Then

$$\xi_{a,b} = \sum_{(i,j) \in S} a^i b^j = a^{-1} + a^{-2}b^{-1} + a^{-2}b^{-3} + \dots.$$

6.3. Jacobi continued fraction for formal power series. In the field $\mathbb{F}((x))$ of formal power series over a field \mathbb{F} , the *Jacobi continued fraction* $J(\mathbf{u}, \mathbf{v})$ defined by two sequences $\mathbf{u} = (u_n)_{n \geq 1}$ and $\mathbf{v} = (v_n)_{n \geq 0}$ with $v_n \neq 0$ for all $n \geq 0$ is the infinite continued fraction

$$J(\mathbf{u}, \mathbf{v}) = \frac{v_0}{1 + u_1x - \frac{v_1x^2}{1 + u_2x - \frac{v_2x^2}{1 + u_3x - \frac{v_3x^2}{\ddots}}}}.$$

The basic properties of Jacobi continued fractions can be found in [16, 32]. The *Hankel determinants* $H_n(J(\mathbf{u}, \mathbf{v}))$ of $J(\mathbf{u}, \mathbf{v})$ can be calculated by means of the following fundamental relation, first stated by Heilermann in 1846 [18]:

$$H_n(J(\mathbf{u}, \mathbf{v})) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}.$$

Assume that \mathbb{F} is the field \mathbb{F}_2 . If the Jacobi continued fraction exists, then $v_j = 1$ for $j \geq 0$. In this case, the Hankel determinants H_n are all equal to 1. By [17, 7], the sequence $(c_n)_{n \geq 0}$ of the coefficients of $J(\mathbf{u}, \mathbf{v}) = \sum_{n \geq 0} c_n x^n$ is *apwenian*. This means that

$$c_0 = 1 \quad \text{and} \quad c_n \equiv c_{2n+1} + c_{2n+2} \pmod{2}, \quad n \geq 0.$$

Consider the Jacobi continued fraction $\omega(x)$ defined by the Thue–Morse sequence

$$\omega(x) = \frac{1}{1 + u_1x + \frac{x^2}{1 + u_2x + \frac{x^2}{1 + u_3x + \frac{x^2}{\ddots}}}},$$

where $(u_1, u_2, \dots) = (1, 0, 0, 1, 0, 1, 1, 0, \dots)$ is the Thue–Morse sequence over $\{0, 1\}$. Since

$$\frac{\omega(z^{-1})}{z} = \xi_{z+1, z},$$

it follows from Theorem 2.1 that

$$g_4 \omega(x)^4 + g_3 \omega(x)^3 + g_2 \omega(x)^2 + g_1 \omega(x) + g_0 = 0,$$

where

$$\begin{aligned} g_0 &= x^5 + x^3 + x^2 + x + 1, \\ g_1 &= x^6 + x^5 + x^4 + x^3 + x^2 + x, \\ g_2 &= x^6 + 1, \\ g_3 &= x^8 + x^7 + x^6 + x^5 + x^4 + x^3, \\ g_4 &= x^{10} + x^9 + x^8 + x^7 + x^5 + x^4. \end{aligned}$$

Consequently, $\omega(x)$ is algebraic of degree 4.

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