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THE TENSION FIELD OF THE CONFORMAL GAUSS MAP

BY

JIANBO FANG (Guiyang) and LIN LIANG (Chuxiong)

Abstract. Let $x : M^m \to S^{m+1}$ be an m-dimensional hypersurface isometrically immersed in an $(m+1)$-dimensional unit sphere. The smooth map

$$\varphi = (H, Hx + e_{m+1}) : M^m \to S_1^{m+2}$$

is called the conformal Gauss map of x, where S_1^{m+2} is the $(m+2)$-dimensional de Sitter space, H the mean curvature and e_{m+1} the local normal frame field of x. Given the Möbius metric on M^m, the harmonicity of φ has some connection with the fact that the immersion x is Willmore. In this paper, by pulling the metric back via x, we study the tension field of the conformal Gauss map, and prove that the conformal Gauss map is harmonic if and only if x is a minimal immersion. Finally, we give some examples for which the conformal Gauss map is harmonic in S^3.

1. Introduction. Given Riemannian manifolds M and N, the *energy* of a C^∞ mapping $f : M \to N$ is defined by

$$E(f) = \frac{1}{2} \int_M \|df\|^2 \, dv_M.$$

The critical points of the energy functional are called *harmonic mappings.*

For various results on harmonic mappings, see [1–4, 6–8, 10, 12, 14–16]. There are two main contexts appearing in the study of harmonic mappings.

One involves the harmonicity of isometric mappings between Riemannian manifolds. In 1984, Peng [13] discussed the relations between minimal immersions and harmonic mappings, pointing out similarities and differences between the two classes. Later, in 1992, Liu [9] studied the isometric immersions of pseudo-Riemannian manifolds, with the same conclusion.

The other context deals with Gaussian mappings between submanifolds. It is well known that classical Gaussian mappings play an important role in the study of surfaces in three-dimensional Euclidean space. Their role is

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also extended to submanifolds with arbitrary dimension and codimension in space forms with constant sectional curvature (see [11] for details).

In conformal geometry, the relationship between submanifolds and harmonic mappings is also of interest. In 2003, Guo [5] considered the conformal Gauss mapping of n-dimensional submanifolds in the spherical space and its harmonic properties. Naturally, the harmonicity of conformal Gauss maps in the de Sitter space is an interesting problem. Let $x : M^m \to S^{m+1}$ be a hypersurface isometrically immersed in an $(m+1)$-dimensional unit sphere. There are two metrics on the manifold M^m: one is the pullback metric of the immersion x, and the other is the Möbius metric on M^m. We also have the conformal Gauss mapping $\varphi : M^m \to S_1^{m+2}$, with the energy function related to the metric of M^m. For the pullback metric on M^m, Guo et al. have studied the variational problem of the energy functional, and for the tension field of the conformal Gauss mapping of the pullback metric on M^m, it is precisely what we want to study.

Based on the above ideas and results, this paper investigates immersed hypersurfaces in the unit sphere S^{m+1} by using the conformal Gauss mapping from a Riemannian manifold to the indefinite-metric space S_1^{m+2}. The main result is the following:

THEOREM 1.1. *Let $x : M^m \to S^{m+1}$ be an isometric immersion. Its conformal Gauss map*

$$\varphi = (H, Hx + e_{m+1}) : M^m \to S_1^{m+2}$$

is harmonic if and only if x is a minimal immersion. Here S_1^{m+2} is the $(m+2)$-dimensional de Sitter space, H the mean curvature and e_{m+1} a local normal frame field of x.

2. Conformal Gauss mapping to the de Sitter space. Let $x : M^m \to S^{m+1}$ be an isometric immersion, $\{e_i\}$ $(1 \le i \le m)$ a local orthogonal tangent frame field on M^m, and $\{\omega_i\}$ $(1 \le i \le m)$ its dual frame. Locally, the metric of M^m can be written as

$$ds_M = dx \cdot dx = \sum \omega_i \otimes \omega_i.$$

Let e_{m+1} be a local normal frame field of x. Then the structural equations of x read

$$\begin{aligned} dx &= \omega_i x_*(e_i), \\ dx_*(e_i) &= \omega_{ij} x_*(e_j) + h_{ij}\omega_j e_{m+1} - \omega_i x, \\ de_{m+1} &= -h_{ij}\omega_j x_*(e_j). \end{aligned}$$

Let Δ_M denote the Laplacian operator of the metric ds_M. We need the well-known formula

$$\Delta_M x = m(He_{m+1} - x), \tag{2.1}$$

where H is the mean curvature of x.

Let R_1^{m+3} be the *Lorentz space* with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle U, V\rangle = -u_{-1}v_{-1} + u_0v_0 + \cdots + u_{m+1}v_{m+1},$$

where $U = (u_{-1}, u_0, \ldots, u_{m+1}), V = (v_{-1}, v_0, \ldots, v_{m+1})$. Let S_1^{m+2} be the $(m+2)$-dimensional *de Sitter space*, defined by

$$S_1^{m+2} = \{U \in R_1^{m+3} \mid \langle U, U\rangle = 1\}.$$

If $f : M^m \to S_1^{m+2}$ is a mapping of class C^2, the metric on M obtained by pullback via f is

$$f_*\langle \cdot, \cdot \rangle = \sum_{i,j} \langle f_*(e_i), f_*(e_j)\rangle \omega_i\omega_j.$$

Define the *energy density* of f by

$$e(f) = \frac{1}{2}\operatorname{tr}(f^*\langle \cdot, \cdot \rangle) = \frac{1}{2}\sum_i \langle f_*(e_i), f_*(e_i)\rangle.$$

Thus the energy of f is expressed as

$$\varepsilon(f) = \frac{1}{2}\int_M e(f)\, dM = \frac{1}{2}\int_M \sum_i \langle f_*(e_i), f_*(e_i)\rangle\, dM.$$

For the sake of convenience, we agree on the ranges of the indices

$$-1 \le \alpha, \beta, \ldots \le m, \qquad 1 \le i, j, k, \ldots \le m.$$

Let $\overline{y}$ be the position vector in S_1^{m+2}, and $\{E_\alpha \mid -1 \le \alpha \le m\}$ an orthogonal basis for the *Lorentz* metric $\langle \cdot, \cdot \rangle$ such that

$$\langle E_{-1}, E_{-1}\rangle = -1, \qquad \langle E_\alpha, E_\beta\rangle = \delta_{\alpha\beta}, \quad \alpha \neq -1. \tag{2.2}$$

Then $\{E_{-1}, \ldots, E_m, \overline{y}\}$ constitutes a local Lorentz moving frame on S_1^{m+2}. Let $\{\theta_\alpha\}$ be the dual basis of $\{E_\alpha\}$, so that

$$\langle \cdot, \cdot \rangle = -\theta_{-1}^2 + \theta_0^2 + \sum_i \theta_i^2. \tag{2.3}$$

We can now write down the structural equations on S_1^{m+2}:

$$d\overline{y} = \theta_{-1}E_{-1} + \theta_0E_0 + \sum_i \theta_iE_i, \tag{2.4}$$

$$dE_{-1} = \sum_\alpha \varphi^*(\theta_{-1,\alpha})E_\alpha + \theta_{-1}\overline{y}, \tag{2.5}$$

$$dE_\alpha = \sum_\beta \theta_{\alpha\beta}E_\beta - \theta_\alpha\overline{y}, \qquad \alpha \neq -1, \tag{2.6}$$

where $\{\theta_{\alpha\beta}\}$ is a set of differential forms satisfying the conditions

$$\theta_{-1,\alpha} - \theta_{\alpha,-1} = 0, \quad \alpha = -1, \ldots, m,$$
$$\theta_{\alpha\beta} + \theta_{\beta\alpha} = 0, \quad \alpha, \beta \neq -1.$$

One calls $\{\theta_{\alpha\beta}\}$ the *connection forms* induced by the metric $\langle\cdot,\cdot\rangle$. Setting $f_*(e_i) = f_{\alpha i}E_\alpha$, we have

$$f^*(\theta_\alpha) = f_{\alpha i}\omega_i.$$

Thus, the energy density of f can be written as

$$\varepsilon(f) = \frac{1}{2}\int_M \Big(-\sum_i f_{-1,i}^2 + \sum_{\alpha=0}^m \sum_i f_{\alpha i}^2 \Big)\, dM.$$

Define the *generalized covariant differential* of $f_{\alpha i}$ as

$$f_{\alpha i,j}\omega_j = df_{\alpha i} + f_{\alpha j}\omega_{ji} + f_{\beta i} f^*\theta_{\beta\alpha},$$

and let

$$\tau_\alpha = \sum_i \tau_{\alpha i,i}.$$

The vector field $\tau(f) = \tau_\alpha E_\alpha$ is called the *tension field* of the mapping f. Similar calculations of variations in classical harmonic mapping theory show that f is a critical point of $\varepsilon(f)$ if and only if $\tau(f)$ is zero, and the mapping f is called *harmonic* if $\tau(f) = 0$.

We define the smooth mapping

$$\varphi = (H, Hx + e_{m+1}) : M^m \to S_1^{m+2},$$

determined by the isometric immersion x, and called the *conformal Gaussian mapping* of x in Möbius geometry. Next we define a traceless tension field T by setting

$$T_{ij} = H\delta_{ij} - h_{ij}.$$

From the structural equation, it is easy to see that

$$d\varphi = (H_i, H_i x + T_{ij} x_*(e_j))\omega_i, \tag{2.7}$$

so that the tangent mapping $\varphi_* : T_pM \to T_{\varphi(p)}S_1^{m+2}$ is characterized by

$$\varphi_*(e_i) = (H_i, H_i x + T_{ij} x_*(e_j)),$$

implying that the metric on M obtained by pullback via f can be written as

$$\varphi^*\langle\cdot,\cdot\rangle = \sum_{i,j} \langle \varphi_*(e_i), \varphi_*(e_j)\rangle \omega_i\omega_j = \sum_{i,j,k} T_{ik}T_{kj}\omega_i\omega_j.$$

This shows that the energy density of the mapping φ is

$$e(\varphi) = \frac{1}{2}\operatorname{tr}(\varphi^*\langle\cdot,\cdot\rangle) = \frac{1}{2}\sum_{i,j}(T_{ij})^2,$$

Thus, the energy of φ can be expressed as

$$\varepsilon(\varphi) = \frac{1}{2}\int_M \sum_{i,j} (T_{ij})^2\, dM.$$

We call the Euclidean invariant $\varepsilon(\varphi)$ the *energy functional* of the conformal Gauss map φ.

Set $Y = (1, x)$, and note that

$$\langle Y, Y\rangle = 0,\ \langle \varphi, Y\rangle = 0,\ \langle \varphi, \Delta_M Y\rangle = 0,\ \langle Y, \Delta_M Y\rangle = -m. \tag{2.8}$$

Next, letting

$$N = -\frac{1}{m}\Delta_M Y - \frac{1}{2m^2}\langle \Delta_M Y, \Delta_M Y\rangle Y, \tag{2.9}$$

one gets

$$\langle N, N\rangle = 0, \quad \langle N, Y\rangle = 1, \quad \langle N, \varphi\rangle = 0.$$

We now set

$$E_{-1} = \frac{1}{\sqrt{2}}(Y - N), \quad E_0 = \frac{1}{\sqrt{2}}(Y + N), \quad E_i = (0, x_*(e_i)).$$

Equation (2.2) and the relation $\langle E_\alpha, \varphi\rangle = 0$ imply that $\{E_{-1}, E_0, E_i, \varphi\}$ constitute a moving frame along φ in S^{m+2}_1, so from (2.3)–(2.4) it is easy to see that

$$d\varphi = \varphi^*(\theta_{-1})E_{-1} + \varphi^*(\theta_0)E_0 + \sum_i \varphi^*(\theta_i)E_i,$$

$$dE_{-1} = \sum_\alpha \varphi^*(\theta_{-1,\alpha})E_\alpha + \varphi^*(\theta_{-1})\varphi, \tag{2.10}$$

$$dE_\alpha = \sum_\beta \varphi^*(\theta_{\alpha\beta})E_\beta - \varphi^*(\theta_\alpha)\varphi, \quad \alpha \neq -1. \tag{2.11}$$

Furthermore, with $\varphi^*(\theta_\alpha) = \varphi_{\alpha i}\omega_i$, (2.7) yields

$$d\varphi = H_i\omega_i Y + T_{ij}\omega_i E_j = \frac{\sqrt{2}}{2}H_i\omega_i E_{-1} + \frac{\sqrt{2}}{2}H_i\omega_i E_0 - T_{ij}\omega_i E_j,$$

and consequently

$$\varphi_{-1,i} = \frac{\sqrt{2}}{2}H_i, \quad \varphi_{0,i} = \frac{\sqrt{2}}{2}H_i, \quad \varphi_{ji} = T_{ij}.$$

Next, we evaluate $\varphi^*(\theta_{\alpha\beta})$. From (2.1), (2.8) and (2.9), one gets

$$N = -\left(\frac{1}{2}(1 + H^2), \frac{1}{2}(-1 + H^2)x + He_{m+1}\right),$$

and

$$E_{-1} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}(3 + H^2), \frac{1}{2}(1 + H^2)x + He_{m+1}\right),$$

$$E_0 = \frac{1}{\sqrt{2}}\left(\frac{1}{2}(1 - H^2), \frac{1}{2}(3 - H^2)x - He_{m+1}\right),$$

while using the structure equations of x, one obtains

$$dE_{-1} = \frac{1}{\sqrt{2}}(H, Hx + e_{m+1})dH + \frac{1}{\sqrt{2}}\Big(0, \Big(\frac{1}{2}(1 + H^2)\delta_{ij} - Hh_{ij}\Big)x_*(e_i)\Big)\omega_j,$$

$$dE_0 = \frac{1}{\sqrt{2}}(H, Hx + e_{m+1})dH + \frac{1}{\sqrt{2}}\Big(0, \Big(\frac{1}{2}(3 - H^2)\delta_{ij} - Hh_{ij}\Big)x_*(e_i)\Big)\omega_j.$$

Set

$$p_{ij} = \frac{1}{2}(1 - H^2)\delta_{ij} + HT_{ij}, \qquad q_{ij} = \frac{1}{2}(3 + H^2)\delta_{ij} - HT_{ij}.$$

Then one can rewrite dE_{-1} and dE_0 as

$$dE_{-1} = \frac{1}{\sqrt{2}}H_i\omega_i\varphi + \frac{1}{\sqrt{2}}p_{ij}\omega_jE_i, \qquad dE_0 = -\frac{1}{\sqrt{2}}H_i\omega_i\varphi + \frac{1}{\sqrt{2}}q_{ij}\omega_jE_i.$$

so that using (2.10)–(2.11), one obtains

$$\varphi^*(\theta_{-1,0}) = 0, \qquad \varphi^*(\theta_{-1,i}) = \frac{1}{\sqrt{2}}p_{ij}\omega_j, \qquad \varphi^*(\theta_{0,i}) = \frac{1}{\sqrt{2}}q_{ij}\omega_j,$$

while noting that

$$dE_i = \omega_{ij}E_j + \omega_j(0, h_{ij}e_{m+1} - \delta_{ij}x),$$

one gets

$$\varphi^*(\theta_{ij}) = \langle dE_i, E_j\rangle = \omega_{ij}.$$

With these preparations, we prove the following lemma.

LEMMA 2.1. *Let $x : M^m \to S^{m+1}$ be an isometric immersion. The components of the tension field with respect to the conformal Gauss map $\varphi = (H, Hx + e_{m+1}) : M^m \to S_1^{m+2}$ are*

$$\tau^{-1} = \frac{\sqrt{2}}{2}(\Delta H + H(s - mH^2)),$$
$$\tau^0 = \frac{\sqrt{2}}{2}(\Delta H + H(s - mH^2)),$$
$$\tau^i = mH_i \qquad (i = 1, \ldots, m-1).$$

Proof. First, one easily verifies that

$$\begin{aligned}\varphi_{-1,ij}\omega_j &= d\varphi_{-1,i} + \varphi_{-1,j}\omega_{ji} + \varphi_{\alpha i}\varphi^*(\theta_{\alpha,-1})\\ &= d\varphi_{-1,i} + \varphi_{-1,j}\omega_{ji} + \varphi_{-1,i}\varphi^*(\theta_{-1,-1}) + \varphi_{0,i}\varphi^*(\theta_{0,-1}) + \varphi_{j,i}\varphi^*(\theta_{j,-1}),\end{aligned}$$

while

$$\varphi_{-1,i} = \frac{\sqrt{2}}{2}H_i, \qquad \theta_{-1,\alpha} = \theta_{\alpha,-1}, \qquad \theta_{-1,j} = \theta_{j,-1}, \qquad \varphi^*(\theta_{-1,0}) = 0,$$

and so one sees that $\varphi^*(\theta_{0,-1}) = \varphi^*(\theta_{-1,0}) = 0$, from which one easily obtains

$$\begin{aligned}\varphi_{-1,ij}\omega_j &= \frac{\sqrt{2}}{2}dH_i + \frac{\sqrt{2}}{2}H_j\omega_{ji} + \varphi_{ji}\varphi^*(\theta_{-1,j})\\ &= \frac{\sqrt{2}}{2}H_i + \frac{\sqrt{2}}{2}dH_j\omega_{ji} + \frac{\sqrt{2}}{2}T_{ji}p_{jk}\omega_k\\ &= \frac{\sqrt{2}}{2}H_i + \frac{\sqrt{2}}{2}dH_j\omega_{ji} + \frac{\sqrt{2}}{2}T_{ik}p_{kj}\omega_j\\ &= \frac{\sqrt{2}}{2}H_{i,j}\omega_j + \frac{\sqrt{2}}{2}T_{ik}p_{kj}\omega_j.\end{aligned}$$

Thus we have

$$\varphi_{-1,ij} = \frac{\sqrt{2}}{2}(H_{ij} + T_{ik}p_{kj}), \quad \text{where} \quad p_{kj} = \frac{1}{2}(1-H^2)\delta_{kj} + HT_{kj},$$

from which one obtains

$$\varphi_{-1,ij} = \frac{\sqrt{2}}{2}\left(H_{ij} + T_{ik}\cdot\frac{1}{2}(1-H^2)\delta_{kj} + HT_{ik}T_{kj}\right),$$

so

$$\tau^{-1} = \sum_{i=j}\varphi_{-1,ii} = \frac{\sqrt{2}}{2}\Big(\Delta H + H\sum_{i,jk}(T_{ik})^2\Big),$$

and

$$\sum_{i,jk}(T_{ik})^2 = \sum(H^2\delta_{ij} - 2Hh_{ij} + h_{ik}h_{kj}) = s - mH^2,$$

where $s = \sum h_{ik}^2$, and consequently

$$\tau^{-1} = \frac{\sqrt{2}}{2}(\Delta H + H(s - mH^2)). \tag{2.12}$$

In the following, we will calculate the component τ^0. One has

$$\begin{aligned}\varphi_{0,ij}\omega_j &= d\varphi_{0,i} + \varphi_{0,j}\omega_{ji} + \varphi_{\alpha i}\varphi^*(\theta_{\alpha,0})\\ &= d\varphi_{0,i} + \varphi_{0,j}\omega_{ji} + \varphi_{0,i}\varphi^*(\theta_{-1,0}) + \varphi_{0,i}\varphi^*(\theta_{0,0}) + \varphi_{j,i}\varphi^*(\theta_{j,0}),\end{aligned}$$

because of the equalities

$$\varphi_{0,i} = \frac{\sqrt{2}}{2}H_i, \qquad \varphi^*\theta_{-1,0} = \varphi^*\theta_{0,-1} = 0, \qquad \theta_{j,0} + \theta_{0,j} = 0,$$

that is, $\varphi^*(\theta_{j,0}) = -\varphi^*(\theta_{0,j})$, and so we obtain

$$\begin{aligned}\varphi_{0,ij}\omega_j &= \frac{\sqrt{2}}{2}dH_i + \frac{\sqrt{2}}{2}H_j\omega_{ji} + \varphi_{ji}\varphi^*(\theta_{j,0})\\ &= \frac{\sqrt{2}}{2}dH_i + \frac{\sqrt{2}}{2}H_j\omega_{ji} + T_{ji}(-\varphi^*(\theta_{0,j}))\\ &= \frac{\sqrt{2}}{2}dH_i + \frac{\sqrt{2}}{2}H_j\omega_{ji} - \frac{\sqrt{2}}{2}T_{ik}q_{jk}\omega_k\\ &= \frac{\sqrt{2}}{2}(dH_i + H_j\omega_{ji}) - \frac{\sqrt{2}}{2}T_{ik}q_{kj}\omega_j\\ &= \frac{\sqrt{2}}{2}H_{i,j}\omega_j - \frac{\sqrt{2}}{2}T_{ik}q_{kj}\omega_j = \frac{\sqrt{2}}{2}(H_{i,j} - T_{ik}q_{kj})\omega_j,\end{aligned}$$

and

$$\varphi_{0,ij} = \frac{\sqrt{2}}{2}(H_{i,j} - T_{ik}q_{kj}), \quad \text{where} \quad q_{kj} = \frac{1}{2}(3 + H^2)\delta_{kj} - HT_{kj},$$

so that $\varphi_{0,ij}$ can be written as

$$\varphi_{0,ij} = \frac{\sqrt{2}}{2}\left(H_{i,j} - \frac{1}{2}(3 + H^2)T_{ik}\delta_{kj} + HT_{ik}T_{kj}\right),$$

and hence we have

$$\tau^0 = \sum_{i=j}\varphi_{0,ii} = \frac{\sqrt{2}}{2}(\Delta H + H(s - mH^2)). \tag{2.13}$$

Finally, we calculate τ^i. We have

$$\begin{aligned}\varphi_{i,jk}\omega_k &= d\varphi_{ij} + \varphi_{ik}\omega_{kj} + \varphi_{\alpha j}\varphi^*(\theta_{\alpha,i})\\ &= d\varphi_{ij} + \varphi_{ik}\omega_{kj} + \varphi_{-1,j}\varphi^*(\theta_{-1,i}) + \varphi_{0,j}\varphi^*(\theta_{0,i}) + \varphi_{kj}\varphi^*(\theta_{ki})\\ &= T_{ijk}\omega_k + \frac{\sqrt{2}}{2}H_j(p_{ik} + q_{ik})\omega_k = (T_{ijk} + H_j\delta_{ik})\omega_k,\end{aligned}$$

and consequently

$$\varphi_{i,jk} = T_{ijk} + H_j\delta_{ik},$$

while, as we know, $T_{ij} = H\delta_{ij} - h_{ij}$, so that

$$T_{ij,k} = H_{,k}\delta_{ij} - h_{ij,k},$$

which gives

$$\varphi_{i,jk} = H_{,k}\delta_{ij} - h_{ij,k} + H_j\delta_{ik},$$

and in particular

$$\varphi_{i,jj} = H_{,j}\delta_{ij} - h_{ij,j} + H_j\delta_{ij} \quad (j = k).$$

Summation of $\phi_{i,jk}$ over $j = k$ yields

$$\tau^i = \sum_{i=j} \varphi_{i,jj} = 2mH_i - mH_i = mH_i \quad (i = 1, \dots, m-1). \tag{2.14}$$

Using (2.12)–(2.14), we complete the proof of Lemma 2.1. ∎

The following main theorem is a direct consequence of Lemma 2.1.

THEOREM 2.2. *Let $x : M^m \to S^{m+1}$ be an isometric immersion. Its conformal Gauss map*

$$\varphi = (H, Hx + e_{m+1}) : M^m \to S_1^{m+2}$$

is harmonic if and only if x is a minimal immersion. Here S_1^{m+2} is the $(m+2)$-dimensional de Sitter space, H the mean curvature and e_{m+1} a local normal frame field of x.

REMARK 2.3. The results show that there are no harmonic hypersurfaces with a conformal Gauss mapping except minimal hypersurfaces.

3. Some examples in S^3. In this section we describe some minimal surfaces in S^3 for which the conformal Gauss mapping is harmonic.

EXAMPLE 3.1. Let

$$X(u,v) = \big(\sqrt{r}\cos(\sqrt{1-r}\,u), \sqrt{r}\sin(\sqrt{1-r}\,u), \sqrt{1-r}\cos(\sqrt{r}\,v), \sqrt{1-r}\sin(\sqrt{r}\,v)\big)$$

be a torus in S^3. The conformal Gauss map is harmonic if and only if X is a minimal torus.

One easily gets

$$X_u = \big(-\sqrt{r}\sqrt{1-r}\sin(\sqrt{1-r}\,u), \sqrt{r}\sqrt{1-r}\cos(\sqrt{1-r}\,u), 0, 0\big),$$
$$X_v = \big(0, 0, -\sqrt{r}\sqrt{1-r}\sin(\sqrt{r}\,v), \sqrt{r}\sqrt{1-r}\cos(\sqrt{r}\,v)\big).$$

Note that

$$\langle N, X\rangle = 0, \quad \langle N, X_u\rangle = 0, \quad \langle N, X_v\rangle = 0,$$

as well as

$$N = \big(-\sqrt{1-r}\cos(\sqrt{1-r}\,u), -\sqrt{1-r}\sin(\sqrt{1-r}\,u), \sqrt{r}\cos(\sqrt{r}\,v), \sqrt{r}\sin(\sqrt{r}\,v)\big),$$

and one easily verifies that

$$X_{uu} = \big(-\sqrt{r}(1-r)\cos(\sqrt{1-r}\,u), \sqrt{r}(1-r)\sin(\sqrt{1-r}\,u), 0, 0\big),$$
$$X_{vv} = \big(0, 0, -r\sqrt{1-r}\cos(\sqrt{r}\,v), -r\sqrt{1-r}\sin(\sqrt{r}\,v)\big),$$

so that

$$E = \langle X_u, X_u \rangle = r(1-r), \quad F = \langle X_u, X_v \rangle = 0, \quad G = \langle X_v, X_v \rangle = r(1-r),$$
$$e = \langle X_{uu}, N \rangle = \sqrt{r}(1-r)^{3/2},\ f = 0,\ g = \langle X_{vv}, N \rangle = -r^{3/2}\sqrt{1-r},$$

while

$$\lambda_1 = \frac{e}{E} = \frac{\sqrt{1-r}}{\sqrt{r}}, \qquad \lambda_2 = \frac{g}{G} = -\frac{\sqrt{r}}{\sqrt{1-r}},$$

which means that

$$H = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\,\frac{1-2r}{\sqrt{r}\sqrt{1-r}}.$$

The components of the tension field of the conformal Gauss map are τ^{-1}, τ^0, τ^1, τ^2. Harmonicity of the conformal Gauss map is equivalent to

$$\tau^{-1} + \tau^0 + \tau^1 + \tau^2 = 0.$$

One can also see that

$$\tau^1 = 2H_{,u} = 0, \qquad \tau^2 = 2H_{,v} = 0,$$

so if $\tau^{-1} = \tau^0 = 0$, we conclude that $H = 0$, which means that x is a minimal torus. In fact,

$$\begin{aligned}\tau^{-1} = \tau^0 &= \frac{\sqrt{2}}{2}(\Delta H + H(s - 2H^2)) \\ &= \frac{\sqrt{2}}{4}\,\frac{1-2r}{\sqrt{r}\sqrt{1-r}}\left(\frac{1-r}{r} + \frac{r}{1-r} - 2\frac{(1-2r)^2}{4r(1-r)}\right) \\ &= \frac{\sqrt{2}(1-2r)}{8r^{3/2}(1-r)^{3/2}} = 0, \quad \text{so} \quad r = \frac{1}{2}.\end{aligned}$$

The next three examples are taken from [8]. For details we refer to the bibliography of [8].

EXAMPLE 3.2 ([8]). $\Gamma_{m,k}$ is associated with a complete, non-singular minimal surface in S^3 which is denoted by $\xi_{m,k}$ (see Figure 1).

EXAMPLE 3.3 ([8]). $\Gamma'_{m,k}$ is associated with a compact, non-singular minimal submanifold $\tau_{m,k}$ (see Figure 2).

EXAMPLE 3.4 ([8]). Indefinite reflection of the surface $\mathfrak{M}_{m,k}$ produces a complete, non-singular minimal submanifold which we denote $\eta_{m,k}$ (see Figure 3).

From Theorem 2.2, the conformal Gauss mapping of $\xi_{m,k}(\tau_{m,k}, \eta_{m,k})$ is harmonic.

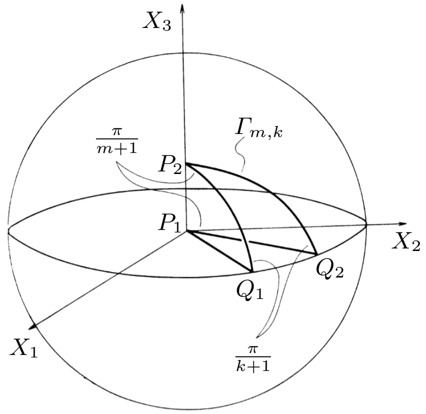

Fig. 1. $\xi_{m,k}$

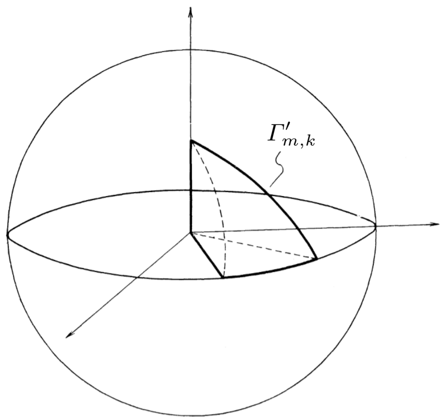

Fig. 2. $\tau_{m,k}$

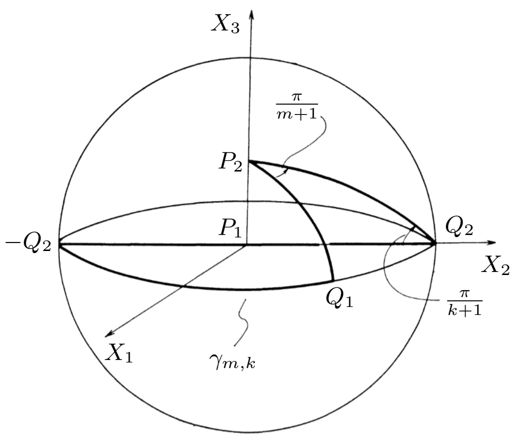

Fig. 3. $\eta_{m,k}$

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Jianbo Fang
School of Mathematics and Statistics
Guizhou University of Finance and Economics
Guiyang, 550025, China
E-mail: fjbwcj@126.com

Lin Liang
Personnel Division
Chuxiong Normal University
Chuxiong, 675000, China
E-mail: cxlianglin@cxtc.edu.cn