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On the characterization of Ricci–Bourguignon solitons

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Abstract. We study solitons associated with the Ricci–Bourguignon flow. Firstly, we prove a result for a Ricci–Bourguignon soliton that generalizes a corresponding result for a Ricci soliton. Then, we study radially-flat gradient Ricci–Bourguignon solitons and infer that they are rigid. Finally, we characterize gradient Ricci–Bourguignon solitons endowed with a non-parallel, closed homothetic vector field.

1. Introduction. A family of metrics $g(t)$ on an n-dimensional Riemannian manifold (M, g) is said to evolve by the *Ricci–Bourguignon flow* if $g(t)$ satisfies the evolution equation

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} - \rho r g), \tag{1.1}$$

where Ric is the Ricci tensor, r is the scalar curvature with respect to g and $\rho \in \mathbb{R}$ is a constant. Ricci–Bourguignon flow was first introduced by Jean-Pierre Bourguignon [1]. The soliton corresponding to the Ricci–Bourguignon flow is defined as follows.

DEFINITION 1.1. Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ and $\rho \in \mathbb{R} \setminus \{0\}$ be a constant. Then M is called a *Ricci–Bourguignon soliton* if there is a smooth vector field V such that

$$\frac{1}{2} L_V g + \mathrm{Ric} = (\rho r + \lambda) g, \tag{1.2}$$

where L_V is the Lie derivative along the vector field V and λ is a constant.

If there exists a smooth function $f : M \to \mathbb{R}$ such that $V = \nabla f$, then equation (1.2) takes the form

$$\mathrm{Hess}\, f + \mathrm{Ric} = (\rho r + \lambda) g, \tag{1.3}$$

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where Hess f is the Hessian of f with respect to g, and M is known as a *gradient Ricci–Bourguignon soliton.* Catino and Mazzieri [2] called Ricci–Bourguignon solitons *ρ-Einstein solitons.* A Ricci–Bourguignon soliton is called *steady* if $\lambda = 0$, and *shrinking* or *expanding* if $\lambda > 0$ or $\lambda < 0$ respectively. For more details on Ricci–Bourguignon solitons, the reader can refer to [5–8]. For $\rho = 0$, (1.1) turns out to be the well-known Ricci flow equation introduced by Hamilton [9], whose corresponding soliton equation is

$$\tfrac{1}{2}L_V g + \mathrm{Ric} = \mu g, \tag{1.4}$$

where μ is a constant. As usual, if $V = \nabla f$, for some smooth function f on M, then the Ricci soliton is said to be a gradient Ricci soliton and equation (1.4) assumes the form

$$\mathrm{Hess}\, f + \mathrm{Ric} = \mu g. \tag{1.5}$$

For details on gradient Ricci solitons, we refer the reader to the work of Petersen and Wylie [11]. For other special values of the parameter ρ, a Ricci–Bourguignon soliton is called

- an *Einstein soliton* if $\rho = \frac{1}{2}$,
- a *traceless Ricci soliton* if $\rho = \frac{1}{n}$,
- a *Schouten soliton* if $\rho = \frac{1}{2(n-1)}$.

In [12], Sharma proved that a Ricci soliton with divergence-free potential vector field V is Einstein and V is Killing. Motivated by this, we study a Ricci–Bourguignon soliton with $\rho \neq \frac{1}{n}$ and divergence-free potential vector field, with the aim of generalizing the above result for the soliton to the Ricci–Bourguignon flow. More precisely, we prove the following

THEOREM 1.1. *Let (M^n, g, V, ρ), $n > 2$, be a Ricci–Bourguignon soliton with $\rho \neq 1/n$ and divergence-free potential vector field V. Then M is Einstein and V is Killing.*

A gradient Ricci soliton is said to be *rigid* if it is isometric to a quotient of $N \times E^k$, where N is an Einstein manifold and the potential function $f = -\frac{\mu}{2}|x|^2$ on the Euclidean factor E^k, i.e., (M, g) is isometric to $N \times_\Gamma E^k$, where Γ acts freely on N and by orthogonal transformations on E^k. For $k = n$, i.e., $N = E^n$, it is just the Gaussian soliton $f = -\frac{\mu}{2}|x|^2$ [11]. It is known that a gradient Ricci soliton is rigid iff it has constant scalar curvature and is radially-flat. This motivates us to study radially-flat gradient Ricci–Bourguignon solitons. In this context, we establish the following result.

THEOREM 1.2. *Let (M^n, g, f, ρ), $n > 2$ and $\rho \neq \{\frac{1}{n}, \frac{1}{2(n-1)}\}$, be a radially-flat gradient Ricci–Bourguignon soliton. Then M is rigid.*

Next, a seminal result of Perelman [10] asserts that a compact Ricci soliton is gradient up to a Killing vector field. In [13], Silva Filho proved the same conclusion for Ricci solitons endowed with a non-parallel, closed homothetic vector field. It was shown by Silva Filho and Sharma [14] that a gradient Ricci soliton admitting a non-parallel, closed homothetic vector field is Ricci-flat. We generalize this result for a non-Schouten gradient Ricci–Bourguignon soliton.

THEOREM 1.3. *Let (M^n, g, f, ρ), $n > 2$, be a gradient Ricci–Bourguignon soliton admitting a non-parallel, closed homothetic vector field. Then any one of the following holds:*

(i) *M reduces to a gradient Schouten soliton;*
(ii) *M is Ricci-flat.*

When M is non-steady, complete and $\rho \neq \dfrac{1}{2(n-1)}$, we have the following consequence.

COROLLARY 1.1. *Let (M^n, g, f, ρ), $n > 2$, be a complete, non-steady gradient Ricci–Bourguignon soliton with $\rho \neq \frac{1}{2(n-1)}$, admitting a non-parallel, closed homothetic vector field. Then M is isometric to the Euclidean space and hence is a Gaussian soliton.*

2. Preliminaries. A vector field U on a Riemannian manifold (M, g) is said to be *conformal* if

$$L_U g = 2\psi g, \tag{2.1}$$

where ψ is the conformal scale function on M. If ψ is a constant, then U is said to be *homothetic*, and *Killing* when $\psi = 0$. Equation (2.1) can be written as

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) = 2\psi g(X, Y) \tag{2.2}$$

for arbitrary smooth vector fields X, Y on M. Now,

$$g(\nabla_X U, Y) = \frac{1}{2}[g(\nabla_X U, Y) + g(\nabla_Y U, X)] + \frac{1}{2}[g(\nabla_X U, Y) - g(\nabla_Y U, X)],$$

and for the 1-form u metrically equivalent to U, we have

$$\begin{aligned} du(X, Y) &= \frac{1}{2}[X(u(Y)) - Y(u(X)) - u[X, Y]] \\ &= \frac{1}{2}[g(\nabla_X U, Y) - g(\nabla_Y U, X)]. \end{aligned}$$

Using the preceding equations and (2.2) gives

$$g(\nabla_X U, Y) = \frac{1}{2}(L_U g)(X, Y) + du(X, Y). \tag{2.3}$$

Since U is conformal, the use of (2.1) in (2.3) entails

$$g(\nabla_X U, Y) = \psi g(X, Y) + du(X, Y). \tag{2.4}$$

Switching X and Y in (2.4) and using the symmetry of g gives

$$g(\nabla_Y U, X) = \psi g(X, Y) + du(Y, X). \tag{2.5}$$

By summing (2.4) and (2.5) and applying (2.2), we observe that

$$du(X, Y) = -du(Y, X).$$

As du is skew-symmetric, we define a tensor field φ of type $(1, 1)$ by

$$du(X, Y) = g(\varphi X, Y). \tag{2.6}$$

The use of (2.6) in (2.4) and then factoring out Y yields

$$\nabla_X U = \psi X + \varphi X. \tag{2.7}$$

We now prove certain equations that hold on a Ricci–Bourguignon soliton.

LEMMA 2.1. *Let (M^n, g, V, ρ), $n > 2$, be a Ricci–Bourguignon soliton. Then:*

(a) $\nabla_X V = -QX + (\rho r + \lambda)X + FX$, *where Q is the Ricci operator such that* $\mathrm{Ric}(X, Y) = g(QX, Y)$ *and F is a skew-symmetric tensor field of type* $(1, 1)$.
(b) $R(Y, X)V = (\nabla_X Q)Y - (\nabla_Y Q)X + \rho\{(Yr)X - (Xr)Y\} + (\nabla_Y F)X - (\nabla_X F)Y$.
(c) $\mathrm{Ric}(X, V) = \left[\frac{1}{2} - \rho(n - 1)\right]Xr + (\operatorname{div} F)X$.

Proof. Equation (1.2) can be written as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2\mathrm{Ric}(X, Y) = 2(\rho r + \lambda)g(X, Y). \tag{2.8}$$

The exterior derivative dv of the 1-form v is given by

$$g(\nabla_X V, Y) - g(\nabla_Y V, X) = 2(dv)(X, Y). \tag{2.9}$$

As dv is skew-symmetric, we define a tensor field of type $(1, 1)$ by

$$dv(X, Y) = g(FX, Y). \tag{2.10}$$

Thus, (2.9) assumes the form

$$g(\nabla_X V, Y) - g(\nabla_Y V, X) = 2g(FX, Y). \tag{2.11}$$

Summing (2.8) and (2.11) and factoring out Y yields

$$\nabla_X V = -QX + (\rho r + \lambda)X + FX,$$

which proves (a). Now, using the formula $R(Y, X)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z$ with $Z = V$ and Lemma 2.1(a) gives

$$\begin{aligned} R(Y, X)V = {} & (\nabla_X Q)Y - (\nabla_Y Q)X + \rho\{(Yr)X - (Xr)Y\} \\ & + (\nabla_Y F)X - (\nabla_X F)Y, \end{aligned}$$

thus proving part (b). Contracting the preceding equation at Y and using the twice contracted Bianchi's second identity shows that

$$\operatorname{Ric}(X,V) = \left[\frac{1}{2} - \rho(n-1)\right] Xr + (\operatorname{div} F)X,$$

which is part (c). This completes the proof.

Next, we prove the following formula for a gradient Ricci–Bourguignon soliton.

LEMMA 2.2. *Let (M^n, g, f, ρ), $n > 2$, be a gradient Ricci–Bourguignon soliton. Then*

$$2\left[\frac{1}{2} - \rho(n-1)\right]\Delta r = g(\nabla r, \nabla f) - 2|Q|^2 + 2r(\rho r + \lambda). \tag{2.12}$$

Proof. For $V = \nabla f$, where f is some smooth function on M, we have $v = df$, and by Poincaré's lemma ($d^2 = 0$), we get $dv = 0$. This, in conjunction with (2.10) shows that $F \equiv 0$. Taking this into consideration and noting that $\operatorname{Ric}(X,Y) = g(QX,Y)$, the equation in Lemma 2.1(c) can be written as

$$Q(\nabla f) = \left[\frac{1}{2} - \rho(n-1)\right]\nabla r.$$

Taking the divergence on both sides of the above equation, using the twice contracted Bianchi's second identity and Lemma 2.1(a) at once gives (2.12), completing the proof.

3. Proofs of the main results

Proof of Theorem 1.1. Contracting the equation in Lemma 2.1(a) gives

$$\operatorname{div} V = -r + (\rho r + \lambda)n.$$

Since V is divergence-free by hypothesis and $\rho \neq \frac{1}{n}$, the preceding equation entails

$$r = \frac{n\lambda}{1-\rho n}. \tag{3.1}$$

Therefore, the scalar curvature is constant and the use of (3.1) in the fundamental equation (1.2) gives

$$\frac{1}{2}L_V g + \operatorname{Ric} = \frac{\lambda}{1-\rho n} g,$$

Using (3.1) in the foregoing equation, we acquire

$$\frac{1}{2}L_V g + \operatorname{Ric} = \frac{r}{n} g, \tag{3.2}$$

which is the Ricci soliton equation (1.4), where $\mu = \frac{r}{n}$ is a constant. Appealing to the following formula (Chow et al. [4]) for a Ricci soliton:

$$L_V r = 2|\mathrm{Ric}|^2 + \Delta r - 2\mu r$$

and noting that r is a constant, we have

$$|\mathrm{Ric}|^2 = \frac{r^2}{n}. \tag{3.3}$$

But we also have the relation

$$|\mathrm{Ric}|^2 - \frac{r^2}{n} = \left|\mathrm{Ric} - \frac{r}{n} g\right|^2. \tag{3.4}$$

Using (3.3) in (3.4) implies that $\mathrm{Ric} = \frac{r}{n} g$, i.e., M is Einstein. Using this in (3.2) shows that V is Killing, completing the proof.

Proof of Theorem 1.2. Let $V = \nabla f$ for some smooth function f on M. Then, in view of (2.10), $F \equiv 0$. We now compute $\nabla_X |\nabla f|^2$ using the equation in Lemma 2.1(a) with $V = \nabla f$:

$$\nabla_X |\nabla f|^2 = 2g(\nabla_X \nabla f, \nabla f) = -2\mathrm{Ric}(X, \nabla f) + 2(\rho r + \lambda) X f.$$

Now, using Lemma 2.1(c) in the above equation provides

$$\nabla_X |\nabla f|^2 = \{2\rho(n-1) - 1\} X r + 2(\rho r + \lambda) X f,$$
$$d(|\nabla f|^2) = \{2\rho(n-1) - 1\} dr + 2(\rho r + \lambda) df,$$

where d denotes the exterior derivative operator.

Applying the operator d in the preceding equation and invoking Poincaré's lemma ($d^2 = 0$) gives

$$dr \wedge df = 0. \tag{3.5}$$

Now, substituting ∇f for V in Lemma 2.1(c) and factoring out X entails

$$Q(\nabla f) = \left[\frac{1}{2} - \rho(n-1)\right] \nabla r.$$

Taking the inner product of the above equation with ∇f and using $\mathrm{Ric}(\nabla f, \nabla f) = 0$ [obtained by contracting $R(\nabla f, X)\nabla f = 0$] we get

$$\left[\frac{1}{2} - \rho(n-1)\right] g(\nabla r, \nabla f) = 0.$$

Since $\rho \neq \frac{1}{2(n-1)}$ by hypothesis, we have

$$g(\nabla r, \nabla f) = 0. \tag{3.6}$$

The use of (3.5) and (3.6) gives $|\nabla r|\,|\nabla f| = 0$. Since a gradient Ricci–Bourguignon soliton is real analytic for $\rho \neq \{\frac{1}{n}, \frac{1}{2(n-1)}\}$ [3], either (i) r is constant on M, or (ii) r is non-constant on M. In case (i), M becomes a gradient Ricci soliton (where μ in (1.5) is $\rho r + \lambda$), hence rigid. In case (ii),

$\nabla f = 0$ on M, which reduces (1.3) to $\mathrm{Ric} = (\rho r + \lambda)g$. Contracting the preceding equation and rearranging the terms yields $r = \frac{n\lambda}{1-\rho n}$, i.e., r is a constant on M, a contradiction. This completes the proof.

Proof of Theorem 1.3. Since the Ricci–Bourguignon soliton is a gradient, $V = \nabla f$ and hence $F \equiv 0$. Thus, the equation in Lemma 2.1(c) can be written as

$$\mathrm{Ric}(X, \nabla f) = \left[\frac{1}{2} - \rho(n-1)\right] Xr. \tag{3.7}$$

Also, as the homothetic vector field U is closed, we have $du = 0$, and in view of (2.6), $\varphi = 0$. Thus, (2.7) becomes

$$\nabla_X U = \psi X. \tag{3.8}$$

Using (3.8) we compute $R(X,Y)U$ and find that $R(X,Y)U = 0$. Contracting the foregoing equation at X and subsequently factoring out Y provides

$$QU = 0. \tag{3.9}$$

Now, setting $X = U$ in (3.7) and using (3.9) implies that

$$\left[\frac{1}{2} - \rho(n-1)\right] Ur = 0,$$

i.e., either (i) $\rho = \frac{1}{2(n-1)}$, in which case M is a gradient Schouten soliton, or (ii) $Ur = 0$. In case (ii), we know that for a homothetic vector field, $Ur = -2\psi r$ (Yano [16]) and therefore $\psi r = 0$. Since U is non-parallel by hypothesis, we have $r = 0$. The use of this in (2.12) shows that M is Ricci-flat. This completes the proof.

Proof of Corollary 1.1. Since $\rho \neq \frac{1}{2(n-1)}$, by Theorem 1.3, M is Ricci-flat and so $r = 0$. Therefore, the fundamental equation (1.3) reduces to $\mathrm{Hess}\, f = \lambda g$. As M is non-steady and complete by hypothesis, invoking a result of Tashiro [15] implies that M is isometric to the Euclidean space. This completes the proof.

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