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ON AN ISOTROPY CRITERION FOR QUADRATIC FORMS OVER FUNCTION FIELDS OF CURVES OVER NON-DYADIC COMPLETE DISCRETE VALUATION RINGS

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Abstract. Harbater, Hartmann and Krashen obtained in 2015 a criterion for the existence of rational points on projective (or principal) homogeneous varieties for rational connected algebraic groups defined over function fields of normal curves over a complete discrete valuation ring in terms of completions of local rings at special points. This was obtained by a reduction via Artin approximation to a related patching problem solved by the same authors in 2009. In the special case of projective quadrics, we present a more elementary reduction in the non-dyadic case. The proof is strongly inspired by the proof of a more Hasse-like local-global principle due to Colliot-Thélène, Parimala and Suresh, and we present a variant of their proof based on the mentioned criterion.

1. Introduction. Let F be the function field of an integral normal two-dimensional scheme $\mathcal{C}$ that is flat and projective over a complete discrete valuation ring T. Let moreover G be a connected rational algebraic group over F, and let H be an F-variety that is a homogeneous variety for G. Assume moreover that for every field extension E/F the action of $G(E)$ on $H(E)$ is transitive, which is for example the case when H is a projective variety (cf. [HHK1, Remark 3.9]). For a point (closed or not) P on the special fibre X of $\mathcal{C} \to \mathrm{spec}(T)$, the field of fraction F_P of the completion $\widehat{\mathcal{O}}_{\mathcal{C},P}$ of the local ring $\mathcal{O}_{\mathcal{C},P}$ of $\mathcal{C}$ at P exists and is naturally a field extension of F (as we will also discuss). The following criterion for the existence of a rational point on H was proven in [HHK2, Theorem 9.1], which we will refer to as the 'special points criterion' (or SPC):

Theorem 1.1 (Harbater, Hartmann, Krashen). *$H(F) \neq \emptyset$ if and only if $H(F_P) \neq \emptyset$ for all $P \in X$.*

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Applying this criterion to the situation of quadrics, seen as projective homogeneous spaces over the special orthogonal group of a defining quadratic form, they deduce an isotropy criterion [HHK2, Theorem 9.3] for quadratic forms in three or more variables in the case where $2 \neq 0$ in T. In my doctoral thesis [G, Theorem 5.2], I independently obtained the same isotropy criterion under the stronger hypothesis that $2 \in T^{\times}$, which we refer to as the *special point criterion for isotropy* (SPCI):

THEOREM 1.2. *Assume that* 2 *is invertible in* T. *Let* q *be a quadratic form of dimension at least* 3 *over* F. *Then* q *is isotropic if and only if* q_{F_P} *is isotropic for each* $P \in X$.

Both the proof of the SPC (Theorem 1.1), as well as my proof of the SPCI (Theorem 1.2), are based on reducing the problem to a related patching problem that was solved in [HHK1, Theorem 3.7]. However, the reduction in the general situation [HHK2, Proposition 5.8] uses Artin approximation. The simplified proof of the SPCI that we present here, is based on the more elementary application of Hensel's Lemma for completions of local rings. The proof is strongly inspired by the proof of the following local-global principle (LGP) obtained in [CTPS, Theorem 3.1]:

THEOREM 1.3 (Colliot-Thélène, Parimala, Suresh). *Assuming that* 2 *is invertible in* T, *any quadratic form over* F *of dimension at least three is isotropic over* F *if and only if it is isotropic over every completion of* F.

This LGP was derived in [CTPS] directly from the 'field patching problem', or better its solution obtained in [HHK1]. By restructuring the ideas of the proof, and showing the SPCI as a first intermediate goal before finally deducing the LGP from it, we hope to present an easy-to-digest portioning of the proof of the LGP (up to understanding the solution of the field patching problem).

This is maybe the main motivation for our proof of the SPCI, since—as it was pointed out in [HHK2, Section 9.2]—the SPCI could conversely be derived from the local-global principle (cf. [HHK2, Proposition 7.4]). We will also take the opportunity to reformulate the local global principle later (cf. Theorem 6.1) in the full strength it has actually been proven (but not stated) in [CTPS, 3.1]. Namely we omit the unnecessary restriction from the original paper that F be the function field of a *smooth* projective curve over the field of fractions of T (note that in positive characteristic, an algebraic function field in one variable might be the function field of a regular projective curve that is not smooth), and we consider as local fields only the completions with respect to certain types of discrete valuations—a subtle strengthening that may however make a difference for applications of the LGP.

2. Completions of noetherian domains. Let A be a noetherian domain and $\mathfrak{a} \subset A$ a proper ideal. Let $\widehat{A}^{\mathfrak{a}} := \varprojlim A/\mathfrak{a}^n$ denote the $\mathfrak{a}$-adic completion of A.

EXAMPLE 2.1. If A is local, then we write $\widehat{A}$ for its completion with respect to its unique maximal ideal $\mathfrak{m}$. In this case, $\widehat{A}$ is again a local ring with maximal ideal $\mathfrak{m}\widehat{A}$ (cf. [Ei, p. 183]). If A is moreover regular local, then so is $\widehat{A}$ (cf. [Ei, Corollary 10.12]). In particular $\widehat{A}$ is an integral domain (cf. [Ei, Corollary 10.14]).

REMARK 2.2. Note that for $m \geq 1$, we have a natural isomorphism $\widehat{A}^{(\mathfrak{a}^m)} \cong \widehat{A}^{\mathfrak{a}}$ given by the identifications $A/(\mathfrak{a}^m)^i \to A/(\mathfrak{a}^{mi})$ for each $i \geq 1$.

By Krull's intersection theorem ([Ei, 5.4]), the natural homomorphism $A \to \widehat{A}^{\mathfrak{a}}$ is an embedding. In particular, any localization of a ring that is of finite type over a noetherian domain (e.g. if it comes from a sheaf of rings of a projective scheme over a discrete valuation ring), embeds into a completion with respect to any of its proper ideals.

We now consider another situation where the completion of a noetherian domain is again an integral domain:

EXAMPLE 2.3. Suppose that A is a normal noetherian domain and that $t \in A$ is contained in a unique prime ideal $\mathfrak{p}$ of height one in A. Then $\widehat{A}^{(t)}$ is an integral domain, since it embeds into the regular local domain $\widehat{A_{\mathfrak{p}}}$:

Note that $tA_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})^r$ for some $r \geq 1$, whereby $\widehat{A_{\mathfrak{p}}} \cong \widehat{A_{\mathfrak{p}}}^{(t)}$. The homomorphism $\widehat{A}^{(t)} \to \widehat{A_{\mathfrak{p}}}^{(t)}$ given by $A/(t^i) \to A_{\mathfrak{p}}/(t^i)$ for all $i \geq 1$ is injective: Suppose an element $f \in A$ is divisible by t^i in $A_{\mathfrak{p}}$, i.e. $f = t^i g$ for some $g \in A \setminus \mathfrak{p}$. To show that f is also divisible by t^i in A, we have to show that $g \in A$. As $\mathfrak{p}$ is the unique prime ideal of height one in A that contains t, we see that t is a unit in the discrete valuation ring $A_{\mathfrak{q}}$ for each height one prime ideal $\mathfrak{q} \neq \mathfrak{p}$, whereby $g \in A_{\mathfrak{q}}$. It follows from [Ei, Corollary 11.4] that $g \in A$.

3. The field-patching criterion for rational points. We first recall some other mathematical objects needed for the formulation of the field patching criterion, given in [HHK1, Theorem 3.7]. Let $T, \mathcal{C}, X, F, F_P, G$ and H be as in Section 1. Let $t \in T$ denote any generator of the the maximal ideal of T, and $k = T/(t)$ the residue field of T.

Example 2.1 gives an easy explanation for the constructibility of the fields F_P (i.e. of the fact that $\widehat{\mathcal{O}}_{\mathcal{C},P}$ is an integral domain) in the case where $\mathcal{C}$ is a regular scheme, which we will assume in Section 6. In the more general case, where $\mathcal{C}$ is only assumed to be normal, it follows from the fact that T is a so called *excellent domain* (which implies excellence for the normal domain $\mathcal{O}_{\mathcal{C},P}$ by [Liu, Theorem 8.2.39]), that $\widehat{\mathcal{O}}_{\mathcal{C},P}$ is also a normal domain and hence in particular an integral domain (cf. [Liu, Proposition 8.2.41]).

We call a nonempty finite set $\mathcal{P} \subset X$ of closed points *admissible* if it contains all points of intersections of distinct irreducible components of X.

In [HH, Proposition 6.6] it was shown that after possibly enlarging $\mathcal{P}$ by a finite number of closed points in X, there exists a finite T-morphism $\mathcal{C} \to \mathbb{P}^1_T$ such that $\mathcal{P}$ is the preimage of a k-rational point on $\mathbb{P}^1_k$. The proof of [HH, Proposition 6.6] is relatively elementary (it actually contains a minor mistake which can be fixed by equally elementary methods—for a fix see e.g. [G, Proposition 4.3.49]).

In fact, in [HHK1], the existence of such a finite T-morphism was required as part of the admissibility condition. It is only in hindsight, by the help of [HHK2, Proposition 3.3], which proves the existence of such a finite T-morphism without the necessity of enlarging the nonempty set $\mathcal{P}$, that now this additional admissibility assumption can (as it holds implicitly) be dropped in all the results and proofs of [HHK1].

For the sake of keeping proofs short and readable, we will assume [HHK2, Proposition 3.3] even though its proof relies on much deeper algebraic geometry than the proof of [HH, Proposition 6.6] (or [G, Proposition 4.3.49]). Since one of our goals is to present an elementary approach where possible, let us stress that the proofs in this paper could be easily adapted to work with the stronger admissibility conditions, using the result of [HH, Proposition 6.6] that any nonempty finite set of closed points containing all intersection points of irreducible components of X can be enlarged in such a way that the existence of the T-finite morphism with the required properties is assured for this larger set.

So, let $\mathcal{P}$ be an admissible set of closed points $\mathcal{P} \subset X$. Let $\mathcal{U}$ denote the set of irreducible components of $X \setminus \mathcal{P}$. For $U \in \mathcal{U}$, we consider the subring of F of germs of functions that are regular at U, namely $\mathcal{R}_U = \varinjlim \mathcal{O}_{\mathcal{C}}(\mathcal{W})$, where the directed system is given by inclusions of the open neighbourhoods $\mathcal{W} \subset \mathcal{C}$ of U (some of which are affine open subset of $\mathcal{C}$, e.g. the pre-image of $\mathbb{A}^1_T$ under some finite T-morphism $\mathcal{C} \to \mathbb{P}^1_T$, whereby in particular F is the field of fractions of $\mathcal{R}_U$). The ring $\mathcal{R}_U$ embeds naturally into the local ring $\mathcal{O}_{\mathcal{C},\eta}$ at the generic point η of U, which is a discrete valuation ring. Since t is contained in a unique height one prime ideal of $\mathcal{R}_U$, we have from Example 2.3 that $\widehat{\mathcal{R}}_U^{(t)}$ is an integral domain, whose field of fractions (which is denoted by F_U) is a field extension of F, the field of fractions of $\mathcal{R}_U$.

In [HHK1, 3.7] the following criterion for the existence of a rational point of a projective of principal homogeneous variety H for a connected rational group G was shown. We will refer to this as the 'field patching criterion' (or FPC):

Theorem 3.1 (Harbater, Hartmann, Krashen). *$H(F) \neq \emptyset$ if and only if $H(F_P) \neq \emptyset$ for all $P \in \mathcal{P}$ and $H(F_U) \neq \emptyset$ for all $U \in \mathcal{U}$.*

As mentioned in the introduction, the reason why we decide to call this criterion for the existence of a rational point the 'field patch criterion' is that its proof relies on the solution of a related patching problem. We omit in this paper any discussion of patching problems, and we refer to the original papers [HHK1] or [HH] for a detailed explanation.

4. Quadratic forms over commutative rings. We assume basic familiarity with the algebraic theory of quadratic forms over fields of characteristic different from 2, in particular the fact that every quadratic form is isometric to a diagonal quadratic form, and that two diagonal quadratic forms whose coefficients only differ by an invertible square are isometric.

In the following we will introduce some ad-hoc notation for diagonal quadratic forms over general rings in which 2 is invertible. Even though similar terminology has been introduced in the setting of particular rings (such as valuation rings), we would like to emphasize that the nature of the notation here only serve the local purpose of abbreviating speech, and do not mark the starting point of a general theory. Most symbols we define are not even stable under all isometries of diagonal forms over general commutative rings. In particular, we do not automatically understand a diagonal quadratic form as an equivalence class of isometric diagonal forms, and we will have to carefully make sure that the isometries (or similarities) that we will use do not change the properties we are interested in.

Let A denote a commutative unital ring in which 2 is invertible. We call a homogeneous quadratic polynomial in n separated variables $\langle a_1, \ldots, a_n\rangle := a_1X_1^2 + \ldots + a_nX_n^2$ with $a_1, \ldots, a_n \in A^\times$ a *regular* quadratic diagonal form in n variables over A. For a proper ideal $\mathfrak{a} \subset A$, we denote the *residue form* by

$$\tilde{q}_{\mathfrak{a}} := \langle \bar{a}_1, \ldots, \bar{a}_n\rangle$$

and we observe that it is regular over $A/\mathfrak{a}$ if q was regular over A.

We call a diagonal quadratic form $q = \langle a_1, \ldots, a_n\rangle$ over A *isotropic over A* if there exists an n-tuple $(x_1, \ldots, x_n) \in A^n$ with $x_i \neq 0$ for some $i \leq n$, such that $a_1x_1^2 + \ldots + a_nx_n^2 = 0$. Note that if the ideal generated by the coefficients $x_1, \ldots, x_n$ of the isotropic n-tuple is the whole ring A, then the residue form $\tilde{q}_{\mathfrak{a}}$ with respect to any proper ideal $\mathfrak{a} \subset A$ is also isotropic, with isotropic n-tuple $(\bar{x}_1, \ldots, \bar{x}_n) \in (A/\mathfrak{a})^n$ whose coefficients generate the whole ring $A/\mathfrak{a}$ as an ideal.

If the diagonal quadratic form $q = \langle a_1, \ldots, a_n\rangle$ is regular over A and has an isotropic n-tuple $(x_1, \ldots, x_n) \in A^n$ such that $x_i \in A^\times$ for some $i \leq n$, then we say that q is *strongly isotropic* over A.

Lemma 4.1. *Suppose that A is complete with respect to a proper ideal $\mathfrak{a} \subset A$ and that 2 is invertible in A. Let $q = \langle a_1, \ldots, a_n\rangle$ be a regular diagonal quadratic form over A. If $\tilde{q}_{\mathfrak{a}}$ is strongly isotropic over $A/\mathfrak{a}$, then q is isotropic over A.*

Proof. Let $(x_1, \ldots, x_n) \in A^n$ be a n-tuple such that $(\bar{x}_1, \ldots, \bar{x}_n) \in (A/\mathfrak{a})^n$ is an isotropic vector for $\tilde{q}_{\mathfrak{a}}$ with $\bar{x}_i \in A^\times$ for some $i \leq n$. After re-ordering the variables we can assume that $i = 1$, and after scaling the tuple by $\bar{x}_1^{-1}$, we can assume that $x_1 = 1$. Now we consider the univariate polynomial $f(Z) = a_1Z^2 + (a_2x_2^2 + \ldots + a_nx_n^2)$, and we observe that $f(1) \equiv 0 \bmod \mathfrak{a}$. Moreover, since $f'(Z) = 2a_1Z$, we have $f'(1) \in A^\times$, and thus Hensel's Lemma [Ei, 7.3] asserts the existence of $z \in A$ with $z \equiv 1 \bmod \mathfrak{a}$ such that $f(z) = 0$. In particular, the tuple $(z, x_2, \ldots, x_n) \in A^n$ is an isotropic tuple for q. ∎

Lemma 4.2. *Let A be a domain and $\mathfrak{p} \subset A$ a regular prime ideal of height one. Let $\pi \in \mathfrak{p}$ be a generator of the maximal ideal of the localization $A_{\mathfrak{p}}$. Let q^1 and q^2 be diagonal forms over A that are regular over $A_{\mathfrak{p}}$, and suppose that $q = q^1 \perp \pi \cdot q^2$ is isotropic over the field of fractions of A. Then either $\tilde{q}_{\mathfrak{p}}^1$ or $\tilde{q}_{\mathfrak{p}}^2$ is isotropic over the field of fractions of $A/\mathfrak{p}$.*

Proof. By assumption, the localization $A_{\mathfrak{p}}$ is a discrete valuation ring. Let $\pi \in A$. Write $q^1 = \langle a_1, \ldots, a_n\rangle$ and $q^2 = \langle b_1, \ldots, b_m\rangle$. Assuming that q is isotropic over the field of fractions F of A, we can find after scaling with an appropriate power of π an isotropic $(n+m)$-tuple $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in A_{\mathfrak{p}}^{n+m}$ for q such that one of the coefficients is a unit in $A_{\mathfrak{p}}$. Suppose first that $x_i \in A_{\mathfrak{p}}^\times$ for some $i \leq n$. Then $\bar{a}_1\bar{x}_1^2 + \ldots + \bar{a}_n\bar{x}_n^2 = 0$ shows that $\tilde{q}_{\mathfrak{p}}^1$ is isotropic over $A_{\mathfrak{p}}/(\pi)$. On the other hand, if none of the x_i are units in $A_{\mathfrak{p}}$, and thus consequently $y_j \in A_{\mathfrak{p}}^\times$ for some $j \leq m$, we can write $x_i = \pi \cdot z_i$ for some $z_i \in A_{\mathfrak{p}}$. Thus

$$\pi^2 \cdot (a_1^2z_1^2 + \ldots + a_n^2z_n^2) + \pi \cdot (b_1^2y_1^2 + \ldots + b_m^2y_m^2) = 0$$

shows that $\tilde{q}_{\mathfrak{p}}^2$ is isotropic over $A_{\mathfrak{p}}/(\pi)$. The statement follows with the observation that $A_{\mathfrak{p}}/(\pi)$ is the field of fractions of $A/\mathfrak{p}$. ∎

REMARK 4.3. If A is a discrete valuation ring and $\mathfrak{m} = (\pi)$, then for every regular quadratic form q over F, one can find two regular diagonal forms q^1 and q^2 over A, such that q has a diagonalization $q^1 \perp \pi \cdot q^2$. Applying both Lemma 4.1 and Lemma 4.2 to the case of a complete discrete valuation ring thus yields what is often referred to as *Springer's theorem* [Lam, Theorem 1.4].

5. Deducing the special point criterion for isotropy from the FPC. We assume for this section again the notation $T, \mathcal{C}, X, F$ and F_P as defined in Section 1 and assume additionally that 2 is invertible in T. The following lemma is the key ingredient in the proof of the SPCI, and has been shown independently and in greater generality in [HHK2, Proposition 5.8], by using Artin approximation instead of Hensel's Lemma.

LEMMA 5.1. *Let Y be an irreducible component of X. Let η denote the generic point for Y in $\mathcal{C}$. Let q be a regular quadratic form over F such that q is isotropic over F_η. Then q is isotropic over F_U for a nonempty open subset $U \subset Y$ that has no intersection with any other irreducible component of X.*

Proof. The point $\eta \in X$ has codimension one in $\mathcal{C}$, hence $\mathcal{O}_{\mathcal{C},\eta}$ is a discrete valuation ring. The field F_η is the completion of F with respect to the corresponding valuation. Its residue field $\kappa_\eta = \mathcal{O}_{\mathcal{C},\eta}/\mathfrak{m}_{\mathcal{C},\eta}$ is a field extension of k, since $\mathcal{O}_{\mathcal{C},\eta}$ dominates T. Let $s \in \mathfrak{m}_{\mathcal{C},\eta}$ such that $\mathfrak{m}_{\mathcal{C},\eta} = (s)$. Then there exists $r \in \mathbb{N}$ and $u \in \mathcal{O}^\times_{\mathcal{C},\eta}$ such that $s^r = ut$. As mentioned in Remark 4.3, one can find regular diagonal forms q^1 and q^2 with respect to $\mathcal{O}_{\mathcal{C},\eta}$ such that $q^1 \perp s \cdot q^2$ is a diagonalization of q, and either the residue form $\tilde{q}^1_{(s)}$ or the residue form $\tilde{q}^2_{(s)}$ over $\mathcal{O}_{\mathcal{C},\eta}/(s)$ is isotropic.

Let us assume, without loss of generality, that $\tilde{q}^1_{(s)}$ is isotropic, and write $q^1 = \langle a_1, \ldots, a_n \rangle$ with $a_i \in \mathcal{O}^\times_{\mathcal{C},\eta}$. Let $x_1, \ldots, x_n \in \mathcal{O}_{\mathcal{C},\eta}$ not all of them in the maximal ideal (without loss of generality let x_1 be a unit) such that $a_1 x_1^2 + \ldots + a_m x_m^2 = ws$ for some $w \in \mathcal{O}_{\mathcal{C},\eta}$. As $\mathcal{O}_{\mathcal{C},\eta} = \varinjlim \mathcal{O}_{\mathcal{C}}(\mathcal{W})$, where $\mathcal{W} \subset \mathcal{C}$ runs over the system of smaller and smaller open neighbourhoods of η in $\mathcal{C}$, we can find a neighbourhood $\mathcal{U}'$ of η, such that

$$a_1, \ldots, a_m, a_1^{-1}, u, u^{-1}, x_1, \ldots, x_m, x_1^{-1}, w, s \in \mathcal{O}_{\mathcal{C}}(\mathcal{U}').$$

Let $\mathcal{U}$ be the open neighbourhood of η in $\mathcal{C}$ that we obtain from $\mathcal{U}'$ after removing all finitely many closed points of intersections of distinct irreducible components of X and set $U = Y \cap \mathcal{U}$. We have $\mathcal{O}_{\mathcal{C}}(\mathcal{U}') \subseteq \mathcal{O}_{\mathcal{C}}(\mathcal{U}) \subseteq \mathcal{R}_U$, and q^1 is a regular quadratic form over $\mathcal{R}_U$ and $\tilde{q}^1_{(s)}$ is strongly isotropic over $\mathcal{R}_U/(s)$. By Lemma 4.1, q is isotropic over $\widehat{\mathcal{R}}_U^{(t)}$. Since we have the equality of ideals $(t) = (s^r)$ in $\mathcal{R}_U$, we have $\widehat{\mathcal{R}}_U^{(t)} = \widehat{\mathcal{R}}_U^{(s)}$. Thus q^1 (and hence q) is isotropic over F_U. ■

We denote by $S^{(i)}$ the set of points of codimension i in a scheme S.

THEOREM 5.2 (SPCI). *A quadratic form in three or more variables defined over F is isotropic over F if and only if it is isotropic over F_P for each $P \in X$.*

Proof. The statement is trivial for non-regular quadratic forms, as they are trivially isotropic over F. Suppose a regular quadratic form q of dimension at least 3 over F is isotropic at F_P for every $P \in X$. Each non-closed point $\eta \in X$ is the generic point of an irreducible component Y_η of X (see [Liu, 8.3.4]). By Lemma 5.1, there exists a nonempty open $U_\eta \subset Y_\eta$ such that $q_{F_{U_\eta}}$ is isotropic for each η and such that U_η contains no point of an irreducible component of X other than Y_η. Set $\mathcal{P} := X \setminus \bigcup_{\eta \in X^{(0)}} U_\eta$. Note that $\mathcal{P}$ is a finite set of closed points that contains in particular all intersection points of distinct irreducible components of X, and is thus admissible for the FPC. Moreover, for every component U of $X \setminus \mathcal{P}$, we have $U = U_\eta$ for some $\eta \in X^{(0)}$, whereby q is isotropic over F_U. The assertion that q is already isotropic over F then follows from the FPC (Theorem 3.1) after recalling that the special orthogonal group of a regular quadratic form over a field of characteristic different from 2 is a connected rational group, more precisely, the classical Cayley transformation (see for example [W, p. 599]) yields a birational map between the special orthogonal group of the quadratic form and the affine space of skew-symmetric (with respect to the adjoint involution) matrices. Note that the associated projective quadric is a homogeneous variety for this group when the quadratic form is in three or more variables (i.e. the projective quadric is of dimension one or more). ∎

6. Deducing the local-global principle from the SPCI. In this section we present a reorganized version of the proof given in [CTPS] for the LGP. The LGP formulated in [CTPS, Theorem 3.1] was formulated with respect to all completions of the function field coming from discrete valuations. Moreover, the function field was assumed to be the function field of a *smooth* curve over a field that is complete with respect to a discrete valuation. In the following, we will state a stronger version of the local-global principle by replacing both hypotheses by weaker ones. As mentioned in the introduction, this stronger version was actually proven in [CTPS], albeit not stated as such.

Let F in this section denote any field extension of finite type and transcendence degree one over the field of fractions K of a complete discrete valuation ring T in which 2 is invertible. We call a discrete valuation on F a *geometric valuation* if its residue field is either a finite extension over K, or a finitely generated extension of transcendence degree one over the residue field of T.

THEOREM 6.1 (LGP). *Any quadratic form in of dimension at least* 3 *over* F *is isotropic if and only if it is isotropic over every completion of* F *with respect to a geometric valuation.*

Let us recall from [Liu, Example 8.3.41] that F admits a regular model over K, that is, an integral regular projective curve C over K whose function field is K-isomorphic to F. Moreover, any valuation ring in F that contains K corresponds to a local ring of C at a codimension one point (i.e. a closed point) on C.

We furthermore recall from [Liu, Remark 10.1.6] that C admits a normal integral model over T (which we recall is complete and thus excellent by [Liu, Theorem 8.2.39]), i.e. a normal 2-dimensional scheme $\mathcal{C}$ together with a flat projective morphism to spec(T) whose generic fibre is isomorphic to C as a curve over K. We call such a $\mathcal{C}$ also an *integral model for* F *over* T.

Moreover, we observe that by Lipman's desingularization result for excellent reduced two-dimensional schemes ([Liu, Corollary 8.3.45]), a normal integral model $\mathcal{C}$ can be chosen that is even regular.

Remark 6.2. The geometric valuations on F are exactly the valuations that come from a point of codimension one on a regular integral model for F over T. Clearly, every codimension one point on a regular model $\mathcal{C}$ gives rise to a discrete valuation on F whose residue field is either an algebraic finite extension of K, when the point lies on the generic fibre C, or an extension of transcendence degree one and of finite type over k, when the point lies on the special fibre X, and is thus the generic point of an irreducible component of X. The converse is also true, as follows for example with [Liu, Theorem 8.3.26], that is, every geometric valuation on F is given by the local ring at some point of codimension one on some regular integral model for F over T, by observing that every discrete valuation ring with field of fractions F necessarily contains T as a subring (cf. [HHK2, Corollary 7.2]).

Remark 6.3. Finally, we recall a result on embedded resolution of singularities of curves on a regular fibred surface over an excellent Dedekind domain ([Liu, Theorem 9.2.26 and Remark 9.2.27]), by which we can assume that for finitely many $f_1, \ldots, f_r \in F^\times$, the regular model $\mathcal{C}$ is such that the effective Weil divisor

$$D_{\{f_1,\ldots,f_r\}} = \sum_{\eta \in \operatorname{supp}(\{f_1,\ldots,f_r\})} \eta$$

has *only normal crossings* on $\mathcal{C}$, where

$$\operatorname{supp}(\{f_1,\ldots,f_r\}) = \{\eta \in \mathcal{C}^{(1)} \mid f_i \in \mathfrak{m}_{\mathcal{C},\eta} \text{ or } f_i^{-1} \in \mathfrak{m}_{\mathcal{C},\eta} \text{ for some } i \leq r\}$$

is the common support for the principal divisors given by $f_1, \ldots, f_r$.

This means that locally at each closed point $P \in \mathcal{C}$, there exist generators x, y of $\mathfrak{m}_{\mathcal{C},P}$ such that for $1 \leq i \leq r$, we have $f_i = u_i x^{\ell_i} y^{s_i}$ for a unit $u_i \in \mathcal{O}_{\mathcal{C},P}^\times$ and some $\ell_i, s_i \in \mathbb{Z}$.

Proof of Theorem 6.1. Obviously, for non-regular forms, the statement of the theorem is trivially true since any such form is isotropic over F. Let q denote a regular quadratic form over F of dimension $n \geq 3$. We can assume (after linear change of variables) that $q = \langle a_1, \ldots, a_n \rangle$ is in diagonal form for some $a_1, \ldots, a_n \in F^\times$. As mentioned before, we can choose a regular integral model $\mathcal{C} \to \operatorname{spec}(T)$ for F such that the effective divisor $D_{\{a_1,\ldots,a_r\}}$ has only normal crossings. Let $X \subsetneq \mathcal{C}$ denote the special fibre of $\mathcal{C} \to \operatorname{spec}(T)$. Suppose q is anisotropic over F. We aim to show that q remains anisotropic over some completion of F with respect to a geometric valuation. By the SPCI (Theorem 1.2), the form q is anisotropic over F_P for some $P \in X$. Let us write $\mathcal{O}$ for $\mathcal{O}_{\mathcal{C},P}$ and $\mathfrak{m}$ for $\mathfrak{m}_{\mathcal{C},P}$.

If P is not a closed point in $\mathcal{C}$, then $P \in \mathcal{C}^{(1)}$ (see e.g. [Liu, 8.3.4]) and thus $\mathcal{O}$ is a discrete valuation ring and F_P is the completion of F at the corresponding discrete valuation v. Moreover, v is a geometric valuation, as its residue field $\mathcal{O}/\mathfrak{m}$ is the function field of Y, and hence an extension of transcendence degree one over k.

If P is a closed point, by the choice of the model $\mathcal{C}$ with respect to the diagonalization of $q = \langle a_1, \ldots, a_n \rangle$, there exist $x, y \in \mathfrak{m}$ such that $\mathfrak{m} = (x, y)$ and $a_i = u_i x^{\ell_i} y^{s_i}$ for some $\ell_i, s_i \in \mathbb{Z}$. After re-ordering and re-scaling the variables of q, we get a new diagonal representation of q over F in the form

$$q \cong q^1 \perp yq^2 \perp xq^3 \perp xyq^4,$$

where the q^i are regular diagonal forms over $\mathcal{O}$ for all $1 \leq i \leq 4$.

We observe that $\varphi^1 := q^1 \perp yq^2$ and $\varphi^2 := q^3 \perp yq^4$ are regular forms over the local ring $\mathcal{O}_{(x)}$, which is a discrete valuation ring of a geometric valuation, since $x \in \mathcal{O}$ defines on some affine neighbourhood of P a point of codimension in $\mathcal{C}$. The aim is to show that q is anisotropic over the field of fractions of the completion of $\mathcal{O}_{(x)}$.

Suppose the contrary is true. Then either $\tilde{\varphi}^1_{(x)}$ or $\tilde{\varphi}^2_{(x)}$ would be isotropic over the field of fractions of $\mathcal{O}/(x)$ by Lemma 4.2. Note that the residue $\bar{y} \in \mathcal{O}/(x)$ generates the unique maximal ideal, which is thus a regular prime ideal of height one, and $(\mathcal{O}/(x))/(\bar{y}) = \mathcal{O}/\mathfrak{m}$.

Let us assume (without loss of generality) that $\tilde{\varphi}^1_{(x)} = \tilde{q}^1_{(x)} \perp \bar{y}\tilde{q}^2_{(x)}$ is isotropic over the field of fractions of $\mathcal{O}/(x)$. Redenoting $\psi^1 := \tilde{q}^1_{(x)}$ and $\psi^2 := \tilde{q}^2_{(x)}$, we observe again by Lemma 4.2 that either $\tilde{\psi}^1_{(\bar{y})}$ or $\tilde{\psi}^2_{(\bar{y})}$ is isomorphic over $(\mathcal{O}/(x))/(\bar{y})$. Let us again assume (without loss of generality) that $\tilde{\psi}^1_{(\bar{y})}$ is isotropic. Since $\tilde{\psi}^1_{(\bar{y})} = \tilde{q}^1_{\mathfrak{m}}$ one obtains with Lemma 4.1 that q^1 is isotropic over $\widehat{O}$, and thus the contradiction that q is isotropic over F_P. ■

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