STUDIA MATHEMATICA
Online First version

Generic continuous Lebesgue measure-preserving interval maps are nowhere monotone but invertible a.e.

by

Jozef Bobok, Jernej Činč, Piotr Oprocha
and Serge Troubetzkoy

Abstract. We consider continuous maps of the interval which preserve the Lebesgue measure. Except for the identity map or $1 - \mathrm{id}$ all such maps have topological entropy at least $\log 2/2$ and generically they have infinite topological entropy. We show that the generic map with respect to the uniform topology has zero measure-theoretic entropy with respect to the Lebesgue measure. This implies that there are dramatic differences in the topological versus measure-theoretic behavior both for injectivity and for the structure of the level sets of generic maps. As a consequence we get a surprising corollary for a family of planar attractors homeomorphic to the pseudo-arc.

1. Introduction. Genericity of zero measure-theoretic entropy has a long history starting with a result of Rokhlin. Let (X, μ) denote a Lebesgue probability space and $\mathrm{Aut}(X, \mu)$ the group of measure-preserving automorphisms of X. Consider the weak topology: a sequence $\{f_i\} \subset \mathrm{Aut}(X, \mu)$ converges to f if for every measurable subset $A \subset X$ we have $\mu(f_i(A) \triangle f(A)) \to 0$ as $i \to \infty$, where $\triangle$ denotes the symmetric difference of sets. This makes $\mathrm{Aut}(X, \mu)$ into a Polish space, i.e., a separable completely metrizable topological space. In 1959 Rokhlin showed that the subset of all transformations $T \in \mathrm{Aut}(X, \mu)$ that have zero entropy with respect to μ contains a dense G_δ subset [24]. Later Rudolph extended this result to countable amenable actions [14] and Bowen further extended it to the nonamenable case [9].

In the late 1960s Katok and Stepin developed a quantitative approximation technique and were able to apply it not only for automorphisms but also for homeomorphisms of a compact manifold M of dimension at least 2 with respect to the uniform topology, i.e., induced by the metric

$$\rho(f, g) := \sup_{x \in M} d(f(x), g(x)),$$

2020 *Mathematics Subject Classification*: Primary 37A05; Secondary 37A35, 37E05.
Key words and phrases: entropy, invertibility, interval map, Lebesgue measure preserving.
Received 13 March 2025; revised 22 January 2026.

DOI: 10.4064/sm250313-9-2

where d is the standard metric on M. In particular, they showed that the generic Lebesgue measure preserving homeomorphisms of M have zero measure theoretic entropy with respect to the Lebesgue measure [17, 18]. These results have modern proofs established by Guihéneuf [16].

Zero entropy systems are invertible almost surely, thus such genericity results as described above might not be surprising for invertible systems. It is less clear that similar results could hold in the noninvertible setting. There are only a few known results for noninvertible maps in this direction. In particular, Parry proved the following results in [21]. In the setting of the one-sided full 2-shift, he considered the weak topology on the set of invariant ergodic probability measures, and showed that the set of zero entropy (ergodic, fully supported, and nonatomic) measures is a dense G_δ set. This result immediately translates to the doubling map on the circle. Then he considered the space S of strictly increasing degree 2 maps of the circle which preserve the Lebesgue measure with the uniform topology. In this setting he showed that the set of ergodic maps whose metric entropy with respect to Lebesgue measure is zero is dense G_δ.

It is clear that Parry's result also holds for full branch unimodal interval maps preserving the Lebesgue measure. The only other result in this direction is the generalization of Parry's result to full branch k-modal interval maps preserving Lebesgue measure by Friedland and Weiss [15].

We consider the generalization of this question when we omit the (quite strong) piecewise monotone restriction. Let $I := [0,1]$. Define C_λ to be the set of continuous maps of I which preserve the Lebesgue measure λ, and denote the measure-theoretic entropy of a map $f \in C_\lambda$ with respect to λ by $h_\lambda(f)$. We consider the uniform topology. The first result of this article is that there is a dense G_δ set of maps in C_λ with zero Lebesgue measure-theoretic entropy; more precisely:

THEOREM 1. *The set* $\{f \in C_\lambda : h_\lambda(f) = 0\}$ *is a dense* G_δ *subset of* C_λ.

Let us note that Bobok and Troubetzkoy [7] proved that $\{f \in C_\lambda : h_\lambda(f) = c\}$ is dense in C_λ, but only for each $c \in (0,\infty]$. The proof of Theorem 1 follows a different path than the proofs of Parry [21] or Friedland and Weiss [15]. By conjugating, we see that the theorem also holds more generally for C_μ where μ is any invariant nonatomic probability measure with full support [6, Remark 1.1].

Adapting various known results and combining them with our results reveals sharp contrasts between the measure-theoretic and topological behavior of generic maps in C_λ. In order to adapt the results of Bruckner and Garg [10], we recall a strong notion of nowhere differentiability: a map f is *of nonmonotonic type* if the upper derivative $\overline{D}f(x)$ equals $+\infty$ and the lower derivative $\underline{D}f(x)$ equals $-\infty$ for every x.

Let $f \in C(I)$. For every $c \in \mathbb{R}$ the set $\operatorname{Lev}_f(c) := \{x : f(x) = c\}$ is called a *level set* of f. Following [10, Definition 1.2] the level sets of a function $f \in C(I)$ are said to be *normal* if there is a countable set $E_f \subset (\min f, \max f)$ such that the level set $\operatorname{Lev}_f(c)$ of f is

- a nonempty perfect set when $c \notin E_f \cup \{\min f, \max f\}$;
- a single point when $c = \min f$ or $c = \max f$;
- the union of a nonempty perfect set P and an isolated point $x \notin P$ when $c \in E_f$.

THEOREM 2.

(I) *Generic maps in C_λ are of nonmonotonic type.*
(II) *The level sets of generic maps in C_λ are normal.*

Clearly there are no continuous invertible interval maps that are nowhere differentiable. A map $f \in C_\lambda$ is called *invertible a.e.* if there is an f-invariant set $X \subset I$ of full Lebesgue measure such that the restriction $f|_X$ is invertible. Also let

$$B_0 := \{x \in I : \# \operatorname{Lev}_f(x) > 1\}.$$

The following corollary contrasts the topological and measure-theoretic behavior of generic maps.

COROLLARY 3. *There is a dense G_δ subset Q of C_λ such that*

(1) *for each $f \in Q$ there exists a set $X_f \subset I$ of Lebesgue measure 1 such that f is a bijection on X_f and $\lambda(M) = \lambda(M \cap X_f) = \lambda(f^{-1}(M \cap X_f))$ for each Borel set $M \subset I$;*
(2) *for each map $f \in Q$ there exists a set $X_f \subset I$ of Lebesgue measure 1 such that*
    (a) *the level set $\operatorname{Lev}_f(c)$ is a perfect set for each $c \in (0,1)$, while*
    (b) *for each $c \in I$ the set $X_f \cap \operatorname{Lev}_f(c)$ contains at most one point;*
(3) *except for the two exceptional maps* id *and* $1-$id *in C_λ the set B_0 contains an interval, and thus we always have a set of positive Lebesgue measure on which f is not one-to-one* (1).

Proof. According to [27, Corollary 4.14.3], $h_\lambda(f) = 0$ implies that f is invertible a.e. Thus combining this with Theorems 1 and 2 yields statement (1). Statement (2) follows immediately by combining (1) with Theorem 2. Statement (3) follows from [5, Corollary 3.2], which says that except for the two exceptional maps id and $1 - $ id the set B_0 contains an interval, and thus we always have a set of positive measure on which f is not one-to-one. ∎

---

(1) On the first page of [7] there is an incorrect statement that any $f \in C_\lambda$ other than id and $1 - $ id satisfies $h_\lambda(f) > 0$. This need not be the case as exhibited by the results of this article.

Unlike the approach given in [21] and [15], it is not clear if one can constructively define the set X_f in Corollary 3 for a generic map in C_λ.

Remark 4. The reverse implication, that f being invertible a.e. implies that $h_\lambda(f) = 0$, is not true in general. It is known that any compact manifold of dimension $d > 1$ which carries a minimal uniquely ergodic homeomorphism also carries a minimal uniquely ergodic homeomorphism with positive topological entropy [1] and the measure can be assumed to be the Lebesgue measure. It seems unknown if this reverse implication holds in C_λ; compare with [23, Theorem 1.9.7] and [27, Corollary 4.18.1].

For comparison we recall other generic properties of maps from C_λ; in our context, items (6) and (8) below are particularly interesting. For the definitions of the notions we mention in the following theorem we refer the reader to the indicated articles.

Theorem. *There is a dense G_δ subset G of C_λ such that for each $f \in G$*

(1) *f has infinite topological entropy* [7]*;*
(2) *f is weakly mixing with respect to λ* [7]*;*
(3) *f is leo and thus satisfies the specification property* [7]*;*
(4) *f has the shadowing property* [3]*;*
(5) *periodic points have Cantor set structure* (*see* [3] *for details*)*;*
(6) *f has λ-a.e. point a knot point* [2]*;*
(7) *the graph of f has Hausdorff dimension and lower box dimension* 1 *and upper box dimension* 2 [7, 25]*;*
(8) *for every $\delta > 0$ there is n such that f^n is δ-crooked* [12].

The leo and specification properties are actually satisfied not only generically but on an open dense subset of C_λ [6].

Two other related results were proven by Sigmund. In his first result he considered the full shift on $\sigma : \mathcal{R}^{\mathbb{Z}} \to \mathcal{R}^{\mathbb{Z}}$ and showed that the probability measures with zero entropy form a dense G_δ in the weak topology. He also considered the set of all subshifts with a fixed finite alphabet with respect to the Vietoris topology and showed that the generic subshift has zero topological entropy [26].

A generalization of Parry's result on genericity of zero entropy measures was obtained by Carvalho and Condori [11]. They considered uniformly continuous maps f defined on a Polish metric space and showed that the set of f-invariant measures with zero measure-theoretic entropy is a G_δ set in the weak topology and that this set is dense and thus generic if the set of f-periodic measures is dense in the set of f-invariant measures.

Let us give an outline of the present article. In Section 2 we prove Theorem 1. In Section 3 we give a construction of Lebesgue measure preserving maps that have measure-theoretic entropy 0 with respect to the Lebesgue measure. In Section 4 we prove Theorem 2, which is the second main result of our article. In Section 5 we provide an application of our main results. Part (8) from the previous theorem implies that the inverse limit of any generic map from C_λ is a curious topological object called the pseudo-arc. We apply Theorem 1 and Corollary 3 to give additional information on some measure-theoretic aspects of the construction of planar homeomorphisms with pseudo-arc attractors from [12].

2. Proof of Theorem 1. Let $\mathcal{P} := \{P_1, \dots, P_k\}$ be a finite partition of I and let

$$H(\mathcal{P}) := -\sum_{i=1}^{k} \lambda(P_i) \log \lambda(P_i).$$

For $f \in C_\lambda$ we denote

$$h_n(f, \mathcal{P}) := \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}\Big).$$

This function is decreasing in n, thus the limit $h(f, \mathcal{P}) := \lim_{n\to\infty} h_n(f, \mathcal{P})$ exists. Finally, we define the *entropy* of f by

$$h_\lambda(f) := \sup\{h(f, \mathcal{P}) : \mathcal{P} \text{ is a finite partition of } I\}.$$

Let $\mathcal{P}_i$ be the partition of I into dyadic intervals $\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right]$ and $\mathcal{A}_i$ the subalgebra generated by $\mathcal{P}_i$. Then $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ and $\bigvee_{n=1}^{\infty} \mathcal{A}_n \mathrel{\mathring{=}} \mathcal{B}$, hence by [27, Theorem 4.22], $h_\lambda(f) = \lim_{i\to\infty} h(f, \mathcal{P}_i)$. In what follows, we denote by $B(f, \delta)$ the ball in C_λ with center at f and of radius δ with respect to the uniform metric.

LEMMA 5. *Fix $f \in C_\lambda$ and choose an arbitrary interval $J = [a, b] \subset I$. Then*

$$\forall \varepsilon > 0\ \exists \delta > 0\colon\ \lambda(f^{-1}(J) \triangle g^{-1}(J)) < \varepsilon \text{ for all } g \in B(f, \delta).$$

Proof. Fix $\varepsilon > 0$ and let $\delta < \varepsilon/4$. If we put $H_\delta = f^{-1}([a+\delta, b-\delta])$, then since $f \in C_\lambda$,

$$\lambda(f^{-1}(J)) = \lambda(J) > \lambda(H_\delta) > \lambda(J) - \varepsilon/2. \tag{1}$$

For each $g \in B(f, \delta)$ we can write analogously

$$\lambda(g^{-1}(J)) = \lambda(J) > \lambda(H_\delta) > \lambda(J) - \varepsilon/2. \tag{2}$$

Clearly $H_\delta \subset (f^{-1}(J) \cap g^{-1}(J))$, so (1) and (2) imply

$$\lambda(f^{-1}(J) \triangle g^{-1}(J)) < \varepsilon. \ \blacksquare$$

Lemma 6. *For $\ell \in \mathbb{N}$ let positive numbers $a_1, \dots, a_\ell$ satisfy $\sum_{i=1}^{\ell} a_i = \eta < 1$. Then*

$$-\sum_{j=1}^{\ell} a_j \log a_j \leqslant \eta \log \ell - \eta \log \eta.$$

Proof. Let $b_i := a_i/\eta$. Then $\sum_{i=1}^{\ell} b_i = 1$ and thus $-\sum_{i=1}^{\ell} b_i \log b_i \leqslant \log \ell$ [27, Corollary 4.2.1], which is equivalent to the announced inequality. ■

Proposition 7. *Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a partition of I consisting of intervals, and $n \in \mathbb{N}$. The map*

$$f \mapsto h_n(f, \mathcal{P}), \qquad f \in C_\lambda,$$

is continuous.

Proof. Fix $\varepsilon > 0$ and $f \in C_\lambda$. Put $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}) = \{A_1, \dots, A_m\}$, where $\lambda(A_j) > 0$ for each $j \in \{1, \dots, m\}$ and $\sum_{j=1}^{m} \lambda(A_j) = 1$. Each set A_j can be written as

$$A_j = P_{i_0(j)} \cap f^{-1}(P_{i_1(j)}) \cap \dots \cap f^{-(n-1)}(P_{i_{n-1}(j)}), \qquad j \in \{1, \dots, m\},$$

and all other $k^n - m$ possible intersections have measure 0.

For g close to f we consider $\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{P})$ and let ℓ_g be the number of positive measure elements of this partition. Enumerate the sets $B_{j,g}$ in $\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{P})$ in such a way that

$$B_{j,g} = P_{i_0(j)} \cap g^{-1}(P_{i_1(j)}) \cap \dots \cap g^{-(n-1)}(P_{i_{n-1}(j)}), \qquad j \in \{1, \dots, m\}.$$

It is clear that $B_{j,f} = A_j$. Furthermore these sets vary continuously with respect to g sufficiently close to f in the following sense. Applying Lemma 5 to the maps $f^0, f^1, \dots, f^{n-1} \in C_\lambda$ we can choose sufficiently small $\delta > 0$ so that each $B_{j,g}$ has positive measure provided that A_j has positive measure for $1 \leq j \leq m$. Hence $\ell_g \geqslant m$, we may assume that $\lambda(B_{j,g}) > 0$ for $1 \leq j \leq \ell_g$ and furthermore (as a consequence of Lemma 5; see also Lemma 6)

$$-\sum_{j=1}^{m} \lambda(A_j) \log \lambda(A_j) + \sum_{j=1}^{m} \lambda(B_{j,g}) \log \lambda(B_{j,g}) < n\varepsilon/2,$$

and $\sum_{j=m+1}^{\ell_g} \lambda(B_{j,g}) = 1 - \sum_{j=1}^{m} \lambda(B_{j,g}) = \eta$, from which

$$\eta \log(k^n - m) - \eta \log \eta < n\varepsilon/2$$

for each $g \in B(f, \delta)$. Since $\ell_g \leqslant k^n$, from Lemma 6 we obtain

$$0 < -\sum_{j=m+1}^{\ell_g} \lambda(B_{j,g}) \log \lambda(B_{j,g}) \leqslant \eta \log(k^n - m) - \eta \log \eta < n\varepsilon/2. \tag{3}$$

From (3) we obtain

$$|h_n(f,\mathcal{P}) - h_n(g,\mathcal{P})| = \left|\frac{1}{n}H\Big(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P})\Big) - \frac{1}{n}H\Big(\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{P})\Big)\right| < \varepsilon. \ \blacksquare$$

Proof of Theorem 1. Fix $\beta > 0$ and let

$$Q_\beta := \bigcap_{k\geqslant 1}\bigcup_{i\geqslant k}\bigcup_{n\geqslant k}\{f \in C_\lambda : h_n(f,\mathcal{P}_i) < \beta\}.$$

We claim that

(i) Q_β a dense G_δ set;
(ii) Q_β coincides with the set of maps with entropy less than β.

Once the claim is established the theorem follows since $Q := \bigcap_{n\geqslant 1} Q_{1/n}$ is a dense G_δ set and it coincides with the set of zero entropy maps.

We turn to the proof of the claim. Let us first prove (ii). If $h_\lambda(f) < \beta$ then $h_n(f,\mathcal{P}_i) < \beta$ for each i for all sufficiently large n, so $f \in Q_\beta$. Conversely, to see that the entropy of the maps in Q_β is at most β we fix $f \in Q_\beta$. By the definition of Q_β, for each $f \in Q_\beta$ there are sequences $i_k \geqslant k$ and $n_k \geqslant k$ such that $h_{n_k}(f,\mathcal{P}_{i_k}) < \beta$ for each $k \geqslant 1$. Since $h_n(f,\mathcal{P})$ is decreasing in n, we conclude $h(f,\mathcal{P}_{i_k}) < \beta$.

Since $\mathcal{P}_{i+1}$ refines $\mathcal{P}_i$, we have $h(f,\mathcal{P}_i)$ monotonic in i and we conclude $h(f,\mathcal{P}_i) < \beta$ for all $i \geq i_1$. Thus $h_\lambda(f) = \lim_i h(f,\mathcal{P}_i) < \beta$, which finishes the proof of (ii).

To prove (i) we remark that the set $\{f \in C_\lambda : h_n(f,\mathcal{P}_i) < \beta\}$ is open by Proposition 7, thus Q_β is a G_δ set.

The set Q_β coincides with the set of maps with entropy less than β, thus its density follows from [7, Proposition 24]. $\blacksquare$

3. Construction of zero entropy maps.

Suppose $f \in C_\lambda$ be such that $\lambda(\mathrm{Per}(f)) = 1$. If we use the ergodic decomposition theorem [27, p. 153] then

$$h_\lambda(f) = \int_{E(I,f)} h_\mu(f)\,d\nu(\mu), \tag{4}$$

where $E(I,f)$ is the set of all ergodic measures invariant for f, and ν is a Borel measure on $E(I,f)$ satisfying

$$\lambda = \int_{E(I,f)} \mu\,d\nu(\mu).$$

Since $\lambda(\mathrm{Per}(f)) = 1$, necessarily ν-a.e. ergodic measure μ is a CO-measure (i.e., a measure on a periodic orbit). In [3, Theorem 2] we constructed a dense set of leo maps for which the periodic points have full measure, and thus they all have zero entropy with respect to Lebesgue measure. This yields an alternative proof of the density of Q from the proof of Theorem 1.

Remark 8. Parry sketched a construction of maps $f \in C_\lambda$ which are piecewise monotone with two full branches which are invertible λ-a.e. [21]; a more detailed version of this construction (with k full branches) was given by Friedland and Weiss [15].

4. Proof of Theorem 2. Let $C(I)$ denote the set of continuous maps $f : I \to I$. A function $f \in C(I)$ is said to be *nondecreasing at a point* $x \in I$ if there exists a $\delta > 0$ such that $f(t) \le f(x)$ when $t \in I \cap (x - \delta, x)$ and $f(t) \ge f(x)$ when $t \in I \cap (x, x - \delta)$. The function f is called *nonincreasing at* x if $-f$ is nondecreasing at x, and f is called *monotone at* x if it is either nondecreasing or nonincreasing at x. If $f \in C(I)$ and $\gamma \in \mathbb{R}$ then we define the map $f_{-\gamma}$ by

$$f_{-\gamma}(x) := f(x) - \gamma x \quad \text{for } x \in I. \tag{5}$$

A map f is *of monotonic type at* x if there exists a real number γ such that $f_{-\gamma}$ is monotone at x. If f is not of monotonic type at any point of I, it will be said to be *of nonmonotonic type.* It is easy to see that f is of nonmonotonic type if and only if for each x, $\overline{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$, where $\overline{D}f$ and $\underline{D}f$ are the upper and lower derivatives.

The proof of Theorem 2 requires two auxiliary lemmas. The first lemma is a minor modification of [4, Lemma 11].

Lemma 9. *Let f be a piecewise affine continuous interval map with nonzero slopes and such that its derivative exists everywhere off a finite set E. Then $f \in C_\lambda$ if and only if*

$$\forall y \in [0,1) \setminus F(E)\colon \quad \sum_{x \in F^{-1}(y)} \frac{1}{|F'(x)|} = 1. \tag{6}$$

A *critical value* of a map f is the value of f at a local maximum point or a local minimum point. A *determining value* of a piecewise affine map f is either one of its critical values or its value at a point at which the derivative of f does not exist.

Lemma 10. *The set of piecewise affine maps with rational determining values is dense in C_λ.*

Proof. Consider a dense collection of piecewise affine maps $\{f_m\} \subset C_\lambda$ (Figure 1, left) with each f_m having finitely many determining points. Fix m and choose $n = n(m)$ so large that each of the intervals $[k/n, k + 1/n]$ contains at most one determining value. For each irrational determining value c_i of f_m we consider the map $g_{n,i} \circ f_m$, where $g_{n,i}$ is the map of Figure 1 (right). The resulting map has the determining value c_i replaced by k/n and

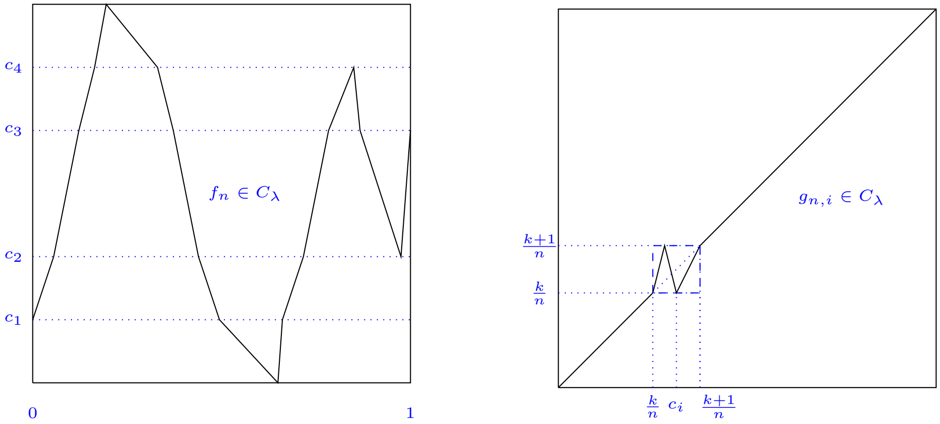

Fig. 1. Removing irrational critical values.

$(k+1)/n$. We consecutively remove all irrational determining values of f_m to obtain a map h_m with only rational determining values. The set $\{h_m\}$ is dense in C_λ since $\rho(f_m, h_m) \to 0$. ■

The proof of Theorem 2 follows the strategy of the proof of [10, Theorem 2.2]. We use the terminology of that proof. In particular, for any $n \in \mathbb{N}$, A_n denotes the set of functions $f \in C_\lambda$ for which there exist $\gamma \in [-n, n]$ and $x \in [0,1]$ such that $f_{-\gamma}(t) \leqslant f_{-\gamma}(x)$ when $t \in [0,1] \cap (x - 1/n, x)$ and $f_{-\gamma}(t) \geqslant f_{-\gamma}(x)$ when $t \in [0,1] \cap (x, x + 1/n)$. Recall that $f_{-\gamma}$ was defined in (5); here we changed the notation of [10] replacing their λ by γ to avoid a collision of terminologies.

Proof of Theorem 2. Except for one step, the strategy of the proof is the same as in [10]; the difference is due to the preservation of Lebesgue measure. The step which differs is showing that each A_n is nowhere dense; this is the most technical step in the proof in [10]. To show this we proceed as illustrated in Figure 2. We know that piecewise affine maps are dense in C_λ [7]. So from now on, the $n \in \mathbb{N}$ defining A_n is fixed.

I. *A choice of f.* Using Lemma 10, given an open set $U \subset C_\lambda$ and $\varepsilon > 0$ we can choose a piecewise affine map $f \in U$ whose determining values are rational such that $B(f, \varepsilon) \subset U$. Let q be the greatest common denominator of the determining values. Let $1/\tau$ be a positive integer multiple of q such that $\tau < \varepsilon/3$. For each integer $1 \leqslant i \leqslant 1/\tau$ consider the interval $J_i = [(i-1)\tau, i\tau]$. These intervals satisfy

(i) $\lambda(J_i) = \tau$ for all i;
(ii) $J_i^\circ \cap J_j^\circ = \emptyset$ for all $i \neq j$;
(iii) no J_i has a determining value in its interior;
(iv) for each i, all points inside a J_i have the same number $n(i)$ of preimages;

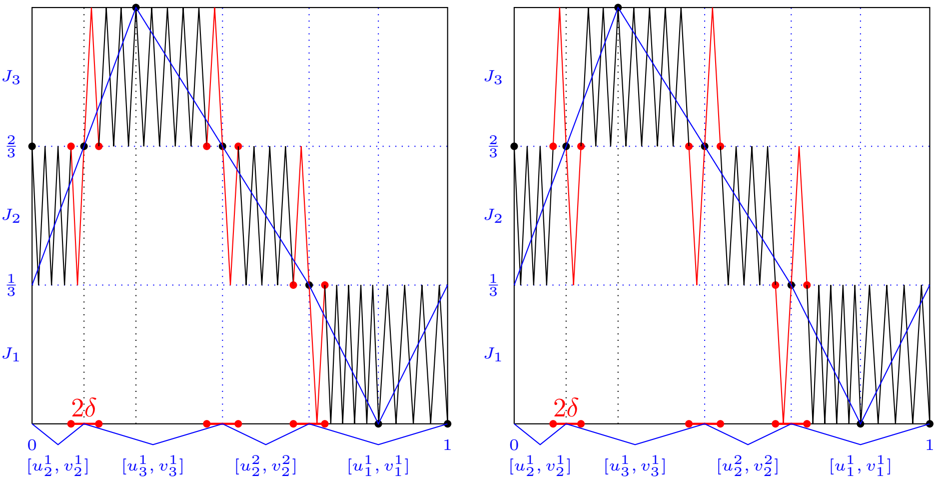

Fig. 2. Left: the map $\tilde{f}$ in step II. Right: the map F in step III.

(v) $f^{-1}(J_i^\circ)$ has $n(i)$ connected components $[u_i^k, v_i^k]$, $1 \leqslant k \leqslant n(i)$;

(vi) $f([u_i^k, v_i^k]) = J_i$ for each k;

(vii) every window perturbation of f on $[u_i^k, v_i^k]$ remains in $B(f, \tau) \subset B(f, \varepsilon/3)$; furthermore, this containment holds if we make a finite number of such window perturbations (see [6, Section 2.1] for the definition of window perturbations).

II. *A perturbation* $\tilde{f}$ *of* f. We choose $\delta > 0$ so small that

$$\delta < \min_{i,k}((v_i^k - u_i^k)/2) \tag{7}$$

and satisfying a further assumption which will be made in the next step.

If x is a common point of two adjacent intervals $[u_i^k, v_i^k]$ then $[x-\delta, x]$ and $[x, x+\delta]$ are called *δ-marked intervals.* Each δ-marked interval is contained in a single interval $[u_i^k, v_i^k]$ and its image is contained in a single J_i, and δ-marked intervals have pairwise disjoint interiors. We consider a (not necessarily regular) piecewise affine window perturbation $\tilde{f} \in B(f, \tau)$ of f on all $[u_i^k, v_i^k]$'s such that (see Figure 2, left)

(viii) if an interval K is contained $[u_i^k, v_i^k]$ and $\lambda(K) = \delta$ then K contains either at least three critical points of $\tilde{f}$, or two critical points of $\tilde{f}$ and one of the endpoints u_i^k, v_i^k;

(ix) if $[u_i^1, v_i^1]$ is the leftmost interval and f is increasing on $[u_i^1, v_i^1]$ then $\tilde{f}(u_i^1) = f(v_i^1)$, $\tilde{f}(v_i^1) = f(v_i^1)$, and the map $\tilde{f}$ has two laps on the δ-marked interval $[v_i^1 - \delta, v_i^1]$;

(x) if $[u_i^{n(i)}, v_i^{n(i)}]$ is the rightmost interval and f is increasing on $[u_i^{n(i)}, v_i^{n(i)}]$ then $\tilde{f}(u_i^{n(i)}) = f(u_i^{n(i)})$, $\tilde{f}(v_i^{n(i)}) = f(u_i^{n(i)})$, and the map $\tilde{f}$ has two laps on the δ-marked interval $[u_i^{n(i)}, u_i^{n(i)} + \delta]$;

(xi) otherwise $\tilde{f}(u_i^k) = f(u_i^k)$ and $\tilde{f}(v_i^k) = f(v_i^k)$ for all i, k, and the map $\tilde{f}$ has two laps on the δ-marked intervals $[u_i^k, u_i^k + \delta]$ and $[v_i^k - \delta, v_i^k]$.

By construction, the map $\tilde{f} \in B(f, \tau) \subset B(f, \varepsilon/3)$ satisfies (6), so $\tilde{f} \in C_\lambda$.

III. *Constructing F from $\tilde{f}$.* Finally, we construct the map F, adapting the map $\tilde{f}$, such that (see Figure 2, right)

(xii) the graphs of $\tilde{f}$ and F coincide outside of the δ-marked intervals;

(xiii) if $J = [d, e]$ is a δ-marked interval, then $\tilde{f}(d) = \tilde{f}(e)$; the graph of $F|_J$ is a copy of the graph $\tilde{f}|_J$ reflected in the line $y = f(d)$, i.e., $F(x) = 2\tilde{f}(d) - \tilde{f}(x)$; equivalently, if $J = [d, e]$ is the union of two consecutive δ-marked intervals, then $\tilde{f}(d) = \tilde{f}(e)$; the graph of $F|_J$ is a copy of the graph $\tilde{f}|_J$ reversed, i.e., $F(x) = \tilde{f}(d + e - x)$.

By construction, the map F satisfies (6), so $F \in C_\lambda$. By (vii), $F \in B(\tilde{f}, 2\tau) \subset B(f, \varepsilon) \subset U$. Let us show that $F \notin A_n$. Notice that by (7) and (viii) the map F has absolute value of the slope greater than $10n$ on each affine part of its graph.

If $x \in [0, 1]$ and F is decreasing on some one-sided neighborhood V of x then $F_{-\gamma}$, $\gamma \in [-n, n]$, has its derivative less than $-9n$ on the same neighborhood V of x, so either there is $t \in (x, x+1/n)$ with $F_{-\gamma}(x) > F_{-\gamma}(t)$ or $t \in (x - 1/n, x)$ for which $F_{-\gamma}(t) > F_{-\gamma}(x)$. This is the case when x is an endpoint or a critical point of F or such that F is decreasing on some two-sided neighborhood of x.

It remains to consider the case when $x \in (0, 1)$ and F is increasing on a two-sided neighborhood of x. Let $x \in [u_i^k, v_i^k]$. We distinguish several possibilities.

For convenience we denote the endpoints of the interval J_i by $a_i = (i-1)\tau$ and $b_i = i\tau$.

We additionally assume that

$$\delta < \frac{\tau}{10n}.$$

The first case is when $F(x) \in J_i$ and $F(x) \leqslant (a_i + b_i)/2$. In this case by (viii) there has to be a point $y \in [u_i^k, x)$ for which $x - y < 3\delta$ and $F(y) = b_i$. Then since

$$3\delta < \frac{3\tau}{10n} < \frac{3}{10n} < \frac{1}{n} \quad \text{and} \quad \frac{F(x) - F(y)}{x - y} < -\frac{\tau}{6\delta} < -\frac{5n}{3},$$

we obtain

$$\frac{F_{-\gamma}(x) - F_{-\gamma}(y)}{x - y} < -\frac{2n}{3} \qquad \text{for } y \in \left(x - \frac{1}{n}, x\right).$$

The second case is $F(x) \in J_i$ and $F(x) > (a_i + b_i)/2$. Again by (viii) there has to be a point $y \in (x, v_i^k)$ for which $y - x < 3\delta$ and $F(y) = a_i$.

Analogously to the previous case,

$$\frac{F(x)-F(y)}{x-y} < -\frac{5n}{3},$$

hence

$$\frac{F_{-\gamma}(x)-F_{-\gamma}(y)}{x-y} < -\frac{2n}{3} \quad \text{for } y \in \left(x, x+\frac{1}{n}\right).$$

The last case is when $F(x) \in J_\ell$ with $\ell \neq i$. Then

- if $\ell = i+1$, one can proceed as above, taking for y the closest critical point to the right of x where the local minimum is attained;
- if $\ell = i-1$, one can proceed as above, taking for y the closest critical point to the left of x where the local maximum is attained.

This finishes the proof of (I). Assertion (II) on level sets follows by using the same arguments as in [10, Theorem 3.3], but replacing their Theorem 2.2 by (I). Note that if $f \in C_\lambda$ then $\min f = 0$ and $\max f = 1$. ■

5. Consequences for a parametrized family of planar attractors.

In this section, we apply Theorem 1 and Corollary 3 to provide a better understanding of some planar attractors from [12]. We find these results surprising, because they highlight the dichotomy between the topological and measure-theoretic perception of these attractors. Before we proceed, we need some additional terminology.

Let X be a compact metric space and let $F : X \to X$ be continuous. Define

$$\hat{X}_F := \varprojlim(X,F) = \Big\{\hat{x} := (x_0,x_1,x_2,\ldots) \in \prod_{i=1}^{\infty} X : x_i = F(x_{i+1}),\ \forall i \geq 0\Big\},$$

where F is called the *bonding map.* We equip the *inverse limit* $\hat{X}_F$ with the subspace metric which is induced by the *product metric* in $\prod_{i=1}^{\infty} X_i$. We can define the *natural extension* $\hat{F} : \hat{X}_F \to \hat{X}_F$ for $(x_0,x_1,x_2,\ldots) \in \hat{X}_F$ by

$$\hat{F}((x_0,x_1,x_2,\ldots)) = (F(x_0),F(x_1),F(x_2),\ldots) = (F(x_0),x_0,x_1,\ldots).$$

Denote by $\pi_n : \hat{X}_F \to X$ the *coordinate projection maps.* Recall also that $\mathcal{B}(X)$ denotes the *σ-algebra of Borel sets in X.*

For the details of what follows we refer the reader to [12]. Let $\mathcal{L}$ be the Lebesgue measure on a topological disk $\mathcal{D}$ and λ the Lebesgue measure on $I = [0,1]$. Suppose that $I \times \{0\} \subset \operatorname{int}(\mathcal{D})$ and let $f : I \to I$ be a continuous map. It can be extended to a map F defined as a uniform limit of homeomorphisms of $\mathcal{D}$. Let us denote $\hat{\mathcal{D}} := \varprojlim(\mathcal{D},F)$; by Brown's theorem, $\hat{\mathcal{D}}$ is again a topological disk. Let $\mathcal{B}(\hat{\mathcal{D}})$ be the smallest σ-algebra on $\hat{\mathcal{D}}$ such that all the projection maps π_i are measurable. Then there exists a unique probability measure $\hat{\mathcal{L}}$ on $\mathcal{B}(\hat{\mathcal{D}})$ such that $\hat{\mathcal{L}}(\pi_n^{-1}(A)) = \mathcal{L}(A)$ for all

$A \in \mathcal{B}(\mathcal{D})$ and each $n \in \mathbb{N}_0$ (where the measure $\hat{\lambda}$ is defined in an analogous way).

Let us recall a result from [22]. Let μ be an F-invariant Borel probability measure on $\mathcal{D}$. The set B_μ which consists of all points $x \in \mathcal{D}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(F^i(x)) = \int g\, d\mu$$

for all continuous maps $g : \mathcal{D} \to \mathbb{R}$ is called the *basin of μ for F*. It is well known that B_μ is a Borel set (see for example [13, Proposition 4.7]). We call the measure μ *physical for F* if $\mathcal{L}(B_\mu) > 0$. An invariant measure $\hat{\nu}$ for the natural extension $\hat{F} : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$ is called an *inverse limit physical measure* if $\hat{\nu}$ has a basin $\hat{B}_{\hat{\nu}}$ such that $\mathcal{L}(\pi_0(\hat{B}_{\hat{\nu}})) > 0$.

By results from [19], if $\mathcal{L}$ is a physical measure for F then the induced measure $\hat{\mathcal{L}}$ on $\hat{\mathcal{D}}$ is an inverse limit physical measure for the natural extension $\hat{F}$. In particular, there is a basin $\hat{B}_{\hat{\mathcal{L}}} := \pi_0^{-1}(B)$ of $\hat{\mathcal{L}}$ for $\hat{F}$ with $\mathcal{L}(B) > 0$. In particular, this statement holds for λ, I and f.

The pseudo-arc is a very curious object arising in continuum theory (see the survey of Lewis [20] and the introduction of [8] for an overview of results involving the pseudo-arc). Its complicated structure is reflected in the fact that it is *hereditarily indecomposable*, i.e., there are no proper subcontinua $A, B \subset H$ such that $A \cup B = H$ for every subcontinuum H of the pseudo-arc P. In this sense its structure is in complete opposition to intuitive perception of an arc. Let us recall two results relating the maps in C_λ to the pseudo-arc and planar attractors.

THEOREM ([12, Theorem 1.1]). *The inverse limit with any $C_\lambda(I)$-generic map as the bonding map is the pseudo-arc.*

THEOREM ([12, Theorem 1.6]). *There exists a dense G_δ subset G of $C_\lambda(I)$ and a parametrized family of homeomorphisms $\{\Phi_f\}_{f\in G} \subset \mathcal{H}(\mathcal{D}, \mathcal{D})$ varying continuously with f and having Φ_f-invariant pseudo-arc attractors $\Lambda_f \subset D$ for every $f \in G$ such that*

1. *$\Phi_f|_{\Lambda_f}$ is topologically conjugate to $\hat{f} \colon \hat{I}_f \to \hat{I}_f$;*
2. *the attractors $\{\Lambda_f\}_{f\in G}$ vary continuously in the Hausdorff topology;*
3. *for each $f \in G$ the measure $\mu_f := \hat{\lambda}$ induced by (λ, f) is a weakly mixing, Φ_f-invariant, inverse limit physical measure supported on the attractor Λ_f (2);*
4. *the measures μ_f vary continuously in the weak* topology.*

---

(2) Here we use the fact that $\mathcal{D}$ is homeomorphic to some $\hat{\mathcal{D}}_{F_f}$, where the maps F_f vary continuously with f.

We can now interpret our results in this setting. By Corollary 3 there is a set X_f of full λ-measure such that $f|_{X_f}$ is invertible. Let

$$Z_f := \varprojlim(X_f, f) = \bigcap_{n=1}^{\infty} \pi_n^{-1}(X_f).$$

Clearly $\hat{\lambda}(Z_f) = 1$ and $\pi_0|_{Z_f}$ is one-to-one onto X_f. Therefore, μ_f is isomorphic to λ, and from the ergodic theory point of view, the dynamics of $(\mathit{\Lambda}_f, \mathit{\Phi}_f, \mu_f)$ and (I, f, λ) are the same. Therefore, on the one hand we observe a very complicated topological structure of the attractor in the disk (pseudo-arc) and a complicated topological dynamics on it (infinite topological entropy and topological mixing), but on the other hand, from the physical perspective we see a simple dynamics governed by λ which is inherited from the interval with zero measure-theoretic entropy with respect to Lebesgue measure (yet weakly mixing).

Acknowledgements. We would like to thank Mark Pollicott for pointing out Parry's article [21] to us.

Funding. The authors gratefully acknowledge the support of the Slovenian research agency ARIS grant J1-4632. J. Činč was partially supported by ARIS project under Contract No. SN-ZRD/22-27/0552. This article has been supported by EU funds under the project "Increasing the resilience of power grids in the context of decarbonisation, decentralisation and sustainable socioeconomic development", CZ.02.01.01/00/23_021/0008759, through the Operational Programme Johannes Amos Comenius.

Co-funded by the European Union

This research is part of J. Činč's project that has received funding from the European Union's Horizon Europe research and innovation programme under the Marie Skłodowska-Curie grant agreement No. HE-MSCA-PF-PFSAIL-101063512.

References

[1] F. Béguin, S. Crovisier and F. Le Roux, *Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the Denjoy–Rees technique*, Ann. Sci. École Norm. Sup. 40 (2007), 251–308.

[2] J. Bobok, *On non-differentiable measure-preserving functions*, Real Anal. Exchange 16 (1990-91), 119–129.

[3] J. Bobok, J. Činč, P. Oprocha and S. Troubetzkoy, *Periodic points and shadowing for generic Lebesgue measure-preserving interval maps*, Nonlinearity 35 (2022), 2535–2558.

[4] J. Bobok, J. Činč, P. Oprocha and S. Troubetzkoy, *s-limit shadowing is generic for continuous Lebesgue measure-preserving circle maps*, Ergodic Theory Dynam. Systems 43 (2023), 78–98.

[5] J. Bobok, J. Činč, P. Oprocha and S. Troubetzkoy, *Interval maps with dense periodicity*, arXiv:2402.05638v1 (2024).

[6] J. Bobok, J. Činč, P. Oprocha and S. Troubetzkoy, *Continuous Lebesgue measure-preserving maps on one-dimensional manifolds: a survey*, Topology Appl. 364 (2025), art. 109101, 18 pp.

[7] J. Bobok and S. Troubetzkoy, *Typical properties of interval maps preserving the Lebesgue measure*, Nonlinearity 33 (2020), 6461–6479.

[8] J. P. Boroński, J. Činč and P. Oprocha, *Beyond* 0 *and* ∞*: a solution to the Barge entropy conjecture*, Trans. Amer. Math. Soc. 379 (2026), 2441–2484.

[9] L. Bowen, *Zero entropy is generic*, Entropy 18 (2016), art. 220, 20 pp.

[10] A. M. Bruckner and K. M. Garg, *The level structure of a residual set of continuous functions*, Trans. Amer. Math. Soc. 232 (1977), 307–321.

[11] S. L. Carvalho and A. Condori, *On the generic behavior of the metric entropy, and related quantities, of uniformly continuous maps over Polish metric spaces*, Math. Nachr. 296 (2023), 980–995.

[12] J. Činč and P. Oprocha, *Parametrized family of pseudo-arc attractors: physical measures and prime end rotations*, Proc. London Math. Soc. (3) 125 (2022), 318–357.

[13] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. 527, Springer, Berlin, 1976.

[14] M. Foreman and B. Weiss, *An anti-classification theorem for ergodic measure preserving transformations*, J. Eur. Math. Soc. 6 (2004), 277–292.

[15] S. Friedland and B. Weiss, *Generalized interval exchanges and the 2 – 3 conjecture*, Centr. Eur. J. Math. 3 (2005), 412–429.

[16] P.-A. Guihéneuf, *Propriétés dynamiques génériques des homéomorphismes conservatifs*, Ensaios Mat. 22, Soc. Brasil. Mat., Rio de Janeiro, 2012.

[17] A. B. Katok and A. M. Stepin, *Approximations of ergodic dynamic systems by periodic transformations*, Dokl. Akad. Nauk SSSR 171 (1966), 1268–1271 (in Russian).

[18] A. B. Katok and A. M. Stepin, *Metric properties of measure preserving homeomorphisms*, Russian Math. Surveys 25 (1970), 191–220.

[19] J. Kennedy, B. E. Raines and D. R. Stockman, *Basins of measures on inverse limit spaces for the induced homeomorphisms*, Ergodic Theory Dynam. Systems 30 (2010), 1119–1130.

[20] W. Lewis, *The pseudo-arc*, in: Continuum Theory and Dynamical Systems (Arcata, CA, 1989), Contemp. Math. 117, Amer. Math. Soc., Providence, RI, 1991, 103–123.

[21] W. Parry, *In general a degree two map is an automorphism*, in: Symbolic Dynamics and Its Applications (New Haven, CT, 1991), Contemp. Math. 135, Amer. Math. Soc., Providence, RI, 1992, 335–338.

[22] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, AMS Chelsea, Providence, RI, 2005.

[23] F. Przytycki and M. Urbański, *Conformal Fractals: Ergodic Theory Methods*, London Math. Soc. Lecture Note Ser. 371, Cambridge Univ. Press, Cambridge, 2010.

[24] V. A. Rokhlin, *Entropy of metric automorphism*, Dokl. Akad. Nauk SSSR 124 (1959), 980–983 (in Russian).

[25] J. Schmeling and R. Winkler, *Typical dimension of the graph of certain functions*, Monatsh. Math. 119 (1995), 303–320.

[26] K. Sigmund, *On the prevalence of zero entropy*, Israel J. Math. 10 (1971), 281–288.

[27] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, New York, 1982.

Jozef Bobok
Department of Mathematics of FCE
Czech Technical University in Prague
166 29 Praha 6, Czech Republic
E-mail: jozef.bobok@cvut.cz

Serge Troubetzkoy
Aix Marseille Univ, CNRS, I2M
Marseille, France
E-mail: serge.troubetzkoy@univ-amu.fr

Jernej Činč
University of Maribor
2000 Maribor, Slovenia
and
Abdus Salam International Centre for Theoretical Physics (ICTP)
Trieste, Italy
E-mail: jernej.cinc@um.si

Piotr Oprocha
Centre of Excellence IT4Innovations
Institute for Research and Applications of Fuzzy Modeling
University of Ostrava
701 03 Ostrava 1, Czech Republic
E-mail: piotr.oprocha@osu.cz