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# FAITHFULLY QUADRATIC RINGS
## A SUMMARY OF RESULTS

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**Abstract.** This is a summary of some of the main results in the monograph *Faithfully Ordered Rings* (Mem. Amer. Math. Soc. 2015), presented by the first author at the ALANT conference, Będlewo, Poland, June 8–13, 2014. The notions involved and the results are stated in detail, the techniques employed briefly outlined, but proofs are omitted. We focus on those aspects of the cited monograph concerning (diagonal) quadratic forms over preordered rings.

**1. Introduction.** The aim of the extended research monograph [DM7] is to lay the groundwork for a theory of quadratic forms over several significant, and quite extensive, classes of preordered rings. By "quadratic forms" we understand, here, *diagonal quadratic forms with unit coefficients*; and "ring" stands for a commutative unitary ring where 2 is invertible.

We achieve this by the use, in the ring context, of our abstract theory of quadratic forms, the theory of *special groups*, expounded in [DM1] [1]. This is done as follows: fix a preordered ring (p-ring) $\langle A, T\rangle$ and for $B \subseteq A$, write $B^{\times}$ for the units of $A$ in $B$; then

(A) We lay down a theory of quadratic forms in $\langle A, T\rangle$ based on the following notion of $T$-isometry:

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[1] Corroborating a development envisaged by Manfred Knebusch at the outset of the theory of special groups. For the axioms of special groups, see [DM1], Def. 1.2, p. 2.

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*Two $n$-dimensional diagonal quadratic forms $\varphi = \sum_{i=1}^{n} a_i X_i^2$, $\psi = \sum_{i=1}^{n} b_i X_i^2$, with $a_i, b_i \in A^{\times}$ are* $T$-isometric, *$\varphi \approx_T \psi$, if there is a sequence $\varphi_0, \varphi_1, \ldots, \varphi_k$ of $n$-dimensional diagonal forms over $A^{\times}$, so that $\varphi = \varphi_0$, $\psi = \varphi_k$ and for every $1 \leq i \leq k$, $\varphi_i$ is* either *isometric to $\varphi_{i-1}$ in the usual sense that there is a matrix $M \in \mathrm{GL}_n(A)$ such that $\varphi_i = M\varphi_{i-1}M^t$,* or *there are $t_1, \ldots, t_n \in T^{\times}$ such that $\varphi_i = \langle t_1x_1, \ldots, t_nx_n\rangle$ and $\varphi_{i-1} = \langle x_1, \ldots, x_n\rangle$.*

(B) To any p-ring, $\langle A, T\rangle$, we associate a structure $G_T(A)$, whose domain is $A^{\times}/T^{\times}$, endowed with the product operation induced by $A^{\times}$, together with a binary relation $\equiv_{G_T(A)}$, defined on ordered pairs of elements of $A^{\times}/T^{\times}$, called *binary isometry*, and having $-1 = -1/T^{\times}$ as a distinguished element. The structure $\langle G_T(A), \equiv_{G_T(A)}, -1\rangle$ *is not* quite a special group in the sense of [DM1], Def. 1.2, p. 2, but it satisfies some of its axioms. However, the ring-theoretic approach, based on the definition in (A), and the formal approach (via $G_T(A)$), though related, *are far from identical.*

(C) We introduce three axioms (see Section 2.1), formulated in terms of $T$-isometry and the *value representation* relation $D^v_T$ on $\langle A, T\rangle$ defined by: for $a$, $b_1, \ldots, b_n \in A^{\times}$,

$$a \in D^v_T(b_1, \ldots, b_n) \Longleftrightarrow \text{There are } t_1, \ldots, t_n \in T \text{ such that } a = \sum_{i=1}^{n} t_i b_i.$$

These axioms express elementary properties of value representation, well-known in the classical theory of quadratic forms over fields. We then show that, when satisfied by $\langle A, T\rangle$, these axioms are sufficient — and under mild assumptions, also necessary — to ensure identity between the ring-theoretic and formal approaches; in fact, we prove:

- (C.i) The structure $G_T(A)$ is a special group (Theorem 3.2 (ii)).
- (C.ii) $T$-isometry and value representation in $\langle A, T\rangle$ are faithfully coded by the corresponding formal notions in $G_T(A)$ (Theorem 3.3 (ii)).

We call $T$-*faithfully quadratic* any p-ring $\langle A, T\rangle$ satisfying these axioms. In fact, this setting, as well as the consequences (C.i) and (C.ii), apply, more generally, to forms with entries in certain subgroups of $A^{\times}$, called $T$-*subgroups*[2], see Definition 2.1 (i) below, and also to the case where $T = A^2$; in this latter case, $T$-isometry is just matrix isometry. It is also worth noticing that, under these axioms, an analog of the classical Pfister local-global principle holds for $T$-isometry (cf. Proposition 3.6 and Definition 3.5 (i)).

Quadratic faithfulness of $\langle A, A^2\rangle$ ensures that the mod 2 $K$-theory of $A$ obtained from that in [Gu] coincides with the $K$-theory of the special group $G(A)$ as defined in [DM2] and [DM5]. In fact, only the simplest of our axioms is needed here.

(D) The axioms for $T$-quadratic faithfulness can be formalized by first-order sentences in the language of unitary rings (consisting of $+$, $\cdot$, 0, 1, $-1$) augmented by a unary predicate symbol for $T$ (not needed if $T = A^2$). These sentences are of a special form — known as *geometric* formulas — automatically guaranteeing preservation of $T$-quadratic

[2] This generalization is not mere *ars gratia artis*; it plays a crucial role in the main result of [DM6]. Notation is adapted to this broader context in a self-explanatory way; cf. Definition 2.1 (iii) and Notation 3.1 (i).

faithfulness under (right-directed) inductive limits. In the case $T = A^2$, the axioms are even *Horn* sentences, a fact ensuring, in addition, preservation under reduced products and, in particular, under arbitrary products. This issue is dealt with in § 2.2.

In Section 4 we collect a number of results proving that certain outstanding classes of p-rings, $\langle A, T\rangle$, are $T$-faithfully quadratic. The considerable effort demanded by some of these proofs (e.g., for the case of $f$-rings) is rewarded by the significance of the results thus reaped, expounded in Section 5. In fact, these establish that the theory of diagonal quadratic forms with invertible entries over several classes of p-rings, important in mathematical practice and although far from being fields, possesses many of the pleasant properties of quadratic form theory previously known to hold only in the case of formally real fields.

## 2. Axioms for faithfully quadratic rings

Definition 2.1. Recall that rings are commutative, unitary and 2 is invertible, and that $T$ denotes a preorder of a ring, $A$, or $T = A^2$.

(i) A set $S \subseteq A^\times$ is called a *$T$-subgroup* if it is closed under product and contains $T^\times \cup \{-1\}$ (if $T = A^2$, $S$ is called a *q-subgroup*).

(ii) (Various notions of representation) Let $T$ be $A^2$ or a preorder of $A$ and let $S$ be a $T$-subgroup of $A$. Let $\varphi = \langle b_1, \ldots, b_n\rangle$ be a $S$-form and let $\varphi^T = \langle b_1^T, \ldots, b_n^T\rangle$ be the corresponding $n$-form in $G_T(S)$ (having as entries the images of the $b_i$'s modulo $T^\times$).

a) $$D^{\text{iso}}_{S,T}(\varphi) = \{a \in S : \exists\, a_2, \ldots, a_n \in S\ (\varphi^T \equiv^S_T \langle a^T, a_2^T, \ldots, a_n^T\rangle)\}.$$

is the set of elements of $S$ *isometry-represented* $\operatorname{mod} T$ *by* $\varphi$ *in* $G_T(S)$ (see Notation 3.1 (i)).

b) $$D^{v}_{S,T}(\varphi) = \Big\{a \in S : \exists\, x_1, \ldots, x_n \in T \quad \Big(a = \sum_{i=1}^{n} x_i b_i\Big)\Big\}$$

is the set of elements of $S$ *value-represented* $\operatorname{mod} T$ *by* $\varphi$.

c) $$D^{t}_{S,T}(\varphi) = \Big\{a \in S : \exists\, z_1, \ldots, z_n \in T^\times \quad \Big(a = \sum_{i=1}^{n} z_i b_i\Big)\Big\}$$

is the set of elements of $S$ *transversally represented* $\operatorname{mod} T$ *by* $\varphi$.

Clearly, $D^t_{S,T}(\varphi) \subseteq D^v_{S,T}(\varphi)$.

d) The set of elements of $S$ *inductively represented by* $\varphi$ $\operatorname{mod} T$, $D^{\text{ind}}_{S,T}(\varphi)$, is defined as follows:

- If $n = 2$, $D^{\text{ind}}_{S,T}(\varphi) = D^v_{S,T}(b_1, b_2)$;
- If $n \geq 3$, $D^{\text{ind}}_{S,T}(\varphi) = \bigcap_{k=1}^{n} \bigcup\{D^v_{S,T}(b_k, u) : u \in D^v_{S,T}(b_1, \ldots, \check{b}_k, \ldots, b_n)\}$.

*Important warning.* These notions of representation *are equivalent* in the case of fields, but *are not* in the realm of rings.

(iii) When applied to forms with entries in a $T$-subgroup, $S$, of a p-ring $\langle A, T\rangle$, the notion of $T$-isometry defined in the Introduction (see item (A)) will be noted by $\approx^S_T$. The forms in the sequence $\varphi_0, \varphi_1, \ldots, \varphi_k$ therein are, of course, required to have their entries in $S$.

**2.1. The axioms.** The axioms for $T$-faithfully quadratic rings (over a $T$-subgroup $S$) are:

[$T$-FQ1] (Transversality for 2-forms) *For* $a, b \in S$, $D^v_{S,T}(a,b) = D^t_{S,T}(a,b)$, *i.e.*,

$$\forall\, x \in S\ \ \forall\, t_1, t_2 \in T\ \big(x = at_1 + bt_2 \implies \exists\, z_1, z_2 \in T^{\times}\ (x = az_1 + bz_2)\big).$$

[$T$-FQ2] *For all* $n \geq 2$ *and all* $n$*-forms* $\varphi$ *with entries in* $S$, $D^v_{S,T}(\varphi) \subseteq D^{\text{ind}}_{S,T}(\varphi)$.
This is a weak version of the inductive definition of (value) representation.

[$T$-FQ3] (Witt-cancellation for $T$-isometry) *For all* $a \in S$ *and all forms* $\varphi, \psi$ *over* $S$ *of the same dimension*, $\langle a\rangle \oplus \varphi \approx^S_T \langle a\rangle \oplus \psi \implies \varphi \approx^S_T \psi$.

**2.2. Logical form and preservation results.** Before proceeding further, we make some observations on the logical form of the preceding axioms and state some related preservation results that are useful in the development of the theory. Here are the relevant notions:

Definition 2.2. Let $L$ be a first-order language with equality.

(i) A formula of $L$, in the free variables $\bar{t}$, is *geometrical* if it is the negation of an atomic formula *or* of the form

$$\forall \overline{v}\ \big(\exists \overline{u}\ \varphi_1(\overline{v}; \overline{u}; \bar{t}) \longrightarrow \exists \overline{w}\ \varphi_2(\overline{v}; \overline{w}; \bar{t})\big),$$

where $\varphi_1, \varphi_2$ are positive and quantifier-free. A *geometrical theory* in $L$ is a theory possessing a set of geometrical axioms.

(ii) A formula in $L$ is a *positive primitive* (*pp-*) *formula* if it is of the form $\exists \overline{v}\ \varphi(\overline{v}; \bar{t})$, where $\varphi$ is a conjunction of atomic formulas.

(iii) A formula in $L$ is *Horn-geometrical* if it is the negation of an atomic formula *or* of the form $\forall \overline{v}\ (\varphi \longrightarrow \psi)$, where $\varphi$ and $\psi$ are pp-formulas. A *Horn-geometrical theory in* $L$ is a theory possessing a set of Horn-geometrical axioms. Clearly, Horn-geometrical formulas are geometrical; they also are logically equivalent to Horn formulas, cf. [CK], p. 407; [DM7], 1.5 (d), pp. 3–4.

The relevance of these notions in applications comes from the following general model-theoretic preservation results.

Proposition 2.3.

(i) ([Mir], Thm. 17.15 (b), p. 172) *The class of models of a geometric first-order theory is closed under right-directed inductive limits.* [3]

(ii) (Keisler, Galvin, Shelah; cf. [CK], Thm. 6.2.5′, p. 445) *The class of models of a first-order theory is closed under arbitrary reduced products if and only if the theory has a Horn axiomatization.*

Theorem 2.4 ([DM7], Thm. 5.2, p. 63).

(i) *The theory of faithfully quadratic rings is Horn-geometrical in the language of unitary rings, having as operations and constants the set* $\{+, \cdot, 0, 1, -1\}$.

(ii) *The theory of $T$-faithfully quadratic rings is geometrical in the language of unitary rings, together with a unary predicate symbol, $T$, which stands for a preorder.*

[3] Conversely, theories preserved under right-directed inductive limits have a geometrical axiomatization, cf. [CK], Ex. 5.2.24, p. 322. This can also be proved using the method of [Ki].

The key to this result is the following proposition, which gives a uniform bound on the length of witnessing sequences for $T$-isometry depending *only* on the dimension of the forms involved, independently of their coefficients and the underlying preordered ring.

PROPOSITION 2.5. *Let $\langle A, T\rangle$ be a p-ring and let $S$ be a $T$-subgroup of $A$. If $S$ is $T$-faithfully quadratic, $\varphi, \psi$ are forms of dimension $n$ over $S$ and $\varphi \approx_T^S \psi$, then there is a witnessing sequence for this $T$-isometry of length not greater than* $\max\{1, 3\,(2^{n-1} - 1)\}$.

As a corollary of Proposition 2.3 (i) and Theorem 2.4, we obtain:

COROLLARY 2.6. *The theories of $T$-faithfully quadratic p-rings (resp., faithfully quadratic rings), in the languages of Theorem* 2.4, *are closed under inductive limits over right-directed posets.*

Another important (and non-trivial) property of $T$-faithfully quadratic p-rings is:

THEOREM 2.7 ([DM7], Thm. 4.6, p. 52). *The theory of $T$-faithfully quadratic p-rings (in the language of Theorem* 2.4 (ii)) *is preserved under* arbitrary *products.*

Since reduced products of arbitrary structures are inductive limits of products along right-directed posets (namely, the filter defining the given reduced product, under inclusion), Theorem 2.7 and Corollary 2.6 together yield:

COROLLARY 2.8. *The theory of $T$-faithfully quadratic rings, where $T$ is a preorder, has an axiomatization by Horn sentences in the language of Theorem* 2.4 (ii).

NOTE. This corollary is only an existential statement. Even though we have a geometric axiomatization of $T$-faithfully quadratic p-rings (Theorem 2.4 (ii)), we *do not* have an explicit axiomatization by Horn sentences and, hence, not a Horn-geometric axiomatization either. However, the natural axiomatization of faithfully quadratic rings in the language of Theorem 2.4 (i) is Horn-geometric.

**3. Faithfully quadratic rings and special groups.** The results stated in this section show that, for $T$-faithfully quadratic rings (and for $T$-subgroups), the intrinsic theory of quadratic forms (via $T$-isometry) coincides with the formal theory (via special groups).

NOTATION 3.1. Let $S$ be a $T$-subgroup of a p-ring $\langle A, T\rangle$.

(i) $G_T(S)$ stands for the structure $\langle S/T^\times, \cdot, 1, -1, \equiv_T^S\rangle$ in the language $L_{SG}$ of special groups ([DM1], p. 217), with product " $\cdot$ ", identity 1 and the constant $-1$ induced from $S$, and the (4-placed) relation $\equiv_T^S$ ("isometry of binary forms") defined by

$$(\equiv_T^S) \qquad \langle a^T, b^T\rangle \equiv_T^S \langle c^T, d^T\rangle \iff a^T b^T = c^T d^T \text{ and } D^v_{S,T}(a,b) = D^v_{S,T}(c,d),$$

where $x^T = x/T^\times$ is the image of $x \in S$ in $S/T^\times$. The relation $\equiv_T^S$ is inductively extended to a binary relation on $G_T(S)^n$, for $n \geq 3$, in a standard way (cf. [DM1], 1.1 (e), p. 2).

(ii) If $\varphi = \langle a_1, \ldots, a_n\rangle$ is a form with entries in $S$, $\varphi^T = \langle a_1^T, \ldots, a_n^T\rangle$ denotes the corresponding form in $G_T(S)$.

Theorem 3.2. *Let $A$ be a ring and $T$ be a preorder of $A$ or $T = A^2$. Let $S$ be a $T$-subgroup of $A$ or a q-subgroup of $A$. If $S$ satisfies* [T-FQ1], $[\text{T-FQ2}]_3$ *and* $[\text{T-FQ3}]_3$ [4]. *Then*:

(i) *For all 3-forms $\varphi, \psi$ over $S$, $\varphi \approx_T^S \psi \iff \varphi^T \equiv_T^S \psi^T$.*

(ii) *$G_T(S)$ is a special group, which is reduced if $T$ is a preorder.*

Theorem 3.3. *Let $A$ be a ring and let $T$ be a preorder of $A$ or $T = A^2$. Let $S$ be a $T$-subgroup of $A$. Assume that $S$ satisfies* [T-FQ1], [T-FQ2] *and* $[\text{T-FQ3}]_3$. *Then*

(i) *For all $S$-forms $\varphi$, $D^{\text{iso}}_{S,T}(\varphi) = D^v_{S,T}(\varphi)$, that is, an element of $S$ is value represented iff it is isometry represented in $G_T(S)$.*

*If, in addition, $S$ satisfies* [T-FQ3] *— i.e., $S$ is $T$-faithfully quadratic — then*

(ii) *For all $S$-forms $\varphi, \psi$ of the same dimension,*

$$\varphi \approx_T^S \psi \iff \varphi^T \equiv_T^S \psi^T.$$

*In other words, the special group $G_T(S)$* faithfully represents *$T$-isometry and value representation of diagonal quadratic forms with entries in $S$.*

The preceding results admit a converse, under the rather mild assumption that axiom [$T$-FQ1] is satisfied.

Proposition 3.4. *Let $A$ be a ring and let $T$ be a preorder of $A$, or $T = A^2$. Let $S$ be a $T$-subgroup of $A$ satisfying* [T-FQ1]. *Then, the following are equivalent*:

(i) *$G_T(S)$ is a special group such that for all $S$-forms of the same dimension, $\varphi, \psi$,*

$$(*) \quad \varphi \approx_T^S \psi \iff \varphi^T \equiv_T^S \psi^T; \qquad (**) \quad D^v_{S,T}(\varphi) = D^{\text{iso}}_{S,T}(\varphi^T).$$

(ii) *$S$ and $T$ satisfy conditions* [T-FQ2] *and* [T-FQ3].

**3.1. The local-global principle.** As stated in the Introduction (see item (C)), a version of Pfister's local-global principle holds for $T$-faithfully quadratic $T$-subgroups of p-rings. The appropriate notion of signature is:

Definition 3.5. Let $\langle A, T\rangle$ be a proper p-ring and let $S$ be $T$-subgroup of $A$.

(i) A *$T$-signature* on $S$ is a group morphism, $\tau : S \longrightarrow \mathbb{Z}_2$, such that $\tau(-1) = -1$ and for all $a \in S$,

$$a \in \ker\tau \implies D^v_{S,T}(1, a) \subseteq \ker\tau.$$

A *$T$-signature* on $A$ is a $T$-signature on $A^\times$.

(ii) If $\varphi = \langle a_1, \ldots, a_n\rangle$ is a form over $S$ and $\tau$ a $T$-signature on $S$, the *integer* $\text{sgn}_\tau(\varphi) := \sum_{i=1}^n \tau(a_i)$ is the *signature of $\varphi$ at $\tau$*.

(iii) Each $\alpha \in \text{Sper}(A, T)$ gives rise to a signature on $S$, $\tau_\alpha : S \longrightarrow \mathbb{Z}_2$, given by

$$\tau_\alpha(x) = \begin{cases} 1 & \text{if } x \in \alpha \setminus (-\alpha); \\ -1 & \text{if } x \in -\alpha \setminus \alpha. \end{cases}$$

Note that if $\alpha \subseteq \beta$ in $\text{Sper}(A, T)$, then $\tau_\alpha = \tau_\beta$. In particular, if $\alpha \in \text{Sper}(A, T)$ and $\rho(\alpha)$ is the unique maximal (i.e., closed) point of $\text{Sper}(A, T)$ extending $\alpha$, then $\tau_\alpha = \tau_{\rho(\alpha)}$; that is, all signatures induced by orderings on $A$ coincide with those induced by closed points of $\text{Sper}(A, T)$.

[4] i.e., the corresponding axioms for forms of dimension $\leq 3$.

PROPOSITION 3.6 (Pfister's local-global principle for $T$-isometry; [DM7], Prop. 3.18, pp. 46–47). *Let $\langle A, T\rangle$ be a p-ring and let $S$ be a $T$-subgroup of $A$. Let $Y_T = \mathrm{Sper}(A, T)$, let $Y_T^*$ be the subspace of closed points in $Y_T$, and let $D$ be a dense subset of $Y_T^*$ in the spectral topology. If $S$ is $T$-faithfully quadratic, then for all forms $\varphi, \psi$ of the same dimension over $S$, the following are equivalent*:

(i) $\varphi \approx_T^S \psi$.

(ii) *For all $T$-signatures $\tau$ on $S$,* $\mathrm{sgn}_\tau(\varphi) = \mathrm{sgn}_\tau(\psi)$.

(iii) *For all $\alpha \in Y_T^*$,* $\mathrm{sgn}_{\tau_\alpha}(\varphi) = \mathrm{sgn}_{\tau_\alpha}(\psi)$.

(iv) *For all $\beta \in D$,* $\mathrm{sgn}_{\tau_\beta}(\varphi) = \mathrm{sgn}_{\tau_\beta}(\psi)$.

**4. Some rings that are ($T$-)faithfully quadratic.** The results collected in this section show that significant classes of p-rings, frequently occurring in various areas of mathematics, are $T$-faithfully quadratic for a variety of natural preorders $T$ (sometimes all of them). The proof of some of these results are non-trivial and occupy a sizeable part of the monograph [DM7].

DEFINITION 4.1. A ring $A$ is called *completely faithfully quadratic* if it is $T$-faithfully quadratic for all preorders $T$ of $A$ and for $T = A^2$.

**A. Rings with many units.** We start by recalling

DEFINITION 4.2. Let $R$ be a ring.

(i) A polynomial $f \in R[X_1, \ldots, X_n]$ with coefficients in $R$ *has local unit values* if for every maximal ideal $\mathfrak{m}$ of $R$, there are $u_1, \ldots, u_n$ in $R$ such that $f(u_1, \ldots, u_n) \notin \mathfrak{m}$.

(ii) $R$ is a *ring with many units* if for all $n \geq 1$, and all $f \in R[X_1, \ldots, X_n]$, if $f$ has local unit values, then there are $r_1, \ldots, r_n \in R$ such that $f(r_1, \ldots, r_n) \in R^\times$.

REMARKS 4.3.

a) For more information on rings with many units the reader is referred to [Mc] (therein called *local-global rings*), [McW], [MW], [Mar] (p. 153), [Wa], Section 4 of [DM5], and [DM6]. Fields, semi-local rings, von Neumann regular rings and arbitrary products of rings with many units have many units.

b) The last two classes of examples mentioned in (a) are special cases of Theorem 3.5 in [DM6]: the ring of global sections of a sheaf of rings over a partitionable space, whose stalks are rings with many units, is also a ring with many units. In this respect, see also Corollary 3.6 in [DM6]. Hence, the ring of global sections of a sheaf of rings over a Boolean space, whose stalks are local rings, is a ring with many units. In particular, the ring of continuous real-valued functions on a Boolean space has many units.

c) Rings of formal power series over a ring with many units also have many units; cf. [DM7], Ch. 6, § 3, pp. 77–79.

THEOREM 4.4 ([DM7], Thm. 6.5, p. 73; see also [DM4], Thm. 3.16, pp. 17–18). *If $A$ is a ring with many units such that every residue field of $A$ modulo a maximal ideal has at least 7 elements, then $A$ is completely faithfully quadratic.*

All preliminary results necessary to prove this appear in [Wa].

Incidentally, we also register:

THEOREM 4.5 ([DM7], Thm. 6.3, p. 71). *Rings with many units are axiomatizable by Horn-geometric sentences in the language of unitary rings with equality.*

**B. Reduced $f$-rings.** For basic definitions concerning lattice-ordered rings and further information on $f$-rings we refer the reader to [BKW], Chs. 8, 9, and to [S].

DEFINITION 4.6 ([BKW], Def. 9.1.1, p. 172). A lattice-ordered ring is called an *$f$-ring* if it is isomorphic to a subdirect product of linearly ordered rings.

REMARKS 4.7.

(a) A $f$-ring is reduced iff it is a subdirect product of linearly ordered domains. Both the classes of $f$-rings and of reduced $f$-rings admit purely algebraic/lattice theoretic characterizations ([BKW], Prop. 9.1.10, p. 175 and Thm. 9.3.1, pp. 178–179).

(b) The partial order of a $f$-ring $A$ will be denoted by $T_\sharp^A$ or, simply, by $T_\sharp$.

(c) The most typical examples of reduced $f$-rings (but by far not the only ones) are the rings of continuous, real-valued functions on a topological space, [GJ].

The following theorem is one of the main results in [DM7].

THEOREM 4.8.

(i) ([DM7], Thm. 8.20, p. 104) *All reduced $f$-rings are $T_\sharp$-faithfully quadratic.*

(ii) *The reduced special group $G_\sharp(A) := A^\times/T_\sharp{}^\times$, $A$ a reduced $f$-ring, is isomorphic to the Boolean algebra $B(A)$ of idempotents of $A$.*

(iii) ([DM7], Thm. 8.21, p. 106) *If $A$ is a reduced $f$-ring and $T$ a preorder of $A$ containing $T_\sharp$, then*:

- (a) *$A$ is $T$-faithfully quadratic.*
- (b) *The reduced special group $G_T(A) := A^\times/T^\times$ is isomorphic to a quotient of $B(A)$.*
- (c) ([DM7], Cor. 8.22, p. 108) *If, in addition, $T$ has bounded inversion* (*i.e.*, $1+T \subseteq A^\times$), *then $G_T(A) \simeq G_{\rm red}(A)\,(:= A^\times/\Sigma {A^\times}^2) \simeq B(A)$.*

*Sketch of the proof.* Firstly, one proves items (i) and (ii); the proof given in [DM7] is intrinsic, i.e., independent of the representation of the given $f$-ring as a subdirect product of linearly ordered domains, and done via the use of suitable sets of idempotents ([DM7], Thm. 8.18, pp. 98–104).

Next, the proof of (iii) is derived from (i) and (ii) by use of the following "lifting" of quadratic faithfulness:

PROPOSITION 4.9 ([DM7], Thm. 3.9 (b), pp. 39–40). *Let $A$ be a ring, let $P = A^2$ or a proper preorder of $A$, and let $T$ be a proper preorder of $A$ containing $P$. Let $S$ be a $T$-subgroup of $A$ that is $P$-faithfully quadratic. The following are equivalent*:

(i) *$S$ is $T$-faithfully quadratic.*

(ii) *For all $x, a_1, \ldots, a_n \in S$, if $x \in D^v_{S,T}(a_1, \ldots, a_n)$, then there are $x_2, \ldots, x_n \in S$ and $t_1, \ldots, t_n \in T^\times$ such that $\langle x, x_2, \ldots, x_n\rangle \approx^S_P \langle t_1a_1, \ldots, t_na_n\rangle$.*

(iii) *For all $S$-forms $\varphi$, $D^v_{S,T}(\varphi) = D^t_{S,T}(\varphi)$.* ∎

REMARK. The *bounded inversion* p-rings occurring in condition (iii.c) of Theorem 4.8 (and appearing in various places below) form a class with good properties. Concerning quadratic form theory, the most important of them is transversality: axiom [$T$-FQ1] holds for forms of arbitrary dimension ([DM7], Thm. 7.6, p. 84). Our proof of this property uses the theory of real semigroups, [DP]. We observe that this result *is not* valid for representation modulo squares ([DM7], Ex. 7.7, p. 85) and that rings with many units *may not* have bounded inversion, even for sums of squares.

Theorem 4.8 has some interesting consequences in the special case of rings of continuous real-valued functions.

COROLLARY 4.10 ([DM7], Prop. 8.25, p. 110 and Cor. 8.26, p. 111). *Let $X$ be a topological space. Then*:

(i) *The ring $\mathbb{C}(X)$ of real-valued continuous functions on $X$ is completely faithfully quadratic.*

(ii) *The reduced special group $G_T(\mathbb{C}(X))$ is a Boolean algebra for any proper preorder $T$ of $\mathbb{C}(X)$.*

(iii) *The ring $\mathbb{C}_b(X)$ of bounded real-valued continuous functions on $X$ is completely faithfully quadratic*; *hence so is the real holomorphy ring of $\mathbb{C}(X)$.*

### C. Other examples

- *Weakly real closed rings* (cf. definition below) — and hence real closed rings — are completely faithfully quadratic.
- *Archimedean p-rings $\langle A, T\rangle$ with bounded inversion* are $T$-faithfully quadratic ([DM7], Thm. 9.9, p. 123).

In particular

- *The real holomorphy ring* of any formally real field, ordered by sums of squares, is faithfully quadratic ([DM7], Cor. 9.12, p. 127).

A ring $A$ is *weakly real closed* if it is reduced, $A^2$ is the positive cone of a partial order on $A$, with which it is a $f$-ring, and for all $a, b \in A$, $0 \le a \le b \implies b$ divides $a^2$. This is a weakening of the notion of a real closed ring, due to Schwartz [Sc]; see also [PS].

**5. The harvest.** The results presented in this section exemplify how information concerning the theory of diagonal quadratic forms with unit coefficients in interesting classes of faithfully quadratic rings may be obtained from the theory of special groups. The same method also yields interesting information about the mod 2 algebraic $K$-theory and the graded Witt ring of some outstanding classes of rings, that we omit in this summary (cf. [DM7], Ch. 10).

THEOREM 5.1 (Arason–Pfister Hauptsatz; [DM7], Thm. 10.4, p. 132). *Let $\langle A, T\rangle$ be a proper p-ring and let $S$ be a $T$-faithfully quadratic $T$-subgroup of $A$. If $\varphi$ is a form over $S$ such that* $\dim(\varphi) < 2^n$ *and $\varphi \in I_T^n(S)$, then $\varphi$ is $T$-hyperbolic. In particular, $\bigcap_{n\ge 1} I_T^n(S) = \{0\}$.*

This comes from the fact that reduced special groups satisfy the Arason–Pfister Hauptsatz; cf. [DM1], Thm. 7.31, p. 171.

Definitions and Notation 5.2. We recall:

(i) Let $\langle A, T\rangle$ be a p-ring, where $T = A^2$ or $T$ is a preorder of $A$. Let $S$ be a $T$-faithfully quadratic $T$-subgroup of $A$. By Theorem 3.3 (ii), $T$-isometry in $S$, $\approx_T^S$, is faithfully reflected by isometry in the special group $G_T(S)$; so (by [DM1], Prop. 1.6 (b), p. 4), Witt-cancellation holds for $\approx_T^S$, and the Witt ring of classes modulo Witt equivalence (with respect to $\approx_T^S$), denoted by $W_T(S)$, can be constructed as usual. Thus, the fundamental ideal $I_T(S)$ of $W_T(S)$, and its powers, make sense and have the same meaning as in the field case (for more details on these constructions in the context of special groups, see [DM1], pp. 20 and 182–208).

(ii) (The Pfister index; [DM3], Def. 2.1.)

- If $\varphi$ is a form over $S$, the *Pfister index of degree $n$* of $\varphi$, $I(n, \varphi, S, T)$, is the least integer $k$ such that $\varphi$ is Witt-equivalent to a linear combination of $k$ Pfister forms of degree $n$, if $\varphi \in I_T^n(S)$, and 0 otherwise.
- For each integer $m \geq 1$, the *$m$-Pfister index of $S$ in degree $n$* is
$$I(n, m, S, T) = \sup\{I(n, \varphi, S, T) : \varphi \text{ a } m\text{-form over } S\} \in \mathbb{N} \cup \{\infty\}.$$
- Similarly, one defines the notions of Pfister index of a form $\varphi$ over a special group $G$, $I(n, m, \varphi, G)$, and $m$-Pfister index of $G$ in degree $n$, $I(n, m, G)$.

Theorem 5.3 ([DM7], Thm. 10.5, p. 132–133). *Let $\langle A, T\rangle$ be a p-ring with $T$ a preorder on $A$, so that either*

(i) *$A$ is an $f$-ring and $T$ a preorder containing the natural partial order $T_\sharp = T_\sharp^A$ of $A$, or*

(ii) *$T$ contains a preorder $P$ such that $\langle A, P\rangle$ is Archimedean with bounded inversion. Then*

(a) *The Arason–Pfister Hauptsatz holds in $\langle A, T\rangle$.*

(b) *For all $a, b \in A^\times$, $\langle 1, a, b, -ab\rangle$ is $T$-isotropic* (cf. [DM1], Prop. 7.17, p. 153).

(c) *With notation as in Definitions and Notation* 5.2 (ii), *for all $n$, $m \geq 1$,*
$$I(n, m, A, T) = \begin{cases} \max\{1, m/2^n\} & \text{if } m \text{ is even;} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

*Hence, for each $m, n \geq 1$, the Pfister index $I(n, m, A, T)$ is uniformly bounded in the class of rings of either type* (i) *or* (ii).

(d) *Let $\langle R, P\rangle$ be another ring of the same type as $\langle A, T\rangle$. If $\langle A, T\rangle \xrightarrow{f} \langle R, P\rangle$ is a p-ring morphism, the following are equivalent, where $f\star\langle a_1, \ldots, a_n\rangle = \langle f(a_1), \ldots, f(a_n)\rangle$:*

(1) *$f$ is a complete embedding, that is, for all $n$-forms $\varphi, \psi$ over $A^\times$,*
$$\varphi \approx_T \psi \iff f \star \varphi \approx_P f \star \psi;$$

(2) *$f^{-1}[P^\times] = T^\times$;*

(3) *$f$ reflects isotropy, i.e., if $\varphi$ is a form over $A^\times$ and $f \star \varphi$ is $P$-isotropic, then $\varphi$ is $T$-isotropic.*

*These conditions are, in turn, equivalent to*

(4) *The induced $\pi$-SG morphism, $f^\pi : G_T(A) \longrightarrow G_P(R)$, is injective.*

Theorem 5.4 (Marshall's signature conjecture; refined version; [DM7], Thm. 10.7, p. 134).

(i) *Let* $\langle A, P\rangle$ *be an Archimedean p-ring with bounded inversion and let* $T$ *be a preorder containing* $P$. *Let* $\varphi$ *be a form over* $A^\times$. *With notation as in Proposition* 3.6, *if for some dense subset* $D \subseteq Y_T^*$ *(in the spectral topology), we have*

$$\textit{for all } \beta \in D,\ \operatorname{sgn}_{\tau_\beta}(\varphi) \equiv 0 \bmod 2^n,$$

*then* $\varphi \in I_T^n(A)$.

(ii) *Let* $A$ *be an* $f$*-ring and let* $T_\sharp$ *be its natural partial order. Let* $T$ *be a preorder on* $A$, *such that* $T_\sharp \subseteq T$. *Let* $\varphi$ *be a form over* $A^\times$. *If for some dense subset* $D$ *of* $Y_T^*$ *we have*

$$\textit{for all } \beta \in D,\ \operatorname{sgn}_{\tau_\beta}(\varphi) \equiv 0 \bmod 2^n,$$

*then* $\varphi \in I_T^n(A)$.

Theorem 5.5 ([DM7], Thm. 10.10, p. 136). *Let* $A$ *be an* $f$*-ring and let* $T_\sharp$ *be its natural partial order. For* $n$*-forms* $\varphi = \langle a_1, \ldots, a_n\rangle$ *and* $\psi = \langle b_1, \ldots, b_n\rangle$ *over* $A^\times$, *the following are equivalent*:

(1) $\varphi \approx_{T_\sharp} \psi$.

(2) [Local-global Sylvester's inertia law] *There is an orthogonal decomposition of* $A$ *into idempotents,* $\{e_1, \ldots, e_m\}$, *such that for every* $1 \le j \le m$, *the following conditions are satisfied*:

(i) *Each entry in* $\varphi$ *and* $\psi$ *is either in* $T_\sharp^\times e_j$ *(i.e., strictly positive in* $Ae_j$*), or in* $-(T_\sharp^\times e_j)$ *(strictly negative in* $Ae_j$*), i.e.,*

$$\begin{aligned} \underline{\boldsymbol{n}} &= \{k \in \underline{\boldsymbol{n}} : a_k e_j >_{T_\sharp} 0\} \cup \{k \in \underline{\boldsymbol{n}} : a_k e_j <_{T_\sharp} 0\} \\ &= \{k \in \underline{\boldsymbol{n}} : b_k e_j >_{T_\sharp} 0\} \cup \{k \in \underline{\boldsymbol{n}} : b_k e_j <_{T_\sharp} 0\}; \end{aligned}$$

(ii) *The number of entries of* $\varphi$ *and* $\psi$ *that are strictly negative in* $Ae_j$ *is the same, i.e.,*

$$\operatorname{card}(\{k \in \underline{\boldsymbol{n}} : a_k e_j <_{T_\sharp} 0\}) = \operatorname{card}(\{k \in \underline{\boldsymbol{n}} : b_k e_j <_{T_\sharp} 0\}).$$

Our proof of Theorem 5.5 passes through the explicit computation of the Horn–Tarski invariants ([DM1], Def. 7.4, p. 140) of the forms $\varphi$ and $\psi$ in the Boolean algebra of idempotents of the $f$-ring $A$ (the equality of these invariants is equivalent to the isometry in item (1); see [DM1], Thm. 7.1, p. 136).

Example. If $A = \mathbb{C}(X)$ is the ring of real-valued continuous functions on a completely regular space $X$, the Boolean algebra of idempotents in $A$ may be identified with $B(X)$, the Boolean algebra of clopens in $X$. With this identification, condition (2) in Theorem 5.5 states that, given forms $\varphi, \psi$ of the same dimension over $A^\times$, there is clopen partition $\mathcal{D}$ of $X$, such that none of the coefficients of $\varphi$ and $\psi$ change sign in any $V \in \mathcal{D}$, and the number of coefficients of $\varphi$ and $\psi$ that have a fixed sign on each $V \in \mathcal{D}$ is the same.

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