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Maximal Haagerup subgroups in $\mathbb{Z}^{n+1} \rtimes_{\rho_n} \mathrm{GL}_2(\mathbb{Z})$

by

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Abstract. For $n \geq 1$, let ρ_n denote the standard action of $\mathrm{GL}_2(\mathbb{Z})$ on the space $P_n(\mathbb{Z}) \simeq \mathbb{Z}^{n+1}$ of homogeneous polynomials of degree n in two variables with integer coefficients. For G a non-amenable subgroup of $\mathrm{GL}_2(\mathbb{Z})$, we describe the maximal Haagerup subgroups of the semidirect product $\mathbb{Z}^{n+1} \rtimes_{\rho_n} G$, extending the classification of Jiang–Skalski (2021) of the maximal Haagerup subgroups in $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. We prove that, for n odd, the group $P_n(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z})$ has infinitely many pairwise non-conjugate maximal Haagerup subgroups which are free groups; and, for n even, $P_n(\mathbb{Z}) \rtimes \mathrm{GL}_2(\mathbb{Z})$ has infinitely many pairwise non-conjugate maximal Haagerup subgroups which are isomorphic to $\mathrm{SL}_2(\mathbb{Z})$.

1. Introduction. For discrete countable groups, the Haagerup property is a weak form of amenability that proved to be useful in many questions in analytical group theory, ranging from K-theory to dynamical systems (see [CC⁺01]). It is not difficult to see that, in a countable group, every Haagerup subgroup is contained in a maximal one (see [JS21, Proposition 1.3] or Lemma 2.1 below). This raises the question, given a group G, of describing the maximal Haagerup subgroups of G.

The study of maximal Haagerup subgroups was initiated by Y. Jiang and A. Skalski [JS21], and we refer to this paper for many interesting and intriguing examples. We mention here Theorem 2.12 in [JS21], where the authors classify maximal Haagerup subgroups of the semidirect product $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. This example is especially interesting in view of a result of Burger [Bu91, Example 2 following Proposition 7]: if G is a non-amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then the pair $(\mathbb{Z}^2 \rtimes G, \mathbb{Z}^2)$ has the relative property (T); in particular, $\mathbb{Z}^2 \rtimes G$ is not Haagerup, in spite of the fact that $\mathbb{Z}^2$ and G are both Haagerup.

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THEOREM 1.1 ([JS21, Theorem 2.12]). *Let H be a maximal Haagerup subgroup of $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. Then there is a dichotomy: either*

(1) *$H = \mathbb{Z}^2 \rtimes C$, where C is a maximal amenable* (1) *subgroup of $\mathrm{SL}_2(\mathbb{Z})$; or*

(2) *$H \cap \mathbb{Z}^2$ is trivial; then H is not amenable. If K denotes the image of H under the quotient map $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ (so that K is isomorphic to H), then $H = \{(b(g), g) : g \in K\}$ where $b : K \to \mathbb{Z}^2$ is a 1-cocycle that cannot be extended to a larger subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

REMARK 1.2. Denote by $L(G)$ the group von Neumann algebra of the group G. In [JS21, Theorem 3.1], Jiang and Skalski prove the stronger result that, if C is a maximal amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $\mathbb{Z}^2 \rtimes C$ has infinite conjugacy classes, then $L(\mathbb{Z}^2 \rtimes C)$ is a maximal Haagerup von Neumann subalgebra of $L(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}))$, where the Haagerup property for finite von Neumann algebras was defined in [Jo02]. Subsequently Y. Jiang [Ji21, Corollary 4.3] showed that $L(\mathrm{SL}_2(\mathbb{Z}))$ is a maximal Haagerup subalgebra in $L(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}))$. It was pointed out to us by A. Skalski that there is no known example of a maximal Haagerup subgroup H in a group G such that $L(H)$ is not maximal Haagerup in $L(G)$.

We now turn to the results of the present paper. Fix $n \geq 1$ and, for $A = \mathbb{Z}, \mathbb{R}, \mathbb{C}$, denote by $P_n(A)$ the set of polynomials in two variables X, Y, with coefficients in A, which are homogeneous of degree n, so that $P_n(A) \simeq A^{n+1}$. It is a classical fact that $\mathrm{GL}_2(\mathbb{R})$ admits an irreducible representation ρ_n on $P_n(\mathbb{R})$ given as follows: for $P \in P_n(\mathbb{R})$ and $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$, set

$$(\rho_n(A)(P))(X, Y) = P((X, Y) \cdot A) = P(a_{11}X + a_{21}Y, a_{12}X + a_{22}Y).$$

Since $\rho_n(\mathrm{GL}_2(\mathbb{Z}))$ leaves $P_n(\mathbb{Z})$ invariant, we may form the semidirect product

$$G_n := P_n(\mathbb{Z}) \rtimes_{\rho_n} \mathrm{GL}_2(\mathbb{Z})$$

(observe that $G_1 = \mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$ contains $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ as a subgroup of index 2). Let G be a non-amenable subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Our goal is to classify the maximal Haagerup subgroups of $P_n(\mathbb{Z}) \rtimes_{\rho_n} G \subset G_n$. Here is our first main result, extending Theorem 1.1:

THEOREM 1.3. *Fix $n \geq 1$ and a non-amenable subgroup G of $\mathrm{GL}_2(\mathbb{Z})$. Let H be a maximal Haagerup subgroup of $P_n(\mathbb{Z}) \rtimes_{\rho_n} G$. Then either*

(1) *H is amenable, in which case $H = P_n(\mathbb{Z}) \rtimes_{\rho_n} C$, with C maximal amenable in G; or*

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(1) We classify those subgroups in Proposition 2.7.

(2) *H is non-amenable and there exists a subgroup $K \subset G$ isomorphic to H, and a 1-cocycle $b \in Z^1(K, P_n(\mathbb{Z}))$ such that*

$$H = \{(b(k), k) : k \in K\}$$

and b cannot be extended to a larger subgroup of G. (In particular, if b is a 1-coboundary, then $K = G$.)

Conversely, any subgroup of $P_n(\mathbb{Z}) \rtimes_{\rho_n} G$ of one of the above two forms defines a maximal Haagerup subgroup in $P_n(\mathbb{Z}) \rtimes_{\rho_n} G$.

Even if the conclusion looks very similar to Theorem 1.1, we emphasize that we had to come up with a totally different argument to show that the intersection $H \cap P_n(\mathbb{Z})$ is either $P_n(\mathbb{Z})$ or trivial.

Theorems 1.1 and 1.3 raise an interesting – and somewhat unusual – question in cohomology of groups: to describe maximal Haagerup subgroups that are non-amenable, we need to describe 1-cocycles that cannot be extended to a larger subgroup. In the case of ρ_1, this question is extensively studied in [JS21, Section 2] (from Lemma 2.14 to Proposition 2.18). We attack the question by observing that, for K a subgroup in $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{GL}_2(\mathbb{Z})$, a cocycle b on K cannot be extended to a larger subgroup if and only if, for any overgroup L of K, the 1-cohomology class of b is not in the image of the restriction map $H^1(L, P_n(\mathbb{Z})) \to H^1(K, P_n(\mathbb{Z}))$. This suggests looking for subgroups K which are maximal, or close to being maximal, in $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{GL}_2(\mathbb{Z})$. Using this approach we prove the following result, which seems to be new even for $n = 1$:

THEOREM 1.4. *Assume that $n \geq 1$ is odd. Then there exists a free subgroup K in $\mathrm{SL}_2(\mathbb{Z})$ such that $P_n(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z})$ contains infinitely many maximal Haagerup subgroups of the form $H = \{(b(k), k) : k \in K\}$, with $b \in Z^1(K, P_n(\mathbb{Z}))$, which are pairwise non-conjugate under $P_n(\mathbb{Z})$. Moreover, the subgroup K may be chosen either with infinite index or with arbitrarily large finite index.*

In the even case we get:

THEOREM 1.5. *Assume that $n \geq 2$ is even. Then the semidirect product G_n contains infinitely many maximal Haagerup subgroups H of the form $H = \{(b(g), g) : g \in \mathrm{SL}_2(\mathbb{Z})\}$, where $b \in Z^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$, which moreover are pairwise non-conjugate under $P_n(\mathbb{Z})$.*

It turns out that the cases of odd and even n's are very different: the reason is that the $\mathrm{GL}_2(\mathbb{Z})$-action on $P_n(\mathbb{Z})$ factors through $\mathrm{PGL}_2(\mathbb{Z})$ for n even, while for n odd it does not.

The paper is organized as follows. Section 2 is devoted to prerequisites, with the exception of Proposition 2.7, describing precisely the maximal amenable subgroups in $\mathrm{SL}_2(\mathbb{Z})$. In Section 3 we prove the analogue of

Burger's aforementioned result, namely that the pair $(G_n, P_n(\mathbb{Z}))$ has the relative property (T) for $n \geq 1$. Theorem 1.3 is proved in Section 4, while cohomological questions are treated in Sections 5 (n odd) and 6 (n even); in particular, Theorem 1.4 is proved in Section 5, and Theorem 1.5 in Section 6. Theorem 1.5 is actually proved by establishing explicit formulae for the ranks (2) of $H^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ and $H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$; those formulae may have their own interest. A combinatorial application of the methods ends the paper.

2. Generalities

2.1. Haagerup property. The following results on countable groups will be used freely:

- The group $\mathrm{GL}_2(\mathbb{Z})$ is Haagerup (see [CC$^+$01, Sections 1.2.2 or 1.2.3]).
- An extension of a Haagerup group by an amenable group is Haagerup; in particular, every amenable group is Haagerup (see [CC$^+$01, Proposition 6.1.5]).

LEMMA 2.1. *Let G be a countable group. Any Haagerup subgroup of G is contained in a maximal Haagerup subgroup.*

Proof. (compare with [JS21, Proposition 1.3]) To apply Zorn's lemma, we must show that being Haagerup is stable by arbitrary increasing unions of subgroups. This follows from the fact that, for countable groups, being Haagerup is a local property, i.e a countable group is Haagerup if and only if every finitely generated subgroup is Haagerup (see [CC$^+$01, Proposition 6.1.1]).

2.2. 1-cohomology of groups. We recall the following facts from the cohomology of groups. For G a group and A a G-module, define the group of 1-cocycles:

$$Z^1(G, A) = \{b : G \to A : b(gh) = gb(h) + b(g) \text{ for all } g, h \in G\};$$

the group of 1-coboundaries:

$$B^1(G, A) = \{b \in Z^1(G, A) : \text{there exists } a \in A \text{ such that } b(g) = ga - a \text{ for all } g \in G\};$$

and the first cohomology group:

$$H^1(G, A) = Z^1(G, A)/B^1(G, A).$$

The following is proved e.g. in [Br82, Proposition 2.3].

(2) By the *rank* of a finitely generated abelian group, we mean the torsion-free rank.

PROPOSITION 2.2. *Let G be a group and let A be a G-module. Splittings of the split extension*

$$0 \to A \to A \rtimes G \to G \to 1$$

are given by $i_b : G \to A \rtimes G : g \mapsto (b(g), g)$ with $b \in Z^1(G,A)$ and are classified up to A-conjugacy by the first cohomology group $H^1(G,A)$. ∎

The first part of the following lemma is a variation on [Gu80, Corollaire 7.2, p. 39].

LEMMA 2.3. *Let G be a group and A be a G-module with cancellation by 2 (i.e. $2x = 0 \Rightarrow x = 0$).*

(1) *Assume G contains a central element z that acts on A by -1. Then $H^1(G,A)$ is a vector space over the field with two elements. Moreover, $b \in B^1(G,A)$ if $b(z)$ belongs to $2A$.*
(2) *Assume that G contains a central element c of order 2 such that the module action on A factors through $G/\langle c\rangle$. Then the map $H^1(G/\langle c\rangle, A) \to H^1(G,A)$ (induced by the quotient map $G \to G/\langle c\rangle$) is an isomorphism.*

Proof. (1) We have to prove that, for every $b \in Z^1(G,A)$, we have $2b \in B^1(G,A)$. But for $g \in G$ we have, using $zg = gz$ in the third equality,

$$b(z) - b(g) = b(z) + zb(g) = b(zg) = b(gz) = gb(z) + b(g).$$

Rearranging gives

$$2b(g) = (1-g)b(z).$$

Hence $2b \in B^1(G,A)$. If we may write $b(z) = 2a$ for some $a \in A$, by cancelling 2 on both sides we get $b(g) = (1-g)a$, so $b \in B^1(G,A)$.

(2) The map $H^1(G/\langle c\rangle, A) \to H^1(G,A)$ is clearly injective. For surjectivity, fix $b \in Z^1(G,A)$ and consider $b(c^2) = b(1) = 0$: using the cocycle relation yields

$$0 = (1+c)b(c) = 2b(c),$$

hence $b(c) = 0$ and b factors through $G/\langle c\rangle$. ∎

2.3. About $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z})$. We denote by C_n the cyclic group of order n, and by D_n the dihedral group of order $2n$. Set

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is well-known (see [Se77, Example 1.5.3]) that $\mathrm{SL}_2(\mathbb{Z})$ admits a decomposition as an amalgamated product:

$$\mathrm{SL}_2(\mathbb{Z}) \simeq C_4 *_{C_2} C_6.$$

with $C_4 = \langle s\rangle$, $C_6 = \langle t\rangle$ and $C_2 = \{1, \varepsilon\}$. It extends to an amalgamated product decomposition of $\mathrm{GL}_2(\mathbb{Z})$ (see [BT16, Section 6]):

$$\mathrm{GL}_2(\mathbb{Z}) \simeq D_4 *_{D_2} D_6, \tag{1}$$

with $D_4 = \langle s, w\rangle$, $D_6 = \langle t, w\rangle$ and $D_2 = \langle \varepsilon, w\rangle$.

LEMMA 2.4. *The amenable radical of* $\mathrm{GL}_2(\mathbb{Z})$ *is the subgroup* C_2.

Proof. This is a very particular case of a result of Cornulier [Co09, Proposition 7]: Let $G = A *_C B$ be an amalgamated product such that $[A : C] \geq 2$ and $[B : C] \geq 3$. Then the amenable radical of G is the largest normal subgroup of C which is amenable and normalized by both A and B. ∎

Our aim now is to describe the maximal amenable subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Recall that an element $A \in \mathrm{SL}_2(\mathbb{R}), A \neq \pm\mathrm{Id}$, is *elliptic* if $|\mathrm{Tr}(A)| < 2$, *parabolic* if $|\mathrm{Tr}(A)| = 2$, and *hyperbolic* if $|\mathrm{Tr}(A)| > 2$. These concepts clearly descend to $\mathrm{PSL}_2(\mathbb{R})$. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we denote by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the image of A in $\mathrm{PSL}_2(\mathbb{R})$.

LEMMA 2.5.

(1) *The maximal amenable subgroups of* $\mathrm{PSL}_2(\mathbb{Z})$ *are isomorphic to* C_3, $\mathbb{Z}$, *or* D_∞ (*the infinite dihedral group*).
(2) *If* $A \in \mathrm{PSL}_2(\mathbb{Z})$ *is parabolic, then* A *is contained in a unique maximal amenable subgroup, isomorphic to* $\mathbb{Z}$.
(3) *If* $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{Z})$ *is hyperbolic, then* A *is contained in a unique maximal amenable subgroup, isomorphic to* $\mathbb{Z}$ *or* D_∞. *The second case happens if* A *is conjugate to a symmetric matrix. If the second case happens, then the integer binary quadratic form*

$$Q_A(x, y) := bx^2 + (d - a)xy - cy^2$$

represents $-b$ *over* $\mathbb{Z}$.

Proof. (1) Since $\mathrm{PSL}_2(\mathbb{Z})$ is the free product $C_2 * C_3$, by the Kurosh theorem any subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ is a free product of a free group with conjugates of the free factors. So a subgroup does not contain a free group if and only if it is isomorphic to one of the following: $C_2, C_3, \mathbb{Z}, C_2 * C_2 = D_\infty$; note that these four groups are amenable. It remains to show that any copy of C_2 is contained in at least one copy of D_∞; but any element of order 2 in $\mathrm{PSL}_2(\mathbb{Z})$ is conjugate to the image of s, and the images of s and tst^{-1} together generate a copy of D_∞.

(2) If A is parabolic then, up to conjugacy in $\mathrm{PSL}_2(\mathbb{Z})$, we may assume that $A = \left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]$ for some $k \in \mathbb{Z}$, $k \neq 0$. Let H be a maximal amenable subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ containing A; by (1), H is isomorphic either to $\mathbb{Z}$ or to D_∞. In both cases H normalizes the subgroup $\langle A\rangle$. But by direct

computation the normalizer of $\langle A\rangle$ is $\left\langle \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]\right\rangle$, which is therefore the unique maximal amenable subgroup contaning A.

(3) Let H be a maximal amenable subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ containing A. Again, H normalizes $\langle A\rangle$. But A is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to $\left[\begin{smallmatrix}\lambda&0\\0&1/\lambda\end{smallmatrix}\right]$ (with $\lambda>1$) whose normalizer in $\mathrm{PSL}_2(\mathbb{R})$ is

$$\mathbb{R}\rtimes C_2=\left\{\begin{bmatrix}e^t&0\\0&e^{-t}\end{bmatrix}:t\in\mathbb{R}\right\}\rtimes\left\langle\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right\rangle.$$

So H is the intersection of $\mathrm{PSL}_2(\mathbb{Z})$ with a given conjugate of $\mathbb{R}\rtimes C_2$, which proves uniqueness of H (and re-proves that H is isomorphic either to $\mathbb{Z}$ or to D_∞).

For the second statement, we may assume that A is symmetric. Then, as the eigenspaces of A are orthogonal in $\mathbb{R}^2$, we see without computation that $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$ conjugates A to A^{-1}, so that A and $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$ together generate a copy of D_∞ (then contained in a maximal one).

For the final statement, observe that the maximal amenable subgroup containing A is isomorphic to D_∞ if and only if there exists an involution $B\in\mathrm{PSL}_2(\mathbb{Z})$ that conjugates A to A^{-1}. Assume such a B exists. Then, denoting by $\overline{A},\overline{B}$ lifts of A,B in $\mathrm{SL}_2(\mathbb{Z})$, we will have $\overline{B}\,\overline{A}\,\overline{B}^{-1}=\pm\overline{A}^{-1}$. Taking the trace of both sides, we see that the minus sign leads to $\mathrm{Tr}(\overline{A})=0$, in contradiction with $|\mathrm{Tr}(\overline{A})|>2$. So $\overline{B}\,\overline{A}\,\overline{B}^{-1}=\overline{A}^{-1}$ or $\overline{B}\,\overline{A}=\overline{A}^{-1}\overline{B}$. Now $\overline{B}$ has order 4 in $\mathrm{SL}_2(\mathbb{Z})$, and therefore $\mathrm{Tr}(\overline{B})=0$. So we may write $\overline{B}=\left(\begin{smallmatrix}x&y\\z&-x\end{smallmatrix}\right)$ and we get

$$\begin{pmatrix}x&y\\z&-x\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\cdot\begin{pmatrix}x&y\\z&-x\end{pmatrix}.$$

Equating the $(1,1)$-coefficients on both sides yields

$$ax+cy=dx-bz\iff z=\frac{(d-a)x-cy}{b}.$$

Taking into account the condition $\det(\overline{B})=1$, i.e. $-1=x^2+yz$, and inserting the previous value of z, gives

$$-1=x^2+y\frac{(d-a)x-cy}{b}\iff Q_A(x,y)=-b,$$

which concludes the proof. Note that if A is symmetric, i.e. $b=c$, then $Q_A(0,1)=-b$. ∎

EXAMPLE 2.6. We claim that the maximal amenable subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ containing $A=\left[\begin{smallmatrix}3&1\\2&1\end{smallmatrix}\right]$ is infinite cyclic. To see this, we check that the quadratic form $Q_A(x,y)=x^2-2xy-2y^2$ does not represent -1. But the equation $x^2-2xy-2y^2=-1$ is equivalent to $(x-y)^2-3y^2=-1$, which leads to

Pell's equation $X^2 - 3Y^2 = -1$; as the fundamental unit $2+\sqrt{3}$ in $\mathbb{Z}[\sqrt{3}]$ has norm 1, this equation has no solution.

We now lift the results of Lemma 2.5 to $\mathrm{SL}_2(\mathbb{Z})$. Since a maximal amenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$ must clearly contain the center, we see that the maximal amenable subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are exactly the pullbacks of maximal amenable subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ by the quotient map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$. Observe that the inverse image of an involution in $\mathrm{PSL}_2(\mathbb{Z})$ has order 4 in $\mathrm{SL}_2(\mathbb{Z})$; consequently, the inverse image in $\mathrm{SL}_2(\mathbb{Z})$ of a copy of D_∞ in $\mathrm{PSL}_2(\mathbb{Z})$ will be isomorphic to the semidirect product $\mathbb{Z} \rtimes C_4$ where a generator of C_4 acts on $\mathbb{Z}$ by $n \mapsto -n$. So we immediately get the following, which improves on [JS21, Proposition 2.13]:

Proposition 2.7.

(1) *The maximal amenable subgroups of* $\mathrm{SL}_2(\mathbb{Z})$ *are isomorphic to* C_6, $\mathbb{Z}\times C_2$, *or* $\mathbb{Z} \rtimes C_4$.
(2) *For parabolic* (*resp. hyperbolic*) *matrices in* $\mathrm{SL}_2(\mathbb{Z})$, *item* (2) (*resp. item* (3)) *of Lemma* 2.5 *applies with the obvious changes.* ∎

3. Relative property (T)

Proposition 3.1. *Let G be a non-amenable subgroup of* $\mathrm{GL}_2(\mathbb{Z})$.

(1) *The restriction of the representation ρ_n to G is irreducible.*
(2) *The pair* $(P_n(\mathbb{Z}) \rtimes_{\rho_n} G, P_n(\mathbb{Z}))$ *has the relative property* (T). *In particular,* $P_n(\mathbb{Z}) \rtimes_{\rho_n} G$ *is not Haagerup.*

Proof. (1) Since $G \cap \mathrm{SL}_2(\mathbb{Z})$ has index at most 2 in G, replacing G by $G \cap \mathrm{SL}_2(\mathbb{Z})$ we may assume that $G \subset \mathrm{SL}_2(\mathbb{Z})$. Let L be the Zariski closure of G in $\mathrm{SL}_2(\mathbb{R})$, so that L is a Lie subgroup of $\mathrm{SL}_2(\mathbb{R})$, hence of dimension 0, 1, 2 or 3. As Lie subgroups of dimension 0, 1, 2 are virtually solvable, hence amenable, L has dimension 3, i.e. G is Zariski dense in $\mathrm{SL}_2(\mathbb{R})$. Since the representation ρ_n is algebraic, irreducibility is preserved by passing to a Zariski dense subgroup ([3]).

(2) Set $V_n = P_n(\mathbb{R})$. By [Bu91, Proposition 7] (see especially [Bu91, Example (2), p. 62]), if G does not fix any probability measure on the projective space $P(V_n^*)$, then the pair $(P_n(\mathbb{Z}) \rtimes_{\rho_n} G, P_n(\mathbb{Z}))$ has the relative property (T). Since the representation ρ_n of $\mathrm{SL}_2(\mathbb{R})$ is equivalent to its contragredient ρ_n^*, it is enough to check that there is no G-fixed probability

([3]) We recall the argument: if W is a $\rho_n(G)$-invariant subspace, and $(f_i)_{i\in I}$ is a set of linear forms such that $W = \bigcap_{i\in I} \ker(f_i)$, then $\rho_n(G)$-invariance of W is equivalent to $f_i(\rho_n(g)w) = 0$ for all $g \in G$, $w \in W$ and $i \in I$. View this as a system of polynomial equations in the matrix coefficients of g; it vanishes on G, hence also vanishes on $\mathrm{SL}_2(\mathbb{R})$ by Zariski density.

measure on the projective space $P(V_n)$. So assume that there is such a measure μ. Then, by [Zi84, Corollary 3.2.2], there are exactly two cases:

- The measure μ is not supported on a finite union of proper projective subspaces. Then the stabilizer $\mathrm{PGL}(V_n)_\mu$ is compact, which contradicts the fact that the image of G in $\mathrm{PGL}(V_n)$ is infinite discrete.
- There exists a proper linear subspace W in V_n such that $\mu([W]) > 0$ (where $[W]$ denotes the image of W in $P(V_n)$), and moreover the orbit of $[W]$ under the stabilizer $\mathrm{PGL}(V_n)_\mu$ is finite. In particular, there is a finite index subgroup of $\mathrm{PGL}(V_n)_\mu$ that leaves $[W]$ invariant. So there is a finite index subgroup G_0 of G that leaves the linear subspace W invariant, a contradiction. ∎

4. Maximal Haagerup subgroups. An interesting question raised in [JS21] is whether every countable group admits a Haagerup radical, i.e. a unique maximal normal subgroup with the Haagerup property. We first show that G_n admits such a Haagerup radical.

PROPOSITION 4.1. *For every $n \geq 1$, the Haagerup radical of G_n is $P_n(\mathbb{Z}) \rtimes_{\rho_n} C_2$.*

Proof. Set $U := P_n(\mathbb{Z}) \rtimes_{\rho_n} C_2$. Let $N \lhd G_n$ be a normal Haagerup subgroup; we want to prove that N is contained in U. We proceed as in [JS21, proof of Proposition 2.10]: The subgroup UN is normal and since $UN/N \simeq U/(U \cap N)$ is amenable, UN is an amenable extension of a Haagerup group, hence UN is Haagerup. Since UN contains U we have in particular $P_n(\mathbb{Z}) \subset UN$, so $UN = \mathbb{Z}^{n+1} \rtimes_{\rho_n} K$ for some normal subgroup $K \lhd \mathrm{GL}_2(\mathbb{Z})$. By Proposition 3.1(2), the subgroup K must be amenable, i.e. $K \subset C_2$ by Lemma 2.4. So $UN \subset U$ and therefore $N \subset U$. ∎

Since $P_n(\mathbb{Z}) \rtimes_{\rho_n} C_2$ is actually amenable, we immediately have:

COROLLARY 4.2. *The amenable radical of G_n is $P_n(\mathbb{Z}) \rtimes_{\rho_n} C_2$.* ∎

We now come to the proof of our first main result.

Proof of Theorem 1.3. Let

$$q_n : P_n(\mathbb{Z}) \rtimes \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}) : (v, S) \mapsto S$$

be the quotient map. Observe that, as $\ker(q_n|_H) = H \cap P_n(\mathbb{Z})$ is abelian, H is amenable if and only if $q_n(H)$ is amenable. We separate the two cases.

(1) If H is amenable, set $C = q_n(H)$, so that H is contained in the amenable subgroup $P_n(\mathbb{Z}) \rtimes_{\rho_n} C$. By maximality, we have $H = P_n(\mathbb{Z}) \rtimes_{\rho_n} C$, and C is maximal amenable in G.

(2) Assume now that H is not amenable, and set $K = q_n(H)$.

CLAIM 1. *The subgroup $H \cap P_n(\mathbb{Z})$ is invariant by $\rho_n(K)$.*

Indeed, fix $h \in H$ and write $h = (v_h, q_n(h))$ as an element of the semidirect product G_n. For $(w, 1) \in H \cap P_n(\mathbb{Z})$, we have, because $P_n(\mathbb{Z})$ is abelian,

$$\begin{aligned}(\rho_n(q_n(h^{-1}))w, 1) &= (0, q_n(h))^{-1}(w, 1)(0, q_n(h)) \\ &= (0, q_n(h))^{-1}(-v_h, 1)(w, 1)(v_h, 1)(0, q_n(h)) \\ &= \big((v_h, 1)(0, q_n(h))\big)^{-1}(w, 1)\big((v_h, 1)(0, q_n(h))\big) \\ &= (v_h, q_n(h))^{-1}(w, 1)(v_h, q_n(h)) = h^{-1}(w, 1)h,\end{aligned}$$

which belongs to $H \cap P_n(\mathbb{Z})$ because the latter is normal in H. This proves Claim 1.

CLAIM 2. *We have* $H \cap P_n(\mathbb{Z}) = \{0\}$.

To see this, let k be the rank of the free abelian group $H \cap P_n(\mathbb{Z})$, so that $0 \le k \le n+1$; we must show that $k = 0$.

If $k = n + 1$, then $H \cap P_n(\mathbb{Z})$ has finite index in $P_n(\mathbb{Z})$, so that H has finite index in $H \cdot P_n(\mathbb{Z})$. By maximality, we must have $H \cdot P_n(\mathbb{Z}) = H$, i.e. $P_n(\mathbb{Z}) \subset H$ and $H = P_n(\mathbb{Z}) \rtimes_{\rho_n} K$; as K is not amenable, this contradicts Proposition 3.1(2).

If $1 \le k \le n$, we denote by W the linear subspace of $P_n(\mathbb{R})$ generated by $H \cap \mathbb{Z}^{n+1}$. By Claim 1, the subspace W is invariant by $\rho_n(K)$, contradicting Proposition 3.1(1). This proves Claim 2.

At this point we know that $q_n|_H$ induces an isomorphism from H onto K, so that by Proposition 2.2 there exists a 1-cocycle $b \in Z^1(K, P_n(\mathbb{Z}))$ such that $H = \{(b(k), k) : k \in K\}$. By maximality of H, the 1-cocycle cannot be extended to a larger subgroup of G. This proves the direct implication of the theorem.

For the converse, if C is a maximal amenable subgroup of G, then $P_n(\mathbb{Z}) \rtimes C$ is Haagerup, and maximality follows immediately from Proposition 3.1(2). If K is a non-amenable subgroup of G and $b \in Z^1(K, P_n(\mathbb{Z}))$ is a 1-cocycle that cannot be extended to a larger subgroup, then $H = \{(b(k), k) : k \in K\}$ is a Haagerup subgroup of $P_n(\mathbb{Z}) \rtimes_{\rho_n} G$, and maximality follows from the dichotomy in the direct implication of the theorem. ∎

5. Cohomological matters: n odd

5.1. Maximal subgroups in $\mathrm{PSL}_2(\mathbb{Z})$. Recall that a subgroup H in a group G is said to be *maximal* if H is maximal among proper subgroups of G.

PROPOSITION 5.1. *The group* $\mathrm{PSL}_2(\mathbb{Z})$ *admits free maximal subgroups of arbitrarily large finite index, and also of infinite index.*

Proof. Since $\mathrm{PSL}_2(\mathbb{Z})$ is isomorphic to the free product $C_2 * C_3$, a subgroup H of $\mathrm{PSL}_2(\mathbb{Z})$ is free if and only it is torsion-free, if and only if it has no element of order 2 or 3, if and only if it does not meet the conjugacy classes of (the images of) s and t in $\mathrm{PSL}_2(\mathbb{Z})$.

(1) *Finite index.* Let p be a prime with $p \equiv 11 \bmod 12$. Consider the congruence subgroup

$$\overline{\Gamma}_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : c \equiv 0 \bmod p \right\}.$$

So $\overline{\Gamma}_0(p)$ is the inverse image, under reduction modulo p, of the upper triangular subgroup in $\mathrm{PSL}_2(p)$. Hence $\overline{\Gamma}_0(p)$ is a maximal subgroup of index $p+1$ in $\mathrm{PSL}_2(\mathbb{Z})$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$, by direct computation we find that the $(2,1)$-entry of gsg^{-1} is c^2+d^2, which is not divisible by p as $p \equiv 3 \bmod 4$; so $\overline{\Gamma}_0(p)$ has no element of order 2. Similarly, the $(2,1)$-entry of gtg^{-1} is $c^2 - cd + d^2$. Assume towards a contradiction that p divides $c^2 - cd + d^2$. Then it also divides $4(c^2 - cd + d^2) = (2c - d)^2 + 3d^2$, from which it follows that the Legendre symbol $\left(\frac{-3}{p}\right)$ equals 1, as one sees by inverting d modulo p. Using quadratic reciprocity, this contradicts $p \equiv 2 \bmod 3$. So $\overline{\Gamma}_0(p)$ has no element of order 3, and therefore is a free group.

(2) *Infinite index.* We will appeal to [FL$^+$22, Theorem E] together with its proof. Consider the following two permutations of $\mathbb{N} = \{1, 2, \ldots\}$:

$$a = (12)(34)(56)(78)\ldots, \qquad b = (123)(456)(789)\ldots.$$

View the symmetric group $\mathrm{Sym}(\mathbb{N})$ as a Polish group with the topology of pointwise convergence. Let $T \subset \mathrm{Sym}(\mathbb{N})$ consist of those $\vartheta \in \mathrm{Sym}(\mathbb{N})$ such that the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{N}$ defined by mapping s to a and t to $\vartheta b \vartheta^{-1}$ is transitive on $\mathbb{N}$. By [FL$^+$22, Lemma 6.2], the set T is a non-empty G_δ in $\mathrm{Sym}(\mathbb{N})$, hence T is itself Polish. Now Theorem E in [FL$^+$22, Section 6.2], together with its proof, shows that for a generic choice of $\vartheta \in T$, the corresponding action of $\mathrm{PSL}_2(\mathbb{Z})$ will be 2-transitive (in fact highly transitive, i.e. n-transitive for every $n \geq 1$). By 2-transitivity, the stabilizer of any point in $\mathbb{N}$ will be a maximal subgroup of infinite index in $\mathrm{PSL}_2(\mathbb{Z})$. Since a and $\vartheta b \vartheta^{-1}$ are permutations without fixed points, that stabilizer avoids the conjugacy classes of s and t, and therefore it is free. ■

5.2. The case $n = 1$. The next result is specific to $n = 1$; it applies in particular to $G = \mathrm{SL}_2(\mathbb{Z})$ and $G = \mathrm{GL}_2(\mathbb{Z})$. For $\mathrm{SL}_2(\mathbb{Z})$, a different proof appears in [JS21, Lemma 2.14].

PROPOSITION 5.2. *Let G be a subgroup of $\mathrm{GL}_2(\mathbb{Z})$ containing an element t_0 of order 6. Let $\rho : G \to \mathrm{GL}_N(\mathbb{Z})$ be a homomorphism such that $\rho(t_0^3) = -1$, and $\rho(t_0)$ does not admit -1 as an eigenvalue. Then $H^1(G, \mathbb{Z}^N) = 0$. This applies in particular to $\rho = \rho_1|_G$.*

Proof. In an amalgamated product of groups, any element of finite order is conjugate to a finite order element in one of the factors (see e.g. [Se77, Section 3, Cor. 1]). So in $\mathrm{GL}_2(\mathbb{Z}) = D_4 *_{D_2} D_6$, conjugating G if necessary, we may assume that $t_0 = t$, so that $\varepsilon := t_0^3$ belongs to G.

Fix $b \in Z^1(G, \mathbb{Z}^N)$. We want to prove that b is a 1-coboundary. By Lemma 2.3 (applied with $z = \varepsilon$) it is enough to prove that $b(\varepsilon) \in 2\mathbb{Z}^N$.

Set $T := \rho(t_0)$. Then

$$0 = T^3 + 1 = (T + 1)(T^2 - T + 1).$$

Since by assumption $T + 1$ is invertible in $M_N(\mathbb{C})$, we have $0 = T^2 - T + 1$, i.e. $2T = T^2 + T + 1$. Now expanding $t_0^3 = \varepsilon$ by the 1-cocycle relation we get

$$b(\varepsilon) = (T^2 + T + 1)b(t_0) = 2Tb(t_0),$$

i.e. $b(\varepsilon) \in 2\mathbb{Z}^N$. For the final statement about ρ_1, it is enough to observe that -1 is not an eigenvalue of $t = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. ∎

We now revisit an example from Jiang–Skalski [JS21, Proposition 2.16]. For $N \geq 2$, define

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N,\ a \equiv d \equiv 1 \bmod N \right\}.$$

Define also the vector $v_N = \left(\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right)$ in $\mathbb{Q}^2$. Then Jiang–Skalski prove that $\Gamma_1(N)$ is exactly the set of elements of $\mathrm{SL}_2(\mathbb{Z})$ such that $b_N(g) = gv_N - v_N$ belongs to $\mathbb{Z}^2$, so that b_N is a 1-cocycle in $Z^1(\Gamma_1(N), \mathbb{Z}^2)$ that does not extend to an overgroup, and $H_N = \{(b_N(k), k) : k \in \Gamma_1(N)\}$ is maximal Haagerup in $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. We make the result more precise by observing that H_N is actually a free group for infinitely many values of N.

Proposition 5.3. *If N is divisible by some prime p with $p \equiv 11 \bmod 12$, then H_N is a free group.*

Proof. We will use the fact that any subgroup of a free group is free. It is enough to show that $\Gamma_1(N)$ is free. Since $\Gamma_1(N) \subset \Gamma_1(p)$, we reduce to the case of $\Gamma_1(p)$. Clearly $\varepsilon \notin \Gamma_1(p)$, so $\Gamma_1(p)$ maps injectively to its image in $\mathrm{PSL}_2(\mathbb{Z})$. Clearly the latter is contained in $\overline{\Gamma}_0(p)$, which is free by the proof of Proposition 5.1. ∎

5.3. n odd, general case. For general odd n, we have the following:

Proposition 5.4. *Let G be a finitely generated subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Assume that G is generated by k elements $g_1, \ldots, g_k$ and contains $\varepsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then, for n odd, $H^1(G, P_n(\mathbb{Z}))$ is a vector space over the field $\mathbf{F}_2$ of two elements, with dimension at most $(k-1)(n+1)$.*

Proof. By Lemma 2.3(1), we know that $H^1(G, P_n(\mathbb{Z}))$ is a vector space over $\mathbf{F}_2$. Now any cocycle $b \in Z^1(G, P_n(\mathbb{Z}))$ is completely determined by the vectors $b(g_1), \dots, b(g_k)$. On the other hand, the boundary map $\partial : P_n(\mathbb{Z}) \to Z^1(G, P_n(\mathbb{Z})) : P \mapsto \partial P$, with $(\partial P)(g) = \rho_n(g)P - P$, is injective. So $H^1(G, P_n(\mathbb{Z}))$ appears as a subquotient of $(P_n(\mathbb{Z}))^{k-1}$, which gives the desired bound on the dimension. ∎

Proof of Theorem 1.4. Let M be a maximal free subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ as in Proposition 5.1. Lifting arbitrarily a free basis of M to $\mathrm{SL}_2(\mathbb{Z})$, we get a free subgroup K in $\mathrm{SL}_2(\mathbb{Z})$, say of rank k (possibly $k = \infty$). In particular, the rank of $H^1(K, P_n(\mathbb{Z}))$ as an abelian group is $(k-1)(n+1)$. Now by maximality of M in $\mathrm{PSL}_2(\mathbb{Z})$, together with the fact that the central extension

$$1 \to \langle \varepsilon \rangle \to \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}) \to 1$$

does not split, we see that the only overgroups of K in $\mathrm{SL}_2(\mathbb{Z})$ are $K \times \langle \varepsilon \rangle$ and $\mathrm{SL}_2(\mathbb{Z})$. For both, H^1 is a torsion group (by Proposition 5.4), so any 1-cocycle b of infinite order in $H^1(K, P_n(\mathbb{Z}))$ will not extend to an overgroup of K, and the subgroup $H = \{(b(k), k) : k \in K\}$ is maximal Haagerup in $P_n(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z})$. ∎

6. Cohomological matters: n even. We will see that for n even the groups $H^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ and $H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ are infinite, so that the situation is completely different from the case of n odd.

Since $\rho_n(\varepsilon) = 1$ as n is even, by the second part of Lemma 2.3 we may replace $H^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ by $H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$, and similarly $H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ by $H^1(\mathrm{PGL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$.

6.1. The case of $\mathrm{SL}_2(\mathbb{Z})$. Set $S = \rho_n(s)$ and $T = \rho_n(t)$, and recall that S, T generate $\rho_n(\mathrm{SL}_2(\mathbb{Z})) \simeq \mathrm{PSL}_2(\mathbb{Z})$.

LEMMA 6.1. *An assignment $S \mapsto b(S)$, $T \mapsto b(T)$ of two vectors in $P_n(\mathbb{Z})$ extends to $b \in Z^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ if and only if $(1+S)b(S) = 0$ and $(1+T+T^2)b(T) = 0$.*

Proof. View $\mathrm{PSL}_2(\mathbb{Z})$ as the free product $\langle S \rangle * \langle T \rangle = \langle S, T \mid S^2 = 1, T^3 = 1 \rangle$. So the assignment $S \mapsto b(S)$, $T \mapsto b(T)$ extends to a 1-cocycle if and only if it satisfies $(S+1)b(S) = 0$ (obtained by expanding $S^2 = 1$ by the 1-cocycle relation) and $(1+T+T^2)b(T) = 0$ (obtained by expanding $T^3 = 1$ by the cocycle relation). ∎

Define, for n even,

$$\eta(n) = \begin{cases} 0 & \text{if } n \equiv 2 \bmod 3, \\ 1 & \text{if } n \equiv 0 \bmod 3, \\ -1 & \text{if } n \equiv 1 \bmod 3. \end{cases}$$

THEOREM 6.2. *For n even:*

$$\operatorname{rk}(H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Z}))) = \frac{n+1+3(-1)^{n/2+1}-4\eta(n)}{6} > 0.$$

In particular, $\operatorname{rk}(H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Z}))) \geq \frac{n-6}{6}$.

Proof. Since we are only interested in the torsion-free part of the group $H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$, we may work rationally and compute the $\mathbb{Q}$-dimension of $H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$.

We define a subgroup of $Z^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$ by

$$Z_0^1 := \{b \in Z^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q})) : b(T) = 0\}.$$

We first observe that every cocycle $b \in H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$ is cohomologous to a cocycle in Z_0^1. This is clear in terms of affine actions: if $\alpha(g)v = gv + b(g)$ (with $v \in P_n(\mathbb{Q})$) is the affine action associated with b, then for any $w \in P_n(\mathbb{Q})$ the vector $w_0 := \frac{1}{3}(w + \alpha(T)w + \alpha(T^2)w)$ is $\alpha(T)$-fixed, so the coboundary $b'(g) = b(g) + gw_0 - w_0$ vanishes on T.

Consequently, setting

$$m_0 := \dim Z_0^1 \quad \text{and} \quad n_0 := \dim(Z_0^1 \cap B^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))),$$

we have

$$\begin{aligned}\dim(H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))) &= \dim(Z_0^1/(Z_0^1 \cap B^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q})))) \\ &= m_0 - n_0,\end{aligned}$$

and it is enough to compute m_0 and n_0 separately. This will be done in three steps. Note that, by Lemma 6.1, the space Z_0^1 can be identified with $\ker(S+1)$.

1. *Computation of $m_0 = \dim(\ker(S+1))$.* Observe that $(SP)(X,Y) = P(Y,-X)$ for $P \in P_n(\mathbb{Z})$. Expanding P as a sum of monomials $P(X,Y) = \sum_{k=0}^n a_k X^{n-k}Y^k$, we get

$$(SP)(X,Y) = \sum_{k=0}^n (-1)^k a_k Y^{n-k}X^k = \sum_{k=0}^n (-1)^k a_{n-k} X^{n-k}Y^k,$$

so that $SP = -P$ if and only if

$$a_k = (-1)^{k+1}a_{n-k} \quad \text{for every } k = 0, 1, \ldots, n. \tag{2}$$

So we may choose $a_0, a_1, \ldots, a_{n/2-1}$ arbitrarily, while $a_{n/2} = 0$ if $n \equiv 0 \bmod 4$, and $a_{n/2}$ can be chosen arbitrarily if $n \equiv 2 \bmod 4$. So

$$m_0 = \frac{n+1+(-1)^{n/2+1}}{2}. \tag{3}$$

2. *We claim that $Z_0^1 \cap B^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$ can be identified with* $\ker(T-1)$. Indeed, $b \in Z_0^1$ is a 1-coboundary if and only if there exists $P \in P_n(\mathbb{Q})$ such that $b(S) = (S-1)(P)$ and $0 = (T-1)(P)$. This

is still equivalent to $b(S) \in (S-1)(\ker(T-1))$. This already identifies $Z_0^1 \cap B^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$ with $(S-1)(\ker(T-1))$.

To prove the claim it remains to show that $(S-1)|_{\ker(T-1)}$ is injective, i.e. $\ker(S-1) \cap \ker(T-1) = \{0\}$. But, as S, T generate $\mathrm{PSL}_2(\mathbb{Z})$, the space $\ker(S-1) \cap \ker(T-1)$ is exactly the space of $\rho_n(\mathrm{SL}_2(\mathbb{Z}))$-fixed vectors in $P_n(\mathbb{Q})$, which is 0 by Proposition 3.1(1).

3. *Computation of* $n_0 = \dim(\ker(T-1))$. Since $T^3 = 1$, the operator T defines a representation σ of the cyclic group C_3, and $n_0 = \dim(\ker(T-1))$ is the multiplicity of the trivial representation in this representation. Recall that C_3 has three irreducible representations, all of dimension 1: the trivial representation χ_0 and the characters $\chi_\pm$ defined by $\chi_\pm(T) = e^{\pm 2\pi i/3}$. Now σ is equivalent to the direct sum of n_0 copies of χ_0 with n_+ copies of χ_+ and n_- copies of χ_-. Note that $n_+ = n_-$ because σ is a real representation. Computing the character of σ we get

$$\begin{aligned} n+1 = \mathrm{Tr}(1) &= n_0\chi_0(1) + n_+\chi_+(1) + n_-\chi_-(1) = n_0 + 2n_+, \\ \mathrm{Tr}(T) &= n_0\chi_0(T) + n_+\chi_+(T) + n_-\chi_-(T) = n_0 - n_+. \end{aligned}$$

Solving this system for n_0 we get

$$\mathrm{rk}(\ker(T-1)) = n_0 = \frac{n+1+2\,\mathrm{Tr}(T)}{3}. \tag{4}$$

It remains to compute $\mathrm{Tr}(T)$ to get the exact value of n_0. We observe that ρ_n extends to a representation of $\mathrm{SL}_2(\mathbb{C})$ on the space $P_n(\mathbb{C})$. Restricting to the compact torus $T = \{a_\vartheta = \left(\begin{smallmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{smallmatrix}\right) : \vartheta \in \mathbb{R}\}$ we have the classical formula (see e.g. [Ha03, (7.26)])

$$\mathrm{Tr}(\rho_n(a_\vartheta)) = \frac{\sin((n+1)\vartheta)}{\sin(\vartheta)}.$$

Since t has order 6 in $\mathrm{SL}_2(\mathbb{Z})$, it is conjugate to $a_{\pi/3}$ in $\mathrm{SL}_2(\mathbb{C})$, and therefore

$$\mathrm{Tr}(T) = \mathrm{Tr}(\rho_n(a_{\pi/3})) = \frac{\sin((n+1)\pi/3)}{\sin(\pi/3)} = \eta(n). \tag{5}$$

Using (3)–(5), we get the desired result. Note that for $n = 2, 4, 6$ we get $m_0 > n_0$ from the following table:

(6)

n	n_0	m_0
2	1	2
4	1	2
6	3	4

■

Corollary 6.3. *For $n \geq 2$ even, let M be a free maximal subgroup of* $\mathrm{PSL}_2(\mathbb{Z})$ *as in Proposition* 5.1*; let $K \simeq M \times \langle \varepsilon \rangle$ denote the inverse image of M in* $\mathrm{SL}_2(\mathbb{Z})$*. Then the semidirect product $P_n(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z})$ contains infinitely many maximal Haagerup subgroups H of the form $H = \{(b(g), g) : g \in K\}$,* *where* $b \in Z^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$, *which moreover are pairwise non-conjugate under* $P_n(\mathbb{Z})$.

Proof. Say that the rank of M is k (possibly $k = \infty$). By Lemma 2.3, we have $\mathrm{rk}(H^1(K, P_n(\mathbb{Z}))) = \mathrm{rk}(H^1(M, P_n(\mathbb{Z}))) = (k-1)(n+1)$ as M is free, So by Theorem 6.2,

$$\mathrm{rk}(H^1(K, P_n(\mathbb{Z}))) \geq n+1 > \frac{n+1+3(-1)^{n/2+1} - 4\eta(n)}{6} = \mathrm{rk}(H^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Z}))).$$

So taking for b cocycles on K whose classes have infinite order in the co-kernel of the restriction map $H^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Z})) \to H^1(K, P_n(\mathbb{Z}))$, we may construct maximal Haagerup subgroups of the desired form. ∎

6.2. The case of $\mathrm{GL}_2(\mathbb{Z})$**.** We recall the notations $S = \rho_n(s)$, $T = \rho_n(t)$, to which we add $W := \rho_n(w)$. Note that, for $P \in P_n(\mathbb{Z})$, we have $(WP)(X,Y) = P(Y,X)$; in particular, $WP = P$ (resp. $WP = -P$) means that P is symmetric (resp. antisymmetric).

To estimate the rank of $H^1(GL_2(\mathbb{Z}), P_n(\mathbb{Z}))$, we will need the following representation-theoretic lemma.

Lemma 6.4. *The* $\mathbb{Q}$*-dimension of the space*

$$P_n^0 := \{P \in P_n(\mathbb{Q}) : P \text{ symmetric},\ TP = P\}$$

is $\frac{n+4+2\eta(n)}{6}$.

Proof. We observe that P_n^0 is invariant under $\rho_n|_{D_3}$, which suggests using representation theory of the dihedral group $D_3 = \langle T, W\rangle$. This group has three irreducible representations defined over $\mathbb{Q}$: the trivial character χ_0, the non-trivial character χ_1 defined by $\chi_1(T) = 1$ and $\chi_1(W) = -1$, and the 2-dimensional irreducible representation on vectors in $\mathbb{R}^3$ whose three coordinates sum to 0. The character table of D_3 is

	e	T	W
χ_0	1	1	1
χ_1	1	1	-1
π	2	-1	0

Write $\rho_n|_{D_3} = n_0\chi_0 \oplus n_1\chi_1 \oplus n_\pi\pi$. From the character table, it follows that the dimension of P_n^0 is exactly the multiplicity n_0 of χ_0 in $\rho_n|_{D_3}$. To compute n_0, we will compute the character of $\rho_n|_{D_3}$. Denote by P_n^s (resp. P_n^a) the subspace of symmetric (resp. antisymmetric) polynomials in $P_n(\mathbb{Q})$. Then

$$\mathrm{Tr}(W) = \dim(P_n^s) - \dim(P_n^a) = (n/2+1) - n/2 = 1,$$

so that the character of $\rho_n|_{D_3}$ is given by

$$\begin{aligned} n+1 = \mathrm{Tr}(1) &= n_0 + n_1 + 2n_\pi, \\ \mathrm{Tr}(T) &= n_0 + n_1 - n_\pi, \\ 1 = \mathrm{Tr}(W) &= n_0 - n_1. \end{aligned}$$

Solving for n_0, and using (5), gives the desired result. ∎

Theorem 6.5. *For even n, we have*

$$\mathrm{rk}(H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))) = \frac{n - 5 + 3(-1)^{n/2+1} - 4\eta(n)}{12}.$$

In particular, $\mathrm{rk}(H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))) \geq \frac{n-12}{12}$.

Proof. As with $\mathrm{SL}_2(\mathbb{Z})$, we will work rationally. We will appeal to a part of the Hochschild-Serre long exact sequence in group cohomology: if G is a group, $N \triangleleft G$ a normal subgroup, and V is a G-module with $V^G = 0$, then the restriction map $H^1(G, V) \to H^1(N, V)^{G/N}$ is an isomorphism (see [Gu80, Section 8.1]).

We apply this to $V = P_n(\mathbb{Q}), G = \mathrm{GL}_2(\mathbb{Z})$ and $N = \mathrm{SL}_2(\mathbb{Z})$, so that $G/N = \langle W \rangle$ has order 2. We must therefore work out the W-invariants on $H^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$. This amounts to finding the W-invariants in $Z^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$ and dividing by the W-invariants in $B^1(\mathrm{PSL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$. Of course we will make use of the fact that the W-invariants in a subspace of $P_n(\mathbb{Q})$ are just the symmetric polynomials in that subspace.

By the beginning of the proof of Theorem 6.2, every cocycle in $Z^1(\mathrm{SL}_2(\mathbb{Z}), P_n(\mathbb{Q}))$ is cohomologous to a cocycle in Z^1_0, which identifies with $\ker(S+1)$. By (2), a polynomial $P(X, Y) = \sum_{k=0}^n a_k X^{n-k} Y^k$ is in $\ker(S+1)$ if and only if $a_k = (-1)^{k+1} a_{n-k}$ for every $k = 0, 1, \ldots, n$. So this polynomial $P(X, Y)$ is symmetric if and only if $a_k = 0$ for even k, and $a_k = a_{n-k}$ for odd k. Hence we get

$$\dim (Z^1_0)^W = \frac{n + 1 + (-1)^{n/2+1}}{4}. \tag{7}$$

Now we compute the dimension of $(Z^1_0 \cap B^1(\mathrm{PGL}_2(\mathbb{Z}), P_n(\mathbb{Q})))^W$. As in the second step of the proof of Theorem 6.2, using $SW = WS$ we identify this space first with $(S-1)(\ker(T-1) \cap \ker(W-1))$ and then with $\ker(T-1) \cap \ker(W-1)$, which is the subgroup P_n^0 from Lemma 6.4; its dimension is $(n + 4 + 2\eta(n))/6$.

Summarizing, by (7) we have

$$\begin{aligned}
&\dim(H^1(\mathrm{GL}_2(\mathbb{Z}),P_n(\mathbb{Z})))\\
&\qquad\qquad = \dim((Z_0^1)^W) - \dim((Z_0^1\cap B^1(\mathrm{PGL}_2(\mathbb{Z}),P_n(\mathbb{Q})))^W)\\
&\qquad\qquad = \frac{n+1+(-1)^{n/2+1}}{4} - \frac{n+4+2\eta(n)}{6}\\
&\qquad\qquad = \frac{n-5+3(-1)^{n/2+1}-4\eta(n)}{12}.
\end{aligned}$$

It is clear that this quantity is bounded below by $\frac{n-12}{12}$. ∎

With this we can prove Theorem 1.5.

Proof of Theorem 1.5. It is enough to show that the cokernel of the restriction map $H^1(\mathrm{GL}_2(\mathbb{Z}),P_n(\mathbb{Z}))\to H^1(\mathrm{SL}_2(\mathbb{Z}),P_n(\mathbb{Z}))$ has rank >0. This follows by comparing the formulae in Theorems 6.2 and 6.5. Indeed, we have

$$\begin{aligned}
&\frac{n+1+3(-1)^{n/2+1}-4\eta(n)}{6} - \frac{n-5+3(-1)^{n/2+1}-4\eta(n)}{12}\\
&\qquad\qquad\qquad\qquad = \frac{n+7+3(-1)^{n/2+1}-4\eta(n)}{12} > 0,
\end{aligned}$$

where the inequality follows from $7+3(-1)^{n/2+1}-4\eta(n)\geq 0$. ∎

Example 6.6. For $n=2,4$, we can give explicit cocycles witnessing the fact that the image of the restriction map $\mathrm{Rest}: H^1(\mathrm{GL}_2(\mathbb{Z}),P_n(\mathbb{Z}))\to H^1(\mathrm{SL}_2(\mathbb{Z}),P_n(\mathbb{Z}))$ has infinite index.

Recall from the proof of Theorem 6.2 that we defined $Z_0^1 := \{b\in Z^1(\mathrm{PSL}_2(\mathbb{Z}),P_n(\mathbb{Z})) : b(T)=0\}$ and identified it naturally with $\ker(1+S)$. For $a\in\mathbb{Z}$, define $b_a\in Z^1(\mathrm{SL}_2(\mathbb{Z}),P_n(\mathbb{Z}))$ by prescribing

$$(b_a(S))(X,Y) = a(X^n-Y^n), \qquad b_a(T)=0;$$

it follows from (2) that $b_a(S)\in\ker(1+S)$, hence $b_a\in Z_0^1$. We first show that, for $a\neq 0$, the cocycle b_a is non-zero in H^1.

Assume that b_a is a coboundary. By the third step of the proof of Theorem 6.2, the space $Z_0^1\cap B^1(\mathrm{SL}_2(\mathbb{Z}),P_n(\mathbb{Z}))$ identifies naturally with $(S-1)(\ker(T-1))$. For $n=2,4$, we have $\mathrm{rk}(\ker(T-1))=1$ by table (6). For $n=2$, the space $\ker(T-1)$ is generated by X^2-XY+Y^2 (a t-invariant quadratic form on $\mathbb{Z}^2$), and by direct computation one gets $(S-1)(X^2-XY+Y^2)=2XY$, forcing $a=0$. For $n=4$, we replace X^2-XY+Y^2 by $(X^2-XY+Y^2)^2$ and proceed analogously.

Finally, as b_a is antisymmetric, for $a\neq 0$ its class in H^1 is certainly not in $H^1(\mathrm{SL}_2(\mathbb{Z}),P_n(\mathbb{Z}))^W$, which is the image of the restriction map

$$H^1(\mathrm{GL}_2(\mathbb{Z}),P_n(\mathbb{Z}))\to H^1(\mathrm{SL}_2(\mathbb{Z}),P_n(\mathbb{Z})).$$

Coming back to Theorem 6.5, we observe that $\mathrm{rk}(H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))) = 0$ for $n = 2, 4, 6, 8, 12$. This means that $H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ is a finite group for these values. We will show that it is not zero.

COROLLARY 6.7. *Fix $n \geq 2$ even. Define*

$$m = \begin{cases} n/4 & \text{if } n \equiv 0 \bmod 4, \\ (n+2)/4 & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Then $H^1(\mathrm{GL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ has order at least 2^m.

Proof. We start with two observations.

1. A 1-cocycle $b \in Z_0^1$ such that $b(S)$ is symmetric extends from $\mathrm{PSL}_2(\mathbb{Z})$ to $\mathrm{PGL}_2(\mathbb{Z})$ by the prescription $b(W) = 0$. Indeed, $\mathrm{PGL}_2(\mathbb{Z}) \simeq D_2 *_{C_2} D_3$ with $C_2 = \langle W \rangle$, $D_2 = \langle S, W \rangle$, $D_3 = \langle T, W \rangle$. So by the cocycle relation, b extends to D_3 by $b|_{D_3} = 0$, and to D_2 by defining $b(SW) = b(S)$, which is indeed equal to $b(WS) = Wb(S)$ as $b(S)$ is symmetric. So b extends to the amalgamated product.

2. If a cocycle $b \in Z^1(\mathrm{PGL}_2(\mathbb{Z}), P_n(\mathbb{Z}))$ with $b(W) = 0$ is a coboundary, then there exists $P \in P_n(\mathbb{Z})$ such that $(W-1)P = 0$ (i.e. P is symmetric) and $b(S) = (1-S)P$. But $SP(X,Y) = P(Y,-X) = P(-X,Y)$ (by symmetry), so writing $P(X,Y) = \sum_{k=0}^{n/2-1} a_k(X^{n-k}Y^k + X^kY^{n-k}) + a_{n/2}X^{n/2}Y^{n/2}$ we get

$$\begin{aligned}((1-S)P)(X,Y) = &\sum_{k=0}^{n/2-1} a_k(1-(-1)^k)(X^{n-k}Y^k + X^kY^{n-k}) \\ &+ a_{n/2}(1-(-1)^{n/2})X^{n/2}Y^{n/2},\end{aligned}$$

so that $(1-S)P$ has even coefficients.

We now turn to the proof proper. For $\varepsilon \in \{0,1\}^m$, define $b_\varepsilon(T) = 0$ and

$$\begin{aligned} &b_\varepsilon(S)(X,Y) \\ &= \begin{cases} \sum_{k=1}^{m} \varepsilon_k(X^{n-2k+1}Y^{2k-1} + X^{2k-1}Y^{n-2k+1}) & \text{if } n \equiv 0 \bmod 4, \\ \sum_{k=1}^{m-1} \varepsilon_k(X^{n-2k+1}Y^{2k-1} + X^{2k-1}Y^{n-2k+1}) & \\ \qquad\qquad + \varepsilon_m X^{n/2}Y^{n/2} & \text{if } n \equiv 2 \bmod 4. \end{cases}\end{aligned}$$

Observe that $(S+1)b_\varepsilon(S) = 0$ by (2), so that by Lemma 6.1 we get a cocycle $b_\varepsilon \in Z_0^1$ with $b_\varepsilon(S)$ symmetric, so by the first observation in the proof we may extend it to $\mathrm{PGL}_2(\mathbb{Z})$ by $b_\varepsilon(W) = 0$.

On the other hand, if $\varepsilon_1, \varepsilon_2$ are distinct elements in $\{0,1\}^m$, then $(b_{\varepsilon_1} - b_{\varepsilon_2})(S)$ has at least one odd coefficient, so $b_{\varepsilon_1} - b_{\varepsilon_2}$ cannot be a coboundary by the second observation above. ∎

6.3. A combinatorial application. We end the paper with an amusing combinatorial consequence of our proof of Theorem 6.2. For n even, the sum

$$\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k},$$

which is an alternating sum on a diagonal in Pascal's triangle, can be computed by combinatorial means (see [BQ08]). We give a representation-theoretic derivation.

Corollary 6.8. *For even n,*

$$\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}=\eta(n).$$

Proof. In view of (5), it is enough to prove that the LHS is equal to $\mathrm{Tr}(T)=\mathrm{Tr}(T^{-1})$. For this we observe that $T^{-1}(P)(X,Y)=P(X-Y,X)$ for $P\in P_n(\mathbb{R})$. So in the canonical basis $X^n,X^{n-1}Y,\dots,XY^{n-1},Y^n$ we have, for $k=0,1,\dots,n$,

$$(T^{-1})(X^{n-k}Y^k)=X^k(X-Y)^{n-k}=\sum_{\ell=0}^{n-k}(-1)^\ell\binom{n-k}{\ell}X^{n-\ell}Y^\ell.$$

In the last sum, the term $X^{n-k}Y^k$ does not appear if $k>n/2$, and it appears with coefficient $(-1)^k\binom{n-k}{k}$ if $0\le k\le n/2$. In other words, the kth diagonal coefficient of the matrix of T^{-1} in the canonical basis is

$$\begin{cases}0 & \text{if } k>n/2,\\ (-1)^k\binom{n-k}{k} & \text{if } k\le n/2.\end{cases}$$

This yields the trace of T^{-1}:

$$\mathrm{Tr}(T^{-1})=\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}.\ \blacksquare$$

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