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Bilinear pairings on two-dimensional cobordisms and generalizations of the Deligne category

by

Mikhail Khovanov (New York, NY) and
Radmila Sazdanovic (Raleigh, NC)

Abstract. The Deligne category of symmetric groups is the additive Karoubi closure of the partition category. It is semisimple for generic values of the parameter t while producing categories of representations of the symmetric group when modded out by the ideal of negligible morphisms when t is a nonnegative integer. The partition category may be interpreted, following Comes, via a particular linearization of the category of two-dimensional oriented cobordisms. The Deligne category and its semisimple quotients admit similar interpretations. This viewpoint coupled to the universal construction of two-dimensional topological theories leads to multi-parameter monoidal generalizations of the partition and the Deligne categories, one for each rational function in one variable.

1. Introduction.

The Deligne category $\mathrm{Rep}(S_t)$ interpolates between the categories of finite-dimensional representations of the symmetric groups S_n, viewed as tensor categories, turning the integer n into an element t of the ground field [D], [CO], [EGNO, Section 9.12.1].

The Deligne category has a diagrammatic description, via the partition category Pa_t, as the Karoubi envelope of the additive closure of Pa_t. When $t = n$ is a nonnegative integer, the Deligne category $\mathrm{Rep}(S_n)$ has a non-trivial ideal of negligible morphisms, and the quotient by this ideal is naturally equivalent to the tensor category of finite-dimensional representations of the symmetric group.

Diagrams commonly used to describe partitions [CO, C, HR, LS] can be thickened to two-dimensional surfaces or cobordisms S between unions of circles. Circles appear as "thickenings" of points on which the partitions are formed. Vice versa, any two-dimensional cobordism S gives rise to a partition

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upon ignoring closed components of S and the genus of each connected component with boundary. This informal correspondence is depicted in Figure 1. Cobordisms boast higher variability than partitions, admitting components without boundary and allowing arbitrary genus of each component.

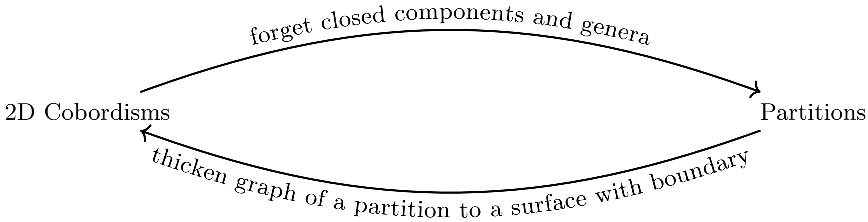

Fig. 1. Schematic correspondence between set partitions and 2D cobordisms.

A precise connection between 2D cobordisms and the partition category was pointed out by Comes [C, Section 2.2]: modding out the cobordism category by the relations that adding a handle is the identity and that a 2-sphere evaluates to t (see Figure 2) produces the partition category, with parameter t corresponding to the 2-sphere. Comes used this observation to derive a set of defining relations for the partition category from that of the cobordism category.

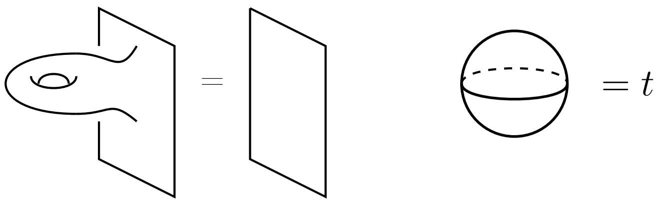

Fig. 2. Handle removal and sphere evaluation skein relations on 2D cobordisms.

A family of 2-dimensional topological theories was recently introduced by one of the authors [Kh2], based on Blanchet, Habegger, Masbaum and Vogel's *universal construction* [BHMV]. It starts with an evaluation of closed oriented surfaces, which may be described by power series

$$Z_\alpha(T) = \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots = \sum_{n\geq 0} \alpha_n T^n \in R[\![T]\!], \tag{1.1}$$

where R is a ground commutative ring or a field $\mathbf{k}$, evaluating a connected component of genus g to α_g. This evaluation gives rise to state spaces $A_\alpha(k)$ for collections of k circles. It can then be extended to produce a category Cob_α with objects nonnegative integers n and hom spaces $\mathrm{Hom}_{\mathrm{Cob}_\alpha}(n,m)$ being R-linear combinations of cobordisms between n and m circles modulo universal relations defined by the sequence $\alpha = (\alpha_0, \alpha_1, \dots)$; also see below.

In the partition category Pa_t, when two partitions are composed, each connected component of the composition that has no boundary points is evaluated to $t \in R$ and removed. In the correspondence between 2D cobordisms and partitions, these components give rise to cobordisms of various genera, which, in general, evaluate to α_g, where g is the genus of the cobordism.

To match the general α-evaluation of 2D cobordisms to the evaluation by powers of t in the partition and Deligne categories, specialize the sequence α to

$$\alpha(t) = (t, t, t, \dots), \quad \alpha_g(t) = t \quad \forall g \in \mathbb{Z}_+. \tag{1.2}$$

Then the additive Karoubi closure of the category $\mathrm{Cob}_{\alpha(t)}$ is equivalent, as a tensor category, to the Deligne category

$$\mathrm{Kar}(\mathrm{Cob}^{\oplus}_{\alpha(t)}) \cong \mathrm{Rep}(S_t), \tag{1.3}$$

for $t \in \mathbf{k} \setminus \mathbb{Z}_+$ (specializing to a characteristic zero field $\mathbf{k}$ as the ground ring). When $t = n \in \mathbb{Z}_+$, the quotient of the Deligne category by the ideal J_n of negligible morphisms produces the category of finite-dimensional representations of the symmetric group S_n, equivalent to the above Karoubi closure for $t = n$,

$$\mathrm{Kar}(\mathrm{Cob}^{\oplus}_{\alpha(n)}) \cong \mathrm{Rep}(S_n)/J_n \cong \mathbf{k}[S_n]\text{-mod}. \tag{1.4}$$

This observation allows us to generalize the Deligne category and its semisimple quotients by taking a more general sequence α of elements of R and then forming the tensor category Cob_α and its additive Karoubi closure $\mathrm{Kar}(\mathrm{Cob}^{\oplus}_\alpha)$, also denoted $\underline{D\mathrm{Cob}}_\alpha$. The latter is given by first allowing finite linear combinations of objects of Cob_α, with suitably defined hom spaces, and then adding all idempotents in endomorphism rings of these linear combinations as additional objects.

When R is a field $\mathbf{k}$, it follows from [Kh2] and goes back to a theorem of Kronecker that hom spaces in Cob_α are finite-dimensional if and only if the power series $Z_\alpha(T)$ in (1.1) can be represented as a rational function,

$$Z_\alpha(T) = \frac{P(T)}{Q(T)}, \tag{1.5}$$

where $P(T), Q(T)$ are coprime polynomials with coefficients in $\mathbf{k}$. To each such rational function we can assign an additive Karoubi-complete tensor (symmetric monoidal) category

$$\underline{D\mathrm{Cob}}_\alpha := \mathrm{Kar}(\mathrm{Cob}^{\oplus}_\alpha) \tag{1.6}$$

with finite-dimensional hom spaces. This category is a natural generalization of the Deligne category $\mathrm{Rep}(S_t)$ for generic t and of its semisimple quotients for $t = n \in \mathbb{Z}_+$. The Deligne category corresponds to the rational function

$$Z_{\alpha(t)}(T) = \frac{t}{1-T} = t + tT + tT^2 + \cdots . \tag{1.7}$$

It should be extremely interesting to extend various results and constructions related to the Deligne category and its semisimple quotients to this large family of tensor categories $\underline{\mathrm{DCob}}_\alpha$ (as well as categories DCob_α defined in Section 4) parametrized by rational functions.

2. Category of two-dimensional cobordisms and its linearization categories

Category Cob_2. Consider the symmetric monoidal category of 2-dimensional oriented cobordisms. We use the skeletal version of this category (one object in each isomorphism class), denoted Cob_2. Its objects are nonnegative integers $n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and morphisms from n to m are diffeomorphism classes rel boundary of compact oriented 2-manifolds S with a fixed diffeomorphism

$$\partial S \cong \Big(-\bigsqcup_n \mathbb{S}^1\Big) \sqcup \Big(\bigsqcup_m \mathbb{S}^1\Big), \tag{2.1}$$

where $\mathbb{S}^1$ is the oriented circle. In other words, the boundary of S is separated into the *bottom* and *top* boundary, and identified, correspondingly, with disjoint unions of n and m circles. Composition is given by concatenation. Cobordisms may have connected components with no boundary. An example of a morphism (cobordism) from 3 to 4 is given in Figure 3.

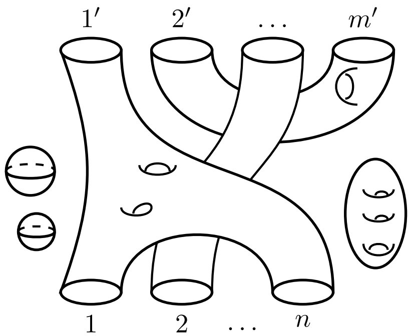

Fig. 3. A morphism in Cob_2. The cobordism is not embedded anywhere, so overlaps of components do not carry any information and can be reversed. We label top circles by $1', 2', \dots, m'$ and bottom circles by $1, 2, \dots, n$. In this example $m = 4$ and $n = 3$.

Morphisms from n to m in Cob_2 can be enumerated as follows. A morphism x may have some number of closed components of various genera. Counting these components gives a sequence $cl(x) = (a_0, a_1, \dots, 0, 0, \dots)$, where a_k is the number of closed components of genus k in x. All but finitely many terms in the sequence $cl(x)$ are zero. Connected components with boundary provide a decomposition of the set of $n+m$ boundary circles into nonempty subsets, where circles from the same subset are the boundaries of the same connected component. Furthermore, each such component has genus zero or higher, which counts the number of handles of the component.

Denote by D_n^m the set of decompositions of $n+m$ circles. A morphism $x \in \mathrm{Hom}_{\mathrm{Cob}_2}(n,m)$ can be described by a decomposition $\lambda \in D_n^m$, an assignment of a nonnegative integer (genus or number of handles) to each set in the decomposition λ and a choice of a sequence $cl(x)$ as above describing genera of closed components of x.

Let us label bottom circles $1,\dots,n$ and top circles $1',\dots,m'$, from left to right. A cobordism x induces a decomposition of the set

$$\mathbb{N}_n^m := \{1,2,\dots,n,1',2',\dots,m'\}. \tag{2.2}$$

For the cobordism x in Figure 3 we have $n=3$, $m=4$, the set is $\mathbb{N}_3^4 = \{1,2,3,1',2',3',4'\}$, and its subsets corresponding to components with boundary are $\{1,3,1'\}$, $\{2,3'\}$, and $\{2',4'\}$. These components have genera $2,0,1$, respectively. The sequence $cl(x)$ is $(2,0,0,1,0,\dots)$, since x has two closed components of genus 0 (2-spheres) and one component of genus 3.

Category $R\mathrm{Cob}_2$. Fix a commutative ring R and consider the pre-additive R-linear category $R\mathrm{Cob}_2$ freely generated by Cob_2. It has the same objects n as Cob_2, and morphisms from n to m in $R\mathrm{Cob}_2$ are linear combinations of morphisms from n to m in Cob_2 with coefficients in R and with composition induced from that in Cob_2. One can think of this construction, for an arbitrary category C, as analogous to passing from a group G to its group algebra $R[G]$ or from a semigroup G to its semigroup algebra. It results in an idempotented ring RC with a collection of mutually orthogonal idempotents (corresponding to identity morphisms), one for each object of C, as a substitute for the unit element; see [KS2].

Category Cob'_α for a sequence α. A more interesting category is obtained if we choose an infinite sequence of elements

$$\alpha = (\alpha_0,\alpha_1,\alpha_2,\dots) \tag{2.3}$$

of R and evaluate each closed component of genus k to α_k. For a closed surface S denote

$$\alpha(S) = \prod_{k\geq 0} \alpha_k^{c_k}, \tag{2.4}$$

where c_k is the number of components of S of genus k. The resulting monoidal category, denoted $\mathrm{Cob}'(\alpha)$ or Cob'_α, has the same objects n as the earlier categories. A morphism from n to m in Cob'_α is an R-linear combination of cobordisms from n circles to m circles in Cob_2 without closed components. Composition is given by concatenation followed by evaluating each closed component of genus k to α_k. We can informally refer to Cob'_α as the *α-prelinearization* of the category Cob_2.

There is the obvious "evaluation" or "reduction" functor $R\mathrm{Cob}_2 \to \mathrm{Cob}'_\alpha$, which is the identity on objects, that evaluates (or reduces) each closed component of genus k to α_k. The hom space $\mathrm{Hom}_{\mathrm{Cob}'_\alpha}(n,m)$ is a free R-module with a basis of cobordisms without closed components. Basis elements are parametrized by partitions in D_n^m with a nonnegative integer (genus) assigned to each part of the partition.

REMARK. Object 0 associated to the empty 1-manifold $\emptyset_1$ is the unit object of the monoidal categories $\mathrm{Cob}_2, R\mathrm{Cob}_2$, and Cob'_α. The commutative monoid of endomorphisms $\mathrm{End}_{\mathrm{Cob}_2}(0)$ is freely generated by isomorphism classes of closed oriented connected surfaces, one for each genus $g \geq 0$, and can be identified with the free abelian monoid on these generators. The commutative ring

$$\mathrm{End}_{R\mathrm{Cob}_2}(0) \cong R[\mathrm{End}_{\mathrm{Cob}_2}(0)]$$

is the semigroup algebra of the monoid $\mathrm{End}_{\mathrm{Cob}_2}(0)$. Also, $\mathrm{End}_{\mathrm{Cob}'_\alpha}(0) \cong R$.

Generating functions and the bilinear form. A sequence α is conveniently encoded by the generating function

$$Z_\alpha(T) = \sum_{n\geq 0} \alpha_n T^n \in R[\![T]\!]. \tag{2.5}$$

For a closely related construction see [Kh2], where to such $Z_\alpha(T)$ there is associated a family of R-modules $A_\alpha(n)$, for each $n \geq 0$, constructed via a bilinear form on the space of linear combinations of oriented 2-manifolds with boundary the disjoint union of n circles $\bigsqcup_n \mathbb{S}^1$. Namely, one considers the free R-module $\mathrm{Fr}(n)$ with a basis $\{[S]\}_S$ of oriented compact surfaces S with $\partial S \cong \bigsqcup_n \mathbb{S}^1$ (with the diffeomorphism fixed). On $\mathrm{Fr}(n)$ there is an R-bilinear form $(\,,)_n$ given on pairs of generators S_1, S_2 by gluing the two surfaces along the common boundary and evaluating via α:

$$([S_1],[S_2])_n = \alpha\Big((-S_1) \sqcup_\partial S_2\Big). \tag{2.6}$$

The state space of n circles is the quotient of $\mathrm{Fr}(n)$ by the kernel of this bilinear form:

$$A_\alpha(n) := \mathrm{Fr}(n)/\ker((\,,)_n). \tag{2.7}$$

This collection of R-modules is naturally a representation of the category Cob'_α, when the latter is viewed as an idempotented R-algebra with a system of mutually-orthogonal idempotents $\{1_n\}_{n\in\mathbb{N}}$. Namely, to the object n of

Cob'_α associate the R-module $A_\alpha(n)$. To a morphism given by a cobordism $x \in \mathrm{Cob}_2$ from n to m associate an R-module map

$$x_\alpha : A_\alpha(n) \to A_\alpha(m), \tag{2.8}$$

obtained directly from the construction in [Kh2], via the evaluations of x capped off by various oriented surfaces with n and m circles as the boundary. These morphisms over all $x \in \mathrm{Cob}_2$ provide a representation of Cob_2 and Cob'_α on the direct sum of R-modules

$$A_\alpha := \bigoplus_{n\geq 0} A_\alpha(n). \tag{2.9}$$

Monoidal structures on Cob_2 and Cob'_α are not used in this construction. A_α is a representation of the idempotented R-algebra underlying the category Cob'_α, in the sense of [KS1, KS2].

Category Cob_α **as a quotient by negligible morphisms.** The action of Cob'_α can be quotiented down to a smaller category. The categories $R\mathrm{Cob}_2$ and Cob'_α admit trace maps. Namely, given an element $x \in \mathrm{Hom}(n,n)$, a finite linear combination of cobordisms with n bottom and n top circles, close up opposite circles i and i', $1 \leq i \leq n$, by annuli to get a linear combination of closed cobordisms $\widehat{x}$ and then evaluate the result via α:

$$\mathrm{Tr}(x) = \alpha(\widehat{x}) \in R; \tag{2.10}$$

see Figure 4.

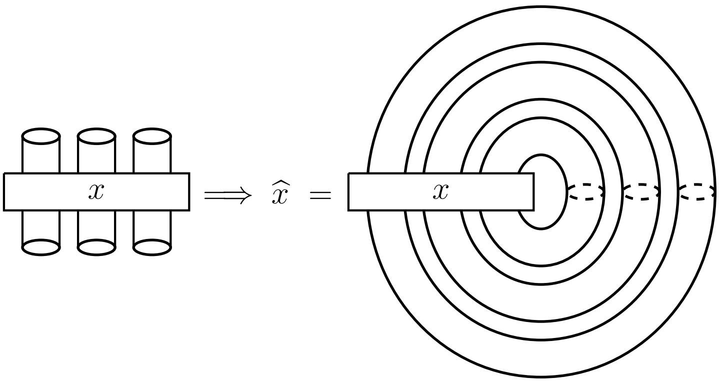

Fig. 4. Closing up a linear combination x of (n,n) cobordisms into a linear combination $\widehat{x}$ of closed cobordisms.

For morphisms $x \in \mathrm{Hom}(n,m)$ and $y \in \mathrm{Hom}(m,n)$ we have $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$.

A morphism $x \in \mathrm{Hom}(n,m)$ in the category Cob'_α (or in $R\mathrm{Cob}_2$) is called *negligible* if $\mathrm{Tr}(yx) = 0$ for any $y \in \mathrm{Hom}(m,n)$. Denote by $J(n,m) \subset \mathrm{Hom}(n,m)$ the subset of negligible morphisms from n to m. This subset is an R-submodule of $\mathrm{Hom}(n,m)$, and the union of $J(n,m)$, over all $n,m \geq 0$, is the tensor ideal J_α of Cob'_α. Define the category Cob_α to be the quotient of Cob'_α by this ideal,

$$\mathrm{Cob}_\alpha := \mathrm{Cob}'_\alpha / J_\alpha. \tag{2.11}$$

This category has objects $n \in \mathbb{Z}_+$, and

$$\mathrm{Hom}_{\mathrm{Cob}_\alpha}(n,m) = \mathrm{Hom}_{\mathrm{Cob}'_\alpha}(n,m)/J(n,m). \tag{2.12}$$

The category Cob_α is an R-linear tensor category with duals and a non-degenerate trace: for any $x \in \mathrm{Hom}_{\mathrm{Cob}_\alpha}(n,m)$, $x \neq 0$, there is $y \in \mathrm{Hom}_{\mathrm{Cob}_\alpha}(n,m)$ such that $\mathrm{Tr}(yx) \neq 0$. For information about ideals of negligible morphisms and corresponding quotient categories we refer the reader to [EO, BW].

Starting with the category $R\mathrm{Cob}_2$ instead of Cob'_α in this construction will result in the quotient category isomorphic to Cob_α.

The functor of modding out an R-linear tensor category with duals by the ideal of negligible morphisms is essentially the same operation as used in the universal construction [BHMV, Kh2], where one mods out by the kernel of the bilinear form. Thus, in the example above, there are isomorphisms of R-modules

$$\mathrm{Hom}_{\mathrm{Cob}_\alpha}(n,m) \cong \mathrm{Hom}_{\mathrm{Cob}_\alpha}(0,n+m) \cong A_\alpha(n+m), \tag{2.13}$$

with the first isomorphism given by bending the n bottom circles up; see Figure 5.

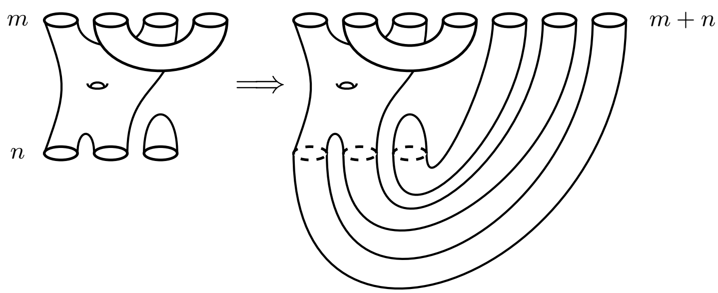

Fig. 5. Turning a morphism in $\mathrm{Hom}(n,m)$ into a morphism in $\mathrm{Hom}(0,n+m)$.

The space $J(0,n+m)$ of negligible morphisms in $\mathrm{Hom}_{\mathrm{Cob}'_\alpha}(0,n+m)$ is exactly the kernel of the bilinear form on $\mathrm{Hom}_{\mathrm{Cob}'_\alpha}(0,n+m)$ constructed via formula (2.6) for $n+m$ boundary circles, implying the second isomorphism above. It is easy to rewrite composition of morphisms in Cob_α via these isomorphisms and suitable cobordism maps.

We refer to the category Cob_α as the *α-linearization* of Cob_2 (and of the related categories $R\mathrm{Cob}_2$ and Cob'_α). These categories are part of the package of the universal construction or pairing (see [BHMV, Kh2] and closely related [FKN$^+$]), and can be defined in any dimension and in a variety of situations, given an evaluation of closed manifolds or similar objects (foams [Kh1, RW], manifolds with embedded submanifolds [FKN$^+$, KR], or other decorations).

When the commutative ring R is a field $\mathbf{k}$, it is observed in [Kh2] that the spaces $A_\alpha(n)$ are finite-dimensional for all n (equivalently, for some $n \geq 1$) iff the generating function (2.5) is a rational function in T. Equivalently, the representation (2.9) of Cob_α is locally finite-dimensional in a similar sense; see [KS1, KS2]. The case when the function $Z_\alpha(T)$ is rational seems especially interesting, for many reasons.

The R-modules $A_\alpha(n)$ and maps between them induced by cobordisms (see the discussion around (2.8)) define a representation of Cob_α viewed as an idempotented ring

$$B_\alpha = \bigoplus_{n,m\geq 0} 1_m \mathrm{Hom}_{\mathrm{Cob}_\alpha}(n,m)1_n; \tag{2.14}$$

see [KS1, KS2] for a general discussion. On the corresponding representation A_α in (2.8) idempotent 1_n acts as the projector onto $A_\alpha(n)$, and an element $x \in \mathrm{Hom}_{\mathrm{Cob}_\alpha}(n,m)$ acts by the corresponding map $A_\alpha(n) \to A_\alpha(m)$. When R is a field and $Z_\alpha(T)$ is rational, this representation is locally finite-dimensional.

Additive closure and the Karoubi envelope. It is useful to consider the additive Karoubi envelope Kob_α of Cob_α. First form the finite additive closure $\mathrm{Cob}_\alpha^\oplus$ of Cob_α by taking formal finite direct sums of objects n of Cob_α, and extending to morphisms in the obvious way. The additive closure has the zero object $\mathbf{0}$ different from the object 0. The latter is associated to the empty 1-manifold and comes from the corresponding object of Cob_α. The endomorphisms of the object $\mathbf{0}$ form the zero R-algebra, while $\mathrm{End}_{\mathrm{Cob}_\alpha^\oplus}(0) = \mathrm{End}_{\mathrm{Cob}_\alpha}(0) \cong R$.

Next, let $\underline{\mathrm{DCob}}_\alpha$ be the Karoubi envelope of $\mathrm{Cob}_\alpha^\oplus$. The six types of categories we have encountered so far are listed below:

$$\mathrm{Cob}_2 \to R\mathrm{Cob}_2 \to \mathrm{Cob}'_\alpha \to \mathrm{Cob}_\alpha \to \mathrm{Cob}_\alpha^\oplus \to \underline{\mathrm{DCob}}_\alpha. \tag{2.15}$$

The first arrow consists in allowing R-linear combinations of cobordisms. In the second arrow we evaluate closed surfaces of genus k to fixed elements α_k of R, over all $k \geq 0$. We can refer to this procedure as *α-prelinearization.* The third arrow consists in modding out Cob'_α by the ideal of negligible morphisms.

Like Cob'_α, the category $R\mathrm{Cob}_2$ also has the ideal I_α of negligible morphisms, via the trace given by α. The composition of the second and third arrows above can also be described as the quotient of $R\mathrm{Cob}_2$ by this ideal,

$$R\mathrm{Cob}_2 \to R\mathrm{Cob}_2/I_\alpha \cong \mathrm{Cob}_\alpha. \tag{2.16}$$

The fourth and the fifth arrows in (2.15) are fully faithful functors. The second, third and fourth categories are pre-additive, the fifth category is additive and the last category is additive and Karoubi-complete. All six categories are tensor (symmetric monoidal) and these five functors are monoidal.

3. Partition category and the Deligne category. Recall that R is a commutative ring. In the context of the partition category and the Deligne category ring R is often taken to be a field $\mathbf{k}$. Fix $t \in R$.

Partition category. The partition category Pa_t extends the notion of the partition algebra that originally appeared in Martin [M] and Jones [J]; see [HR, LS] for more information and references.

Objects n of the partition category are nonnegative integers and morphisms from n to m are R-linear combinations of decompositions D_n^m (also called partitions) of the set $\mathbb{N}_n^m$; see discussion around formula (2.2).

Diagrammatically, partitions are often denoted by marking n points on a horizontal line in the plane and m points on a parallel line above it. One connects these $n+m$ points by arcs, and the connected components of the resulting graph are the parts of the partition. Intersections of arcs are ignored. A partition usually has more than one such diagram. For instance, if $\{1,3,1'\}$ is a part of the partition, it can be described by two arcs $(1,3),(1,1')$ or two arcs $(1,3),(3,1')$, or all three arcs; see also Figure 9 (left) below demonstrating this indeterminacy. This diagrammatic description of partitions is standard in papers on the partition algebra and category; see for instance [LS]. Two examples of diagrammatic presentations are given in Figure 6.

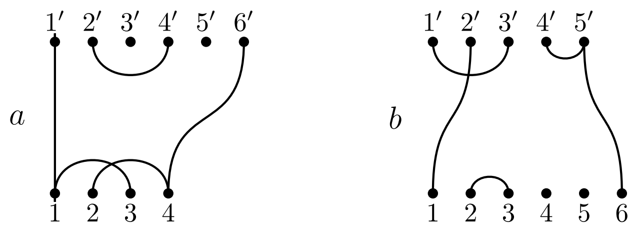

Fig. 6. Partitions $a = \{\{1,3,1'\},\{2,4,6'\},\{2',4'\},\{3'\},\{5'\}\} \in P_4^6$ and $b = \{\{1,2'\}, \{2,3\}, \{4\}, \{5\}, \{6,4',5'\}, \{1',3'\}\} \in P_6^5$. Notice multiple ways to display the same partition. For the subset $\{1,3,1'\}$ we depicted edges $(1,3)$ and $(1,1')$. Another possibility is to depict edges $(1,3)$ and $(3,1')$ or edges $(1,1')$ and $(3,1')$. In choosing a diagram for a partition it is natural to at least minimize the number of bottom-top edges, showing only one such edge for each subset that contains both bottom and top points.

Composition is given by concatenating diagrams (see Figure 7) and treating points in the middle that connect to bottom or top as 'pass through' points that vanish from the concatenation but are used before that to create the new partition. If there exist a connected component that consists entirely of points in the middle part of the diagram, it is removed and what is left is multiplied by t. This procedure is iterated until no such components are left.

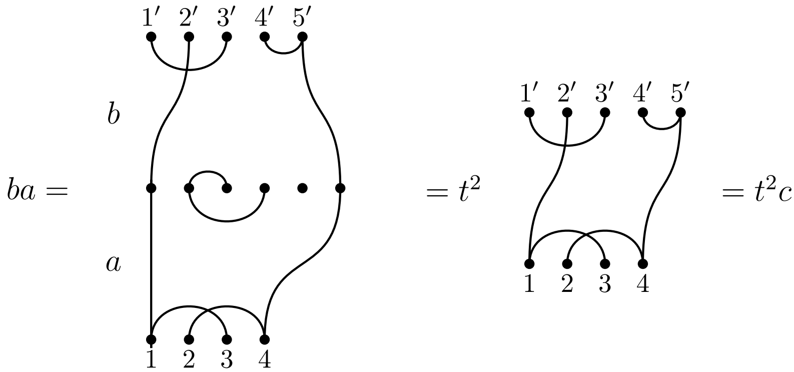

Fig. 7. Composition $ba = t^2c$, where the partition c is shown on the right, $c = \{\{1,3,2'\},\{2,4,4',5'\},\{1',3'\}\}$. The coefficient t^2 comes from removing two connected components in the middle of the ba diagram that connect to neither bottom nor top points.

Composition is then extended bilinearly to R-linear combinations of partitions. The resulting R-linear category Pa_t is symmetric monoidal, with the tensor product given on partitions by placing their diagrams in parallel.

Noah Snyder's diagrammatics for the partition category. A long time ago Noah Snyder [S] pointed out to one of us an alternative diagrammatics for the partition category [H]. Figure 8 shows conventional diagrammatics versus the Snyder diagrammatics for the standard generating morphisms of the partition category. One difference is the use of a trivalent vertex to depict the morphism from 2 to 1 corresponding to the partition $\{1,2,1'\}$ and the dual morphism from 1 to 2. This trivalent vertex as well as other configurations can be freely rotated in the plane. As in the usual diagrammatics, one allows intersections of distinct parts of the partition, thinking of them as virtual intersections.

Figure 9 displays one benefit of the Snyder calculus: the generating morphism (a) has an essentially unique minimal presentation.

It is convenient to introduce cup and cap diagrams (as additional generators), defined in the top row of Figure 10 via the original generators. Isotopy

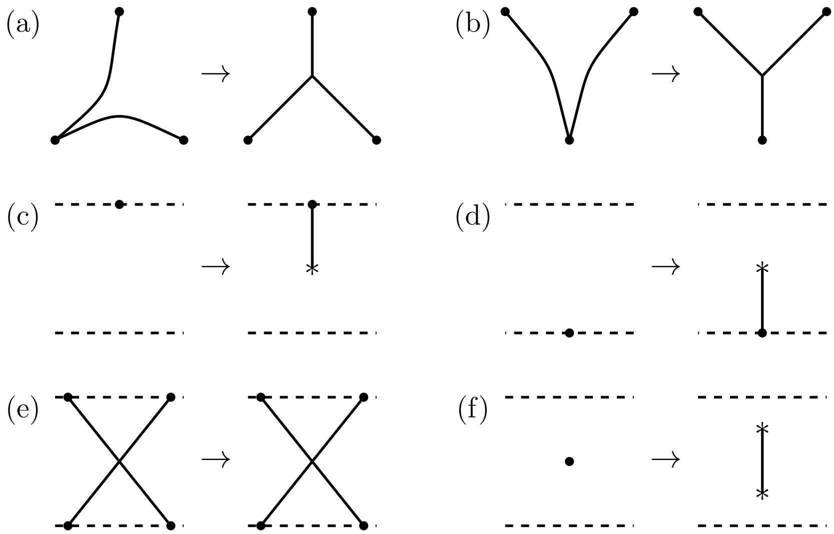

Fig. 8. Conventional and Snyder's generators for the partition category. Object 0 is shown by a dashed line without dots on it. In (c) and (d) we indicated the loose end of a strand by the $*$ symbol; other ways to depict the end are fine too. Likeng–Savage [LS] use a similar notation for the generators (c), (d). The element shown in (f) is a suitable composition of the generators (c) and (d) and evaluates to t in the partition category. The top and bottom dashed lines in the depiction of a diagram are optional and are not shown in the top row diagrams.

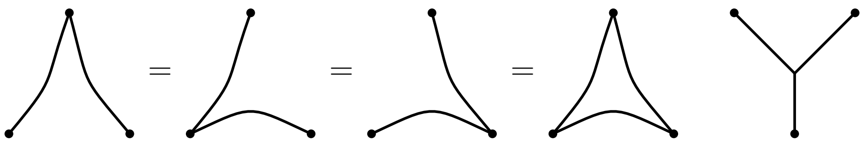

Fig. 9. Multiple ways to depict the generating morphism (a) in Figure 8 versus unique up to isotopy diagram in the Snyder graphical calculus.

relations on the generators are shown in the next two rows of Figure 10. Some other defining relations are shown in Figure 11. We leave it to the reader to convert a full set of relations as found in [C, Theorem 1] or [LS] into defining relations for the Snyder calculus.

The relation between 2D cobordisms and partitions is especially easy to see in the Snyder calculus. Thickening Snyder's trivalent graphs when viewed as graphs in $\mathbb{R}^3$ rather than in $\mathbb{R}^2$ results in a surface with boundary that corresponds to the partition. Intersection points of different components of the graph should be disregarded, as before, for instance by pulling the components slightly apart in $\mathbb{R}^3$ before thickening (the embedding into $\mathbb{R}^3$ is then forgotten). An example of a matching between Snyder's relations and diffeomorphisms of surfaces is shown in Figure 12.

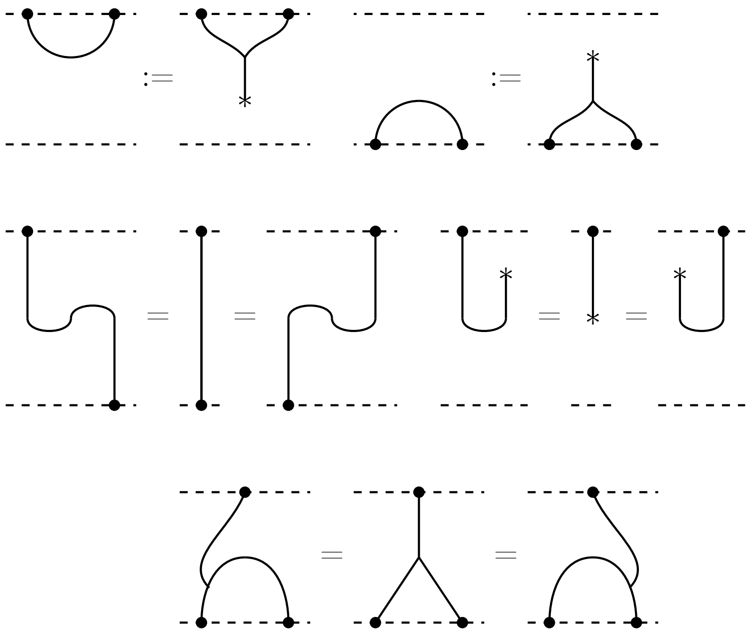

Fig. 10. Cup and cap diagrams and some isotopy relations in Snyder's diagrammatics.

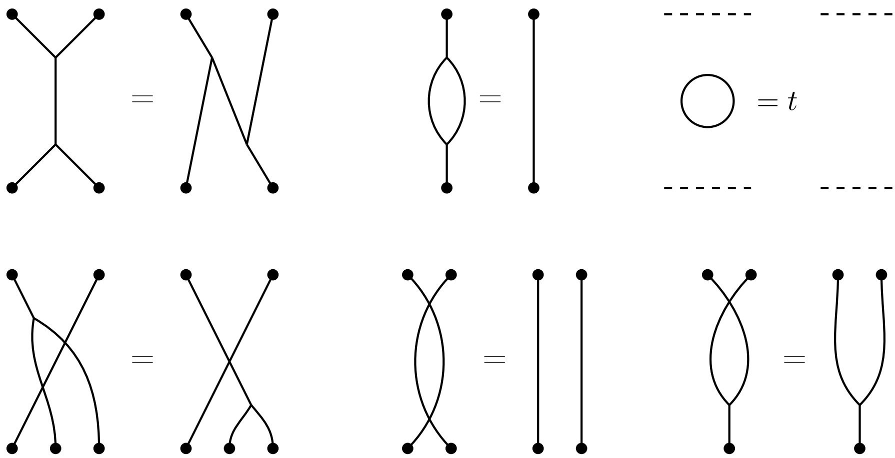

Fig. 11. Some other defining relations in the Snyder calculus.

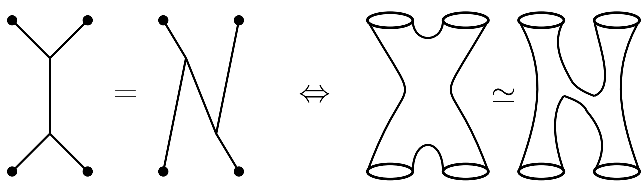

Fig. 12. One of the defining relations versus surface diffeomorphism.

From graphs to surfaces. The lifting, discussed in this paper and in [C], from partitions, which are graph-like objects, to two-dimensional cobordisms is analogous, in some rather naive way, to passing from Feynman diagrams (graph-like objects) to strings (two-dimensional objects):

$$\text{Feynman diagrams} \longrightarrow \text{Strings},$$
$$\text{Partition diagrams} \longrightarrow \text{2D cobordisms},$$

Of course, the complexity of mathematics hidden in the top arrow structures is orders of magnitude higher than those in the bottom arrow, discussed in the present paper.

Deligne category. Let us specialize the ground ring R to be a field $\mathbf{k}$ of characteristic 0. The Deligne category $\mathrm{Rep}(S_t)$ is the additive Karoubi envelope of the partition category Pa_t, $t \in \mathbf{k}$,

$$\mathrm{Rep}(S_t) = \mathrm{Kar}(\mathrm{Pa}_t^{\oplus}). \tag{3.1}$$

It is known to be semisimple when $t \notin \mathbb{Z}_+$. When $t = n \in \mathbb{Z}_+ \subset \mathbf{k}$, the Deligne category admits a nontrivial ideal J_n that consists of the *negligible morphisms*. A morphism $x \in \mathrm{Hom}(a,b)$ is negligible if $\mathrm{Tr}(yx) = 0$ for any $y \in \mathrm{Hom}(b,a)$. The category $\mathrm{Rep}(S_t)$ is a tensor category with duals, and the trace is straightforward to define. The trace in Pa_t on a diagram $\lambda \in D_m^m$ is given by identifying points i and i', $1 \le i \le m$. If r is the number of components in the resulting diagram, then $\mathrm{Tr}(\lambda) = t^r$.

The quotient $\mathrm{Rep}(S_n)/J_n$ is equivalent, as a tensor category, to the category $\mathbf{k}[S_n]$-mod of finite-dimensional representations of the symmetric group S_n,

$$\mathrm{Rep}(S_n)/J_n \cong \mathbf{k}[S_n]\text{-mod}. \tag{3.2}$$

The ideal J_t of negligible morphisms in $\mathrm{Rep}(S_t)$ is zero if $t \notin \mathbb{Z}_+$.

4. Generalized Deligne categories

Cobordisms and partitions. Consider the category Cob_2 of two-dimensional cobordisms. Given a morphism x from n to m, disregard its closed components and ignore genera of connected components with boundary. A connected component with boundary defines a subset among the set of boundary circles of x. The latter set can be identified with $\{1,\dots,n,1',\dots,m'\}$ (see Figure 3). Consequently, the union of connected components of x that have a non-empty boundary determines a partition in D_n^m. To a cobordism x from n to m we associate this partition in D_n^m, denoted $p(x)$.

To extend this assignment to a functor

$$F : R\mathrm{Cob}_2 \to \mathrm{Pa}_t \tag{4.1}$$

let $|cl(x)|$ be the number of connected components of x without boundary (closed components). The functor F is the identity on objects $n \in \mathbb{Z}_+$ of $R\mathrm{Cob}_2$ and Pa_t and

$$F(x) = t^{|cl(x)|}p(x) \tag{4.2}$$

on cobordisms. It is then extended R-linearly to linear combinations of cobordisms. Notice that F ignores the genera of all components of x. Clearly, F is a tensor (symmetric monoidal) functor. This construction can be found in Comes [C, Section 2.2]. One can think of Pa_t as the quotient of $R\mathrm{Cob}_2$ by skein relations in Figure 2.

Recall the categories Cob'_α and Cob_α introduced earlier and associated to a sequence α, where a closed surface of genus g evaluates to $\alpha_g \in R$. Let $\alpha(t) = (t, t, t, \dots)$ be the constant sequence associated to $t \in \mathbf{k}$. The evaluation $\alpha(t)$ associates t to any oriented connected closed surface irrespectively of its genus. The relations in Figure 2 hold in the category $\mathrm{Cob}_{\alpha(t)}$, and they hold in $\mathrm{Cob}'_{\alpha(t)}$ when restricted to closed components. Consequently, there are natural tensor functors

$$\mathrm{Cob}'_{\alpha(t)} \xrightarrow{F'_t} \mathrm{Pa}_t \xrightarrow{F''_t} \mathrm{Cob}_{\alpha(t)} \tag{4.3}$$

between these three categories. These functors are the identities on objects, $F'_t(n) = n$, $F''_t(n) = n$. The first functor forgets about handles of each component of a cobordism S, evaluates each closed component to t, and associates a partition of the set $\mathbb{N}_n^m$ in (2.2) to S according to subsets of boundary circles bounded by connected components of S.

The second functor F''_t exists by an earlier discussion, due to the definition of $\mathrm{Cob}_{\alpha(t)}$ via the quotient by the kernel of a bilinear form. It identifies $\mathrm{Cob}_{\alpha(t)}$ with the quotient of Pa_t by the ideal of negligible morphisms.

The Deligne category. Starting with the functor F''_t and passing to additive Karoubi closures results in a functor

$$F_t : \mathrm{Rep}(S_t) \to \mathrm{Kob}_{\alpha(t)} \tag{4.4}$$

from the Deligne category to $\mathrm{Kob}_{\alpha(t)} = \mathrm{Kar}(\mathrm{Cob}^{\oplus}_{\alpha(t)})$, the additive Karoubi closure of the category $\mathrm{Cob}_{\alpha(t)}$. From the structure theory of the Deligne categories we can conclude that F_t consists of taking the quotient of $\mathrm{Rep}(S_t)$ by the ideal J_t of negligible morphisms and induces an equivalence

$$\mathrm{Rep}(S_t)/J_t \xrightarrow{\cong} \mathrm{Kob}_{\alpha(t)}. \tag{4.5}$$

Notice that there is a difference in the order in which we take the additive Karoubi closure and mod out by the negligible morphisms. On the Deligne category side, one first forms the additive Karoubi closure and then mods out by the negligible morphisms. On the $\mathrm{Kob}_{\alpha(t)}$ side, one first mods out by the negligible morphisms to get the category $\mathrm{Cob}_{\alpha(t)}$ and then forms the additive Karoubi closure. It is not clear whether this may produce a discrepancy in more general cases, but for Deligne categories (and with R a field $\mathbf{k}$ of characteristic 0) this change of order results in equivalent categories and makes no difference.

Generalizations. We obtain an immediate generalization of the categories $\mathrm{Rep}(S_t)/J_t$ by changing from the constant sequence $\alpha(t)$ in (1.2) to a more general sequence α. The most interesting case is when the generating function $Z_\alpha(T)$ of α (see (1.1)) is a rational function, a ratio of two coprime polynomials

$$Z_\alpha(T) = \frac{P(T)}{Q(T)} \tag{4.6}$$

with coefficients in $\mathbf{k}$. In this case the categories Cob_α and $\underline{D\mathrm{Cob}}_\alpha$ have finite-dimensional hom spaces. We can view $\underline{D\mathrm{Cob}}_\alpha$ as a natural generalization of the quotient category $\mathrm{Rep}(S_t)/J_t$. For generic t, the ideal J_t is zero, and then the quotient category is the Deligne category.

Theorem 4.1. *The categories $\underline{D\mathrm{Cob}}_\alpha$ are tensor $\mathbf{k}$-linear Karoubi-closed additive categories. When $Z_\alpha(T)$ is rational, morphism spaces in $\underline{D\mathrm{Cob}}_\alpha$ are finite-dimensional.*

It is an interesting project to investigate the categories $\underline{D\mathrm{Cob}}_\alpha$ when the generating function $Z_\alpha(T)$ is rational. Deligne category quotients are recovered for the rational function in (1.7).

Notice that the categories $\underline{D\mathrm{Cob}}_\alpha$ deliver generalizations of the quotients $\mathrm{Rep}(S_t)/J_t$ rather than of the Deligne categories $\mathrm{Rep}(S_t)$ themselves. To remedy this discrepancy, we instead pass from Cob'_α to $\underline{D\mathrm{Cob}}_\alpha$ in one more step, when $R = \mathbf{k}$ is a field and the partition function $Z_\alpha(t)$ is rational as in (4.6). Let

$$N = \deg(P(T)), \qquad M = \deg(Q(T)), \qquad K = \max(N+1, M), \tag{4.7}$$

$$Q(T) = 1 - e_1 T + e_2 T^2 + \cdots + (-1)^M e_M T^M, \qquad e_i \in \mathbf{k}, \tag{4.8}$$

as in [Kh2, Section 2.4]. Then in the state space $A_\alpha(1)$ of a circle we have the equality

$$x^K - e_1 x^{K-1} + e_2 x^{K-2} - \cdots + (-1)^M e_M x^{K-M} = 0, \tag{4.9}$$

where x denotes a 2-torus with one boundary component. The power x^k of x represents a surface of genus k with one boundary component, with multiplication in $A_\alpha(1)$ given by the pants cobordism [Kh2]. Equation (4.9) gives a skein relation in the category Cob_α which reduces a collection of K handles on a single component to a linear combination of collections of $K-1$, $K-2, \dots, K-M$ handles.

For rational α, start with the pre-additive category Cob'_α of Section 2, and diagram (2.15) that shows the position of Cob'_α in the chain of categories and functors associated with α. In Cob'_α only closed components are reduced to elements α_k of $\mathbf{k}$. Hom spaces $\mathrm{Hom}(n,m)$ in Cob'_α are infinite-dimensional $\mathbf{k}$-vector spaces, unless $n=m=0$, with a basis of diffeomorphism classes rel boundary of all cobordisms without closed components. Thus, a basis element is described by a decomposition in D_n^m and a choice of genus for each connected component.

Define the category PCob_α to have the same objects $n \geq 0$ as Cob'_α and morphism spaces to be quotients of those in Cob'_α by the skein relations corresponding to equation (4.9). That is, we set this linear combination of morphisms (cobordisms) to zero in the quotient category. Applying this relation we reduce a component which contains at least K handles to components with fewer handles. In particular, any morphism in Cob'_α reduces to a $\mathbf{k}$-linear combination of cobordisms with no closed components and at most $K-1$ handles on each connected component. Diffeomorphism classes rel boundary of these cobordisms are in bijection with elements of the set $D_n^m(<K)$ of partitions in D_n^m with an integral weight between 0 and $K-1$ associated to each part (number of handles of the component, on the cobordism side). Recall that to a partition x we associated a cobordism $p(x)$ (see discussion preceding formula (4.2). We can now extend this association, also denoted p, and assign to a partition x with nonnegative integral weights of its parts the cobordism $p(x)$ by starting with the cobordism for the partition without weights and adding the number of handles equal to the weight to each two-sphere with boundary holes.

Computations in [Kh2, Section 2.4] imply that relation (4.9) is compatible with evaluation α applied to closed cobordisms. In particular, no additional relations on cobordisms appear and elements of the set $D_n^m(<K)$, converted to cobordisms, provide a basis of $\mathrm{Hom}_{\mathrm{PCob}_\alpha}(n,m)$.

PROPOSITION 4.2. *The hom space* $\mathrm{Hom}(n,m)$ *in* PCob_α *has a basis* $\{p(x)\}$*, over all* $x \in D_n^m(<K)$*.*

In particular, hom spaces in PCob_α are finite-dimensional. We can now insert category PCob_α into the chain of six categories in (2.15):

$$\mathrm{Cob}_2 \to R\mathrm{Cob}_2 \to \mathrm{Cob}'_\alpha \to \mathrm{PCob}_\alpha \to \mathrm{Cob}_\alpha \to \mathrm{Cob}^{\oplus}_\alpha \to \underline{D\mathrm{Cob}}_\alpha. \tag{4.10}$$

It fits in between Cob'_α and Cob_α. The category PCob_α is the quotient of Cob'_α by the skein relation (4.9). Like any other category in this chain, it is tensor (symmetric monoidal). The trace form on Cob'_α descends to that on PCob_α. The quotient of PCob_α by the ideal of negligible morphisms relative to this trace form is isomorphic to Cob_α (isomorphic and not just equivalent, since these categories are essentially skeletal and have very few objects). As already mentioned, this insertion is possible when $Z_\alpha(T)$ is a rational function and R is a field.

The category PCob_α generalizes the partition category Pa_t. The partition category Pa_t is isomorphic to $\mathrm{PCob}_{\alpha(t)}$ for the constant sequence $\alpha(t) = (t, t, \dots)$. More generally, choosing K in (4.7) fixes the size of homs in PCob_α, analogously to independence of dimensions of homs in Pa_t on t, given by the number of partitions (the Bell number). When $K = 1$, the dimensions of hom spaces in PCob_α are also given by the number of partitions in D_n^m, and for $K > 1$ by the number of weighted partitions as discussed above.

To get from PCob_α to the analogue of the Deligne category, pass to the additive Karoubi closure to get an additive and idempotent-complete category

$$\mathrm{DCob}_\alpha := \mathrm{Kar}(\mathrm{PCob}_\alpha^\oplus). \tag{4.11}$$

Chain (4.10) of functors can be upgraded to a commutative diagram of functors

$$\begin{array}{ccccccccccc} \mathrm{Cob}_2 & \xrightarrow{e} & R\mathrm{Cob}_2 & \longrightarrow & \mathrm{Cob}'_\alpha & \longrightarrow & \mathrm{PCob}_\alpha & \longrightarrow & \mathrm{PCob}_\alpha^\oplus & \longrightarrow & \mathrm{DCob}_\alpha \\ & & & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & & & \mathrm{Cob}_\alpha & \longrightarrow & \mathrm{Cob}_\alpha^\oplus & \longrightarrow & \underline{D\mathrm{Cob}}_\alpha \end{array} \tag{4.12}$$

where the chain (4.10) is given by the (left, left, left, down, left, left) sequence of arrows. Two new categories are added in the upper right. Vertical down arrows are quotients by the ideals of negligible morphisms. Both squares in the diagram are commutative.

Notice that first modding out PCob_α by the negligible morphisms to get Cob_α and then taking the Karoubi envelope $\underline{D\mathrm{Cob}}_\alpha$ compared to first taking the Karoubi envelope DCob_α and then modding out by the negligible morphisms does not produce any extra idempotents. This is due to the easy-to-check idempotent lifting property that holds for any finite-dimensional algebra B over $\mathbf{k}$ and any 2-sided ideal $J \subset B$ (not necessarily nilpotent). Any idempotent in B/J lifts to an idempotent in B. Endomorphism algebras of objects in $\mathrm{PCob}_\alpha^\oplus$ are finite-dimensional over $\mathbf{k}$. For the ideal J one would take the ideal of negligible endomorphisms of an object in $\mathrm{PCob}_\alpha^\oplus$.

To summarize, the chain of three categories and two functors (the partition category, the Deligne category, and its quotient by the negligible morphisms)

$$\mathrm{Pa}_t \to \mathrm{Rep}(S_t) \to \mathrm{Rep}(S_t)/J_t \tag{4.13}$$

generalizes to a similar chain

$$\mathrm{PCob}_\alpha \to \mathrm{DCob}_\alpha \to \underline{D\mathrm{Cob}}_\alpha \tag{4.14}$$

for any sequence α with rational power series $Z_\alpha(T)$. Specializing to the constant series $\alpha(t)$ and rational function $t/(1-T)$ recovers the original setup (4.13).

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Mikhail Khovanov
Department of Mathematics
Columbia University
New York, NY 10027, USA
E-mail: khovanov@math.columbia.edu

Radmila Sazdanovic
Department of Mathematics
North Carolina State University
Raleigh, NC 27696-8205, USA
E-mail: rsazdan@ncsu.edu