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PACKING OF NON-BLOCKING SQUARES INTO THE UNIT SQUARE

BY

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Abstract. Any collection of non-blocking squares with total area not greater than 5/9 can be packed into the unit square.

1. Introduction. Let S_n be a square, for $n = 1, 2, \dots$, and let I be a square of sidelength 1. We say that the squares $S_1, S_2, \dots$ can be *packed* into I if it is possible to apply translations and rotations to the sets S_n so that the resulting translated and rotated squares are contained in I and have mutually disjoint interiors. The packing is *parallel* if every side of each packed square is parallel to a side of I.

Moon and Moser [MM67] showed that the squares $S_1, S_2, \dots$ can be packed parallel into I provided that the total area of the squares is not greater than $1/2$. This upper bound is tight: it is impossible to pack two squares of sidelengths greater than $1/2$ (and consequently of total area greater than $1/2$) into I; they block each other. Two questions arise: (1) how big a square can still be packed there; (2) what happens if the squares could not block each other. The answer to the first problem can be found in [JZ23b].

Denote by a_n the sidelength of S_n for $n = 1, 2, \dots$. We say that the squares $S_1, S_2, \dots$ are *non-blocking* if $a_i + a_j \le 1$ for any $i \ne j$. The aim of this note is to show that any collection of non-blocking squares can be packed parallel into I provided that the sum of their areas is not greater than $5/9$. This upper bound is tight for parallel packing: five squares of sidelengths greater than $1/3$ cannot be packed parallel into I (see Fig. 1, left). However, some non-parallel packing is possible (see Fig. 1, right, or see [G79]). Five squares of sidelengths $(4 - \sqrt{2})/7 \approx 0.369$ can be packed into I, but a side of one packed square is parallel to a diagonal of I. The three-dimensional version of this problem is discussed in [JZ23c].

The packing method presented in Section 2 is based on the well-known method of Moon and Moser [MM67]. First, the squares (from the finite

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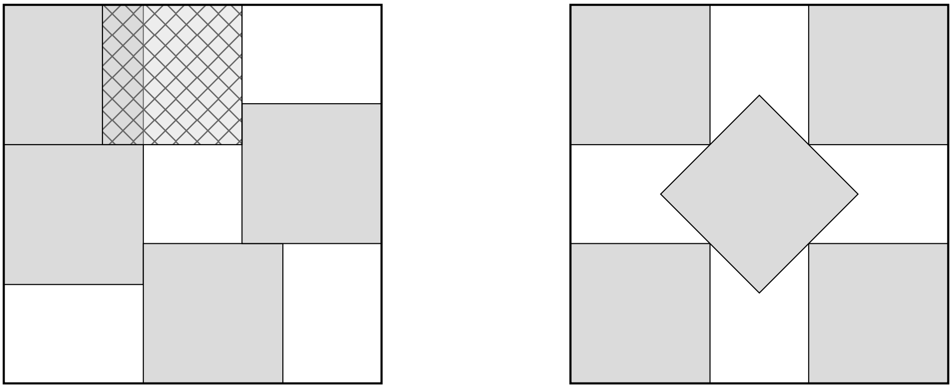

Fig. 1. Five squares of sidelengths $(4-\sqrt{2})/7$

or infinite collection) are arranged by size, starting with the largest one. Then the squares are packed in successive layers. In our main method, unlike the method presented in [MM67], the first three squares are packed in the upper right corner of I. Moreover, different layers can have bases of different lengths.

2. Packing method. We denote by $[b_1, b_2] \times [c_1, c_2]$, where $b_1 < b_2$ and $c_1 < c_2$, the rectangle $\{(x, y) : \ b_1 \le x \le b_2, c_1 \le y \le c_2\}$. The packing method is a small modification of the algorithm of Moon and Moser [MM67]. There are two differences: the squares S_1, S_2 and S_3 are packed in the upper right corner of I. In addition, not all layers need to have bases of equal length.

Let $I = [0,1] \times [0,1]$. Moreover, let S_n be a square of sidelength a_n, where $a_n \ge a_{n+1}$ for $n = 1, 2, \ldots$ and let $a_1 + a_2 \le 1$.

Description of the method MM^+.

[1] The square S_1 is packed into the upper right corner of I.
[2] The square S_2 is packed along the right side of I as near to the top as possible.
[3] S_3 is packed along the top of I as far to the right as possible.

The next squares will be packed in two different ways depending on the sidelength of S_4.

[4a] If $a_4 \le 1/3$, then $S_4, S_5, \ldots$ are packed into I in layers (as the darker squares in Fig. 2) similarly to the method of Moon and Moser [MM67]. The first layer has height a_4 and base equal to the base of I. The squares $S_4, S_5, \ldots$ are packed into I along the base of the first layer from left to right. Let S_{n_1} be the first square that cannot be packed in that way. The new layer, of height a_{n_1}, is created directly above the first layer. The

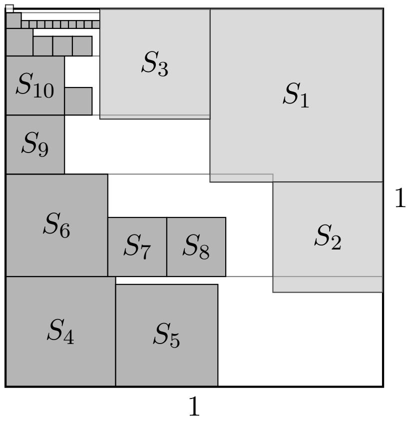

Fig. 2. Packing method for $a_4 \leq 1/3$

squares $S_{n_1}, S_{n_1+1}, \ldots$ are packed into I along the base of the second layer from left to right. If S_{n_2} is the first square that cannot be packed in that way in the second layer, then the new layer, of height a_{n_2}, is created directly above the second layer, and so on.

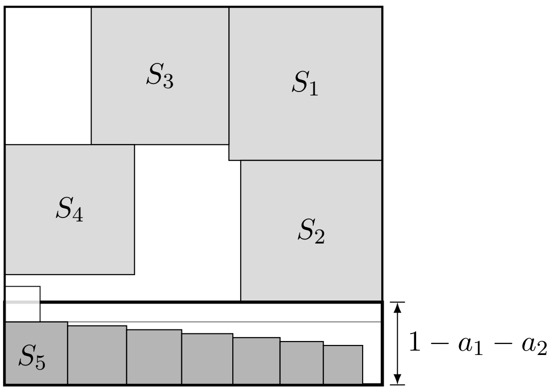

Fig. 3. Packing method for $a_4 > 1/3$

[4b] If $a_4 > 1/3$, then (see Fig. 3)

- S_4 is packed along the left side of I as near to the top as possible;
- the remaining squares are packed in corresponding layers into the rectangle $[0,1] \times [0, 1 - a_1 - a_2]$ according to the method of Moon and Moser [MM67].

Example 1. In Fig. 2, the squares S_4 and S_5 ($a_4 < 1/3$) are packed in the first layer. Since S_6 cannot be packed next to S_5 (the square S_2 blocks such packing), a new layer of height a_6 is created directly above the first layer to pack S_6, S_7 and S_8. Since S_9 cannot be packed next to S_8 (the square S_2 is blocking), a new layer of height a_9 is created directly above the second layer to pack S_9. It is impossible to pack S_{10} in this layer next to S_9 (now S_3 is blocking), so a new layer is created to pack S_{10} and S_{11}. In the fifth layer we pack S_{12}, S_{13}, S_{14} and S_{15}. The squares $S_{16}, \dots, S_{26}$ are packed in the sixth layer. The next square S_{27} cannot be packed into that layer. Moreover, there is no empty space in I to create a new layer of height a_{27}. Therefore S_{27} cannot be packed by the method MM^+. Clearly, $n_1 = 6$, $n_2 = 9$, $n_3 = 10$, $n_4 = 12$, $n_5 = 16$ and $n_6 = 27$.

Example 2. In Fig. 3 ($a_4 > 1/3$), the squares $S_5, S_6, \dots, S_{11}$ are packed in the first layer of $[0,1] \times [0, 1 - a_1 - a_2]$. Since S_{12} cannot be packed in this layer, we try to create a new layer of height a_{12} directly above the first one. It is impossible, so S_{12} stops the packing process.

3. Main result

Theorem 3.1. *Any finite or infinite collection of non-blocking squares with total area no greater than* $5/9$ *can be packed into the unit square.*

Proof. Denote by $S_1, S_2, \dots$ the squares in the collection. Without loss of generality we can assume that $a_1 \ge a_2 \ge \cdots$, where a_n is the sidelength of S_n for $n = 1, 2, \dots$.

We will show that if the squares cannot be packed into I by the method MM^+, then $a_1^2 + a_2^2 + \cdots > 5/9$, which is a contradiction. Let S_{n_t} be the first square from the collection that cannot be packed into I by the method MM^+. Consider two cases depending on the size of S_4.

Case 1: $a_4 \le 1/3$. In this case we pack $S_4, S_5, \dots$ into the layers as shown in Fig. 2, where $n_t = 27$. Put

$$\begin{aligned} a &= a_4, \\ n_0 &= 4, \\ b_k &= \sum_{i=0}^{k} a_{n_i} && \text{for } k = 0, 1, \dots, t, \\ R_k &= [0,1] \times [b_{k-1}, b_k] && \text{for } k = 1, \dots, t-1, \\ R_t &= [0,1] \times [b_{t-1}, 1]. \end{aligned}$$

In other words, R_k for $k = 1, \dots, t-1$ is the rectangle of sidelengths 1 and a_{n_k} contained in I and, at the same time, containing S_{n_k} (see Fig. 4, right).

Since S_{n_1} cannot be packed in the first layer, it follows that the distance λ_1 (see Fig. 5, left) between the right side of $S_{n_1 - 1}$ and either the left side of S_2 (provided that $a_{n_1} > 1 - a_1 - a_2$, as on Fig. 5) or the right side of I (provided that $a_{n_1} \le 1 - a_1 - a_2$) is smaller than a_{n_1}. Consequently, the sum

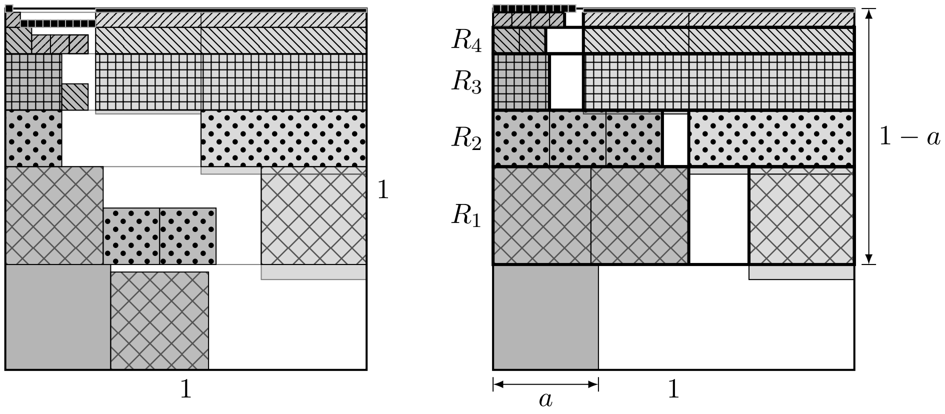

Fig. 4. Layers

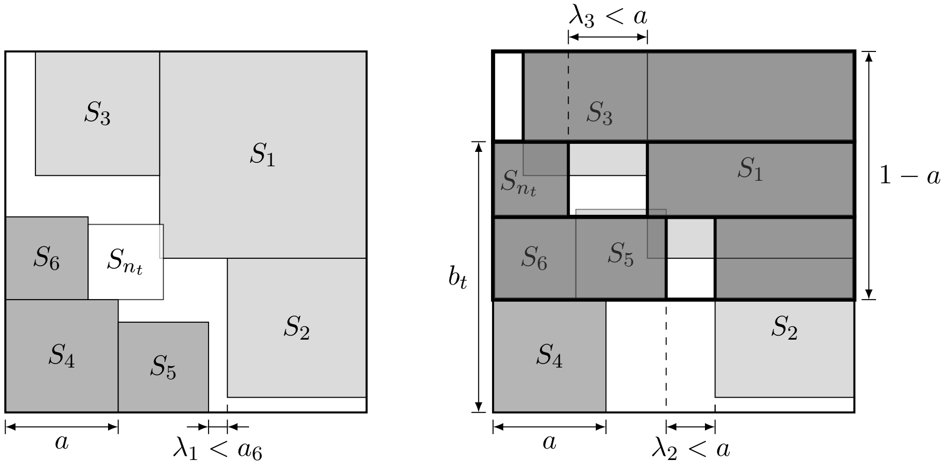

Fig. 5. Case 1, $a_4 \leq 1/3$

of the sidelengths of $S_5, \dots, S_{n_1}$ is greater than $1-a-a_2$ if $a_{n_1} > 1-a_1-a_2$, or is greater than $1-a$ if $a_{n_1} \leq 1-a_1-a_2$. This implies that if $a_{n_1} > 1-a_1-a_2$, then the sum of the sidelengths of $S_5, \dots, S_{n_1}$ plus a_2 is greater than $1-a$ ($\lambda_2 = a_4 - a_6 + \lambda_1 < a_4 = a$ in Fig. 5).

Thus the sum of the areas of $S_{n_{k-1}+1}, \dots, S_{n_k}$ plus the area of the part of $S_1 \cup S_2 \cup S_3$ lying in R_k is greater than $(1-a)a_{n_k}$ for $k = 1, \dots, t-1$. The sum of the areas of $S_{n_{t-1}+1}, \dots, S_{n_t}$ plus the area of the part of $S_1 \cup S_2 \cup S_3$ lying in R_t is greater than $(1-a)(1-b_{t-1})$ (see Fig. 4, right).

First assume that

$$a_{n_1} + \dots + a_{n_t} > 1 - a$$

(as in Fig. 4). Clearly, $a_{n_1} + \dots + a_{n_t-1} + 1 - b_{t-1} = 1 - a$. If the squares cannot be packed, then the sum of their areas is greater than

$$\begin{aligned} a^2 + (1-a)a_{n_1} + (1-a)a_{n_2} + \cdots + (1-a)a_{n_t-1} + (1-a)(1-b_{t-1}) \\ = a^2 + (1-a)(a_{n_1} + \cdots + a_{n_t-1} + 1 - b_{t-1}) \\ = a^2 + (1-a)^2 \geq 5/9 \end{aligned}$$

for $a \leq 1/3$, which is a contradiction.

Now assume that

$$a_{n_1} + \cdots + a_{n_t} \leq 1 - a$$

(as in Fig. 5). Then $\sum_{k=1}^{t} a_{n_k} = b_t - a$. Since $a_1 + a_3 > 1 - a$, the shaded part of the rectangle $[0,1] \times [b_t, 1]$ has area greater than $(1-a)(1-b_t)$. If the squares cannot be packed, then the sum of their areas is greater than

$$a^2 + (1-a)(b_t - a) + (1-a)(1-b_t) = a^2 + (1-a)^2 \geq 5/9,$$

which is a contradiction.

CASE 2: $a_4 > 1/3$. In [MM67] it is proved that any collection of squares of sidelengths at most d with total area no greater than $d^2 + (b-d)(c-d)$ can be packed into a rectangle of sidelength b and c provided $d \leq \min(b,c)$. Since $S_5, S_6, \ldots$ are packed into $[0,1] \times [0, 1-a_1-a_2]$ by the method of Moon and Moser, if the squares cannot be packed then the sum of their areas is greater than

$$\begin{aligned} &(a_1^2 + a_2^2 + a_3^2 + a_4^2) + (a_5^2 + (1-a_5)(1-a_1-a_2-a_5)) \\ &\quad = a_1^2 - (1-a_5)a_1 + a_2^2 - (1-a_5)a_2 + a_3^2 + a_4^2 + a_5^2 + (1-a_5)(1-a_5) \\ &\quad = \left(a_1 - \frac{1-a_5}{2}\right)^2 - \left(\frac{1-a_5}{2}\right)^2 + \left(a_2 - \frac{1-a_5}{2}\right)^2 - \left(\frac{1-a_5}{2}\right)^2 \\ &\qquad + a_3^2 + a_4^2 + a_5^2 + (1-a_5)(1-a_5) \\ &\quad > -\left(\frac{1-a_5}{2}\right)^2 - \left(\frac{1-a_5}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + a_5^2 + (1-a_5)(1-a_5) \\ &\quad = \frac{3}{2}a_5^2 - a_5 + \frac{13}{18} \\ &\quad \geq \frac{3}{2} \cdot \frac{1}{9} - \frac{1}{3} + \frac{13}{18} = \frac{5}{9}, \end{aligned}$$

which is a contradiction. ■

4. Questions. Five congruent squares of sidelengths $(4-\sqrt{2})/7 \approx 0.369$ (of total area $(90-40\sqrt{2})/49 \approx 0.68$) can be packed into I (see [G79] or see Fig. 1, right). Stromquist claims to have proved that six squares of sidelength $1/3$ (of total area $2/3$) can be packed into I while six squares with larger sidelength cannot (see [F98, Introduction]).

QUESTION 4.1. *Can any collection of non-blocking squares of total area not greater than* $2/3$ *be packed into* I *(rotations are allowed)?*

Moon and Moser [MM67] proved that any family of squares with total area not greater than $1/2$ can be packed into a unit square, which is tight. Moser (see [M66, Problem LM3] or [CFG91, Section D5]) conjectured that a similar result may be obtained for packing rectangles of sidelengths at most 1. This has been confirmed in [J00] (see also [J02]). Much earlier, Groemer [G82] showed that any collection of rectangles of sidelengths not greater than 1 whose total area is smaller than or equal to $s^2 - 2s + 1$ can be packed into the square of sidelength $s \geq 3$. Assuming $s = 3$ and using a homothety with ratio $1/3$ we find that any collection of rectangles of sidelengths not greater than $1/3$ whose total area is not greater than $4/9$ can be packed into I.

Denote the sidelengths of a rectangle R_n by a_n and b_n, where $a_n \leq b_n$. We say that the rectangles $R_1, R_2, \ldots$ are *non-blocking* if $a_i + a_j \leq 1$ for any $i \neq j$, and $b_i \leq 1$ for any i.

QUESTION 4.2. *Can any collection of non-blocking rectangles of total area not greater than* $5/9$ *be packed parallel into* I*?*

In the *online* packing problem each square S_i is given without any information on the next squares. Moreover, each successive square S_i, where $i > 1$, is given only after S_{i-1} has been packed. The placement of any packed square cannot be changed afterwards. By λ*-packing* (see [JZ22] or [JZ23a]) we mean a packing in which squares from the finite sequence $S_1, \ldots, S_z$ are grouped in batches in the following way. Squares arrive one by one (online) and they are stored in a buffer until either the total area of the stored squares is greater than or equal to λ or all squares from the sequence have already arrived. More precisely, if squares $S_1, \ldots, S_{n-1}$ are stored in the buffer and if either S_n is a square such that the sum of the areas of $S_1, \ldots, S_n$ is greater than or equal to λ or S_n is the last item in the sequence ($n = z$), then $S_1, \ldots, S_n$ are packed offline into I and the buffer is emptied. The next squares $S_{n+1}, \ldots$ (if any) are packed into I in the same way.

QUESTION 4.3. *What is the smallest* λ *such that any finite collection of non-blocking squares of total area not greater than* $5/9$ *can be* λ*-packed parallel into* I*?*

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